

# Characterizing Ideal Weighted Threshold Secret Sharing

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## Abstract

Weighted threshold secret sharing was introduced by Shamir in his seminal work on secret sharing. In such settings, there is a set of users where each user is assigned a positive weight. A dealer wishes to distribute a secret among those users so that a subset of users may reconstruct the secret if and only if the sum of weights of its users exceeds a certain threshold. A secret sharing scheme is ideal if the size of the domain of shares of each user is the same as the size of the domain of possible secrets (this is the smallest possible size for the domain of shares). The family of subsets authorized to reconstruct the secret in a secret sharing scheme is called an access structure. An access structure is ideal if there exists an ideal secret sharing scheme that realizes it. It is known that some weighted threshold access structures are not ideal, while other nontrivial weighted threshold access structures do have an ideal scheme that realizes them. In this work we characterize all weighted threshold access structures that are ideal. We show that a weighted threshold access structure is ideal if and only if it is a hierarchical threshold access structure (as introduced by Simmons), or a tripartite access structure (these structures, that we introduce here, generalize the concept of bipartite access structures due to Padró and Sáez), or a composition of two ideal weighted threshold access structures that are defined on smaller sets of users. We further show that in all those cases the weighted threshold access structure may be realized by a linear ideal secret sharing scheme. The proof of our characterization relies heavily on the strong connection between ideal secret sharing schemes and matroids, as proved by Brickell and Davenport.

## 1 Introduction

A *threshold secret sharing scheme* enables a dealer to distribute a secret among a set of users, by giving each user a piece of information called a *share*, such that only large sets of users will be able to reconstruct the secret from the shares that they got, while smaller sets gain no information on the secret. Threshold secret sharing schemes were introduced and efficiently implemented, independently, by Blakley [5] and Shamir [25]. Efficient threshold secret sharing schemes were used in many cryptographic applications, e.g., Byzantine agreement [23], secure multiparty computations [3, 10], and threshold cryptography [12].

In this paper we deal with *weighted* threshold secret sharing schemes. In these schemes, considered already by Shamir [25], the users are not of the same status. That is, each user is assigned a positive weight and a set can reconstruct the secret if the sum of weights assigned to its users exceeds a certain threshold. As a motivation, consider sharing a secret among the shareholders of some company, each holding a different amount of shares. Such settings are closely related to the concept of *weighted threshold functions*, which play an important role in complexity theory and learning theory.<sup>1</sup>

Ito, Saito, and Nishizeki [13] generalized the notion of secret sharing such that there is an arbitrary monotone collection of authorized sets, called the *access structure*. The requirements are that only sets in the access structure are allowed to reconstruct the secret, while sets that are not in the access structure should gain no information on the secret. A simple argument shows that in every secret sharing scheme, the domain of possible shares for each user is at least as large as the domain of possible secrets (see [16]). Shamir's threshold secret sharing scheme is *ideal* in the sense

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<sup>1</sup>A weighted threshold function is a boolean function  $f : \{0, 1\}^n \rightarrow \{0, 1\}$  where each variable is assigned a positive weight and  $f(x_1, \dots, x_n) = 1$  iff the sum of weights that are assigned to the variables whose value is 1 exceeds a given threshold.

that the domain of shares of each user coincides with the domain of possible secrets. Ideal secret sharing schemes are the most space-efficient schemes. Some access structures do not have any ideal secret sharing schemes that realizes them [4]. Namely, some access structures demand share domains that are larger than the domain of secrets. Access structures that may be realized by an ideal secret sharing scheme are called ideal. Ideal secret sharing schemes and ideal access structures have been studied in, e.g., [1, 6, 7, 14, 17, 18, 20, 22, 24, 29, 32]. Ideal access structures are known to have certain combinatorial properties. In particular, there is a strong relation between ideal access structures and matroids [7].

While threshold access structures are ideal, weighted threshold access structures are not necessarily so. For example, the access structure on four users with weights 1, 1, 1, and 2, and a threshold of 3, has no ideal secret sharing scheme (see Example 4.9 for a proof). Namely, in any perfect secret sharing scheme that realizes this access structure, the share domain of at least one user is larger than the domain of secrets. On the other hand, there exist ideal weighted threshold access structures, other than the trivial threshold ones. For example, consider the access structure on nine users, where the weights are 16, 16, 17, 18, 19, 24, 24, 24, and 24 and the threshold is 92. Even though this access structure seems more complicated than the above access structure, it has an ideal secret sharing scheme (see Example 7.5). Another example of an ideal weighted threshold access structure is the one having weights 1, 1, 1, 1, 1, 3, 3, and 3 and threshold 6 (see Example 8.10).

In this paper we give a combinatorial characterization of ideal weighted threshold access structures. We show that if a weighted threshold access structure is ideal then one of the following statements hold:

1. It is a *multilevel* or *hierarchical* threshold access structure. This type of access structures was introduced by Simmons in [26]. In such settings, the users are divided into a hierarchy of levels, and each level has an associated threshold. A set of users is authorized if it has a subset whose size is at least the threshold that corresponds to the level of the lowest member in that subset.
2. It is a *tripartite* access structure. In such access structures, the set of users is partitioned into three subsets, and a given set is authorized iff the number of users that it has from each of the three subsets satisfy some threshold conditions.
3. The set of users is composed of *strong* users (users with larger weights) and *weak* users (users with smaller weights) and the access structure is a combinatorial composition of two access structures, one that is defined over the subset of strong users and another that is defined over the subset of weak users.

(Formal definitions are given in Section 3.)

Most of the access structures that play part in our characterization are well known [22, 26, 32]. As for the tripartite access structures, to the best of our knowledge they are introduced herein for the first time and we design ideal secret sharing schemes for their realization.

The present study generalizes the work of Morillo, Padró, Sáez, and Villar [20] who characterized the ideal weighted threshold access structures in which all the minimal authorized sets have at most two users. The proof of our characterization relies heavily on the strong connection between ideal secret sharing schemes and matroids, as presented in [7]. We utilize results regarding the structure of matroids in order to understand the structure of ideal weighted threshold access structures. An important tool in our analysis is composition of ideal access structures. When composing two access structures, defined on two disjoint sets, one gets an access structure on a larger set of users. The resulting access structure is ideal if and only if the original two are ideal. This enables us to characterize ideal weighted threshold access structures in a recursive manner, as described above. Composition of access structures was studied, e.g., in [1, 4, 8, 11, 19, 31].

**Related Work.** Secret sharing schemes for general access structures were defined by Ito, Saito, and Nishizeki in [13]. More efficient schemes were presented in, e.g., [4, 6, 15, 28]. We refer the reader to [27, 30] for extensive surveys on secret sharing schemes. However, for most access structures the known secret sharing schemes are highly inefficient, that is, the size of the shares is exponential in  $n$ , the number of users. It is not known whether better schemes exist. For weighted threshold access structures the situation is somewhat better. Shamir [25] proposed a weighted threshold secret sharing scheme that is based on his ideal threshold secret sharing scheme. In that scheme, the ratio between the share size of each user and the size of the secret equals the weight assigned to that user, which may be exponential in the number of users  $n$ . In a recent work [2], secret sharing schemes were constructed for arbitrary weighted threshold access structures in which the shares are of size  $O(n^{\log n})$ . Furthermore, under reasonable computational

assumptions, a secret sharing scheme with computational security was constructed in [2] for every weighted threshold access structure with a polynomial share size.

**Organization.** We begin in Section 2 by supplying the necessary definitions. Then, in Section 3, we state our characterization theorem and outline its proof. We proceed to describe in Section 4 the connection between matroids and ideal secret sharing, and then prove, in Section 5, several properties of matroids that are associated with weighted-threshold access structures. Thereafter, we discuss the connection between ideal weighted threshold access structures and two families of access structures: hierarchical threshold access structures in Section 6, and tripartite access structures in Section 7. Finally, in Section 8 we complete the proof of the characterization theorem by proving that if an ideal weighted threshold access structure is not hierarchical nor tripartite then it is a composition of two access structures on smaller sets of users.

## 2 Definitions and Notations

### 2.1 Secret Sharing

**Definition 2.1 (Access Structure)** Let  $U = \{u_1, \dots, u_n\}$  be a set of users. A collection  $\Gamma \subseteq 2^U$  is monotone if  $B \in \Gamma$  and  $B \subseteq C$  imply that  $C \in \Gamma$ . An access structure is a monotone collection  $\Gamma \subseteq 2^U$  of non-empty subsets of  $U$ . Sets in  $\Gamma$  are called authorized, and sets not in  $\Gamma$  are called unauthorized. A set  $B$  is called a minterm of  $\Gamma$  if  $B \in \Gamma$ , and for every  $C \subsetneq B$ , the set  $C$  is unauthorized. A user  $u$  is called self-sufficient if  $\{u\} \in \Gamma$ . A user is called redundant if there is no minterm that contains it. An access structure is called connected if it has no redundant users.

**Definition 2.2 (Secret-Sharing Scheme)** Let  $S$  be a finite set of secrets, where  $|S| \geq 2$ . An  $n$ -user secret-sharing scheme  $\Pi$  with domain of secrets  $S$  is a randomized mapping from  $S$  to a set of  $n$ -tuples  $\prod_{i=1}^n S_i$ , where  $S_i$  is called the share-domain of  $u_i$ . A dealer shares a secret  $s \in S$  among the  $n$  users of some set  $U$  according to  $\Pi$  by first sampling a vector of shares  $\Pi(s) = (s_1, \dots, s_n) \in \prod_{i=1}^n S_i$ , and then privately communicating each share  $s_i$  to the user  $u_i$ . We say that  $\Pi$  realizes an access structure  $\Gamma \subseteq 2^U$  if the following two requirements hold:

**CORRECTNESS.** The secret  $s$  can be reconstructed by any authorized set of users. That is, for any set  $B \in \Gamma$  (where  $B = \{u_{i_1}, \dots, u_{i_{|B|}}\}$ ), there exists a reconstruction function  $\text{RECON}_B : S_{i_1} \times \dots \times S_{i_{|B|}} \rightarrow S$  such that for every  $s \in S$  and for every possible value of  $\Pi_B(s)$ , the restriction of  $\Pi(s)$  to its  $B$ -entries,

$$\text{RECON}_B(\Pi_B(s)) = s.$$

**PRIVACY.** Every unauthorized set can learn nothing about the secret (in the information theoretic sense) from their shares. Formally, for any set  $C \notin \Gamma$ , for every two secrets  $a, b \in S$ , and for every possible  $|C|$ -tuple of shares  $\langle s_i \rangle_{u_i \in C}$ :

$$\Pr[\Pi_C(a) = \langle s_i \rangle_{u_i \in C}] = \Pr[\Pi_C(b) = \langle s_i \rangle_{u_i \in C}].$$

In every secret-sharing scheme, the size of the domain of shares of each user is at least the size of the domain of the secrets [16], namely  $|S_i| \geq |S|$  for all  $i \in [n]$ . This motivates the next definition.

**Definition 2.3 (Ideal Access Structure)** A secret-sharing scheme with domain of secrets  $S$  is ideal if the domain of shares of each user is  $S$ . An access structure  $\Gamma$  is ideal if for some finite domain of secrets  $S$  there exists an ideal secret sharing scheme realizing it.

In the ideal schemes for weighted threshold access structures that we construct in this paper, the size of the domain of secrets is at most  $2^{\text{poly}(n)}$ . This is a comfortable bound since it implies that both the secret and the shares may be represented by  $\text{poly}(n)$  bits.

Most previously known secret sharing schemes are *linear*. The concept of linear secret sharing schemes was introduced by Brickell [6] in the ideal setting and was later generalized to non-ideal schemes. Linear schemes are equivalent to monotone span programs [15]. For simplicity we only define ideal linear schemes.

**Definition 2.4 (Ideal Linear Secret Sharing Scheme)** Let  $\mathbb{F}$  be a finite field. An ideal linear secret sharing scheme over  $\mathbb{F}$  takes the following form: The domain of secrets and shares is  $S = \mathbb{F}$ . The scheme is specified by  $n + 1$  vectors in  $\mathbb{F}^t$  for some integer  $t$ : a vector  $\mathbf{T}_i$  per user  $u_i \in U$ ,  $i \in [n]$ , and a so-called target vector  $\mathbf{T}$ . To share a secret  $s \in \mathbb{F}$ , the dealer chooses a random vector  $\mathbf{R} \in \mathbb{F}^t$  such that  $\mathbf{R} \cdot \mathbf{T} = s$  and then the share of user  $u_i$  is  $\mathbf{R} \cdot \mathbf{T}_i$ .

The next theorem characterizes the access structure that is realized by a linear secret sharing scheme.

**Theorem 2.5 ([6, 15])** A linear secret sharing scheme with vectors  $\{\mathbf{T}_i\}_{i \in [n]}$  and  $\mathbf{T}$  realizes the following access structure  $\Gamma$ :

$$A \in \Gamma \text{ if and only if } \mathbf{T} \in \text{span} \{\mathbf{T}_i : u_i \in A\}.$$

## 2.2 Weighted Threshold Access Structures

In this paper we concentrate on special access structures, so-called weighted threshold access structures, that were already introduced in [25].

**Definition 2.6 (Weighted Threshold Access Structure – WTAS)** Let  $w : U \rightarrow \mathbb{N}$  be a weight function on  $U$  and  $T \in \mathbb{N}$  be a threshold. Define  $w(A) := \sum_{u \in A} w(u)$  and  $\Gamma = \{A \subseteq U : w(A) \geq T\}$ . Then  $\Gamma$  is called a weighted threshold access structure (WTAS) on  $U$ .

It is easy to see that Definition 2.6 does not restrict generality when assuming that the weights and the threshold are integers. In other words, given a WTAS with real positive weights and threshold, there exist integer weights and threshold that induce the very same access structure.

## 2.3 Terminology and Notations

Throughout this paper we assume that the users are ordered in a nondecreasing order according to their weights, i.e.,

$$w(u_1) \leq w(u_2) \leq \dots \leq w(u_n).$$

Let  $A = \{u_{i_j}\}_{1 \leq j \leq k}$  be an ordered subset of  $U$ , where  $1 \leq i_1 < \dots < i_k \leq n$ . In order to avoid two-levelled indices, we will denote the users in such a subset with the corresponding lower-case letter, namely,  $A = \{a_j\}_{1 \leq j \leq k}$ . We denote the first (lightest) and last (heaviest) users of  $A$  by  $A_{\min} = a_1$  and  $A_{\max} = a_k$  respectively. For an arbitrary ordered subset  $A$  we let  $A_{s,t} = \{a_j\}_{s \leq j \leq t}$  denote a run-subset. If  $s > t$  then  $A_{s,t} = \emptyset$ . Two types of runs that we shall meet frequently are prefixes and suffixes. A prefix of a subset  $A$  is a run-subset of the form  $A_{1,\ell}$ , while a suffix takes the form  $A_{\ell,k}$ , where  $1 \leq \ell \leq k$ . A suffix  $A_{\ell,k}$  is a proper suffix of  $A_{1,k}$  if  $\ell > 1$ .

We conclude this section by introducing the precedence relation  $\prec$ . When applied to users,  $u_i \prec u_j$  indicates that  $i < j$  (and, in particular,  $w(u_i) \leq w(u_j)$ ). This relation induces the following lexicographic order on subsets of  $U$ :

- $\emptyset \prec A$  for all  $A \subseteq U$  such that  $A \neq \emptyset$ .
- If  $a_1 \prec b_1$  then  $A \prec B$ ; if  $b_1 \prec a_1$  then  $B \prec A$ ; otherwise,  $A \prec B$  if and only if  $(A \setminus \{a_1\}) \prec (B \setminus \{b_1\})$ .

## 3 Characterizing Ideal WTASs

The main result of this paper is a combinatorial characterization of ideal WTASs. We define in Definitions 3.1–3.4 the building blocks that play an essential role in this characterization. Using these definitions, we state Theorem 3.5, our main result, that characterizes ideal WTASs. We then outline the proof of that theorem, where the full proof is given in the subsequent sections.

### 3.1 Building Blocks

**Definition 3.1 (Hierarchical Threshold Access Structure – HTAS)** Let  $m$  be an integer,  $U = \bigcup_{i=1}^m L_i$  be a partition of the set of users into a hierarchy of  $m$  disjoint levels, and  $\{k_i\}_{1 \leq i \leq m}$  be a sequence of decreasing thresholds,  $k_1 > k_2 > \dots > k_m$ . These hierarchy and sequence of thresholds induce a hierarchical threshold access structure (HTAS) on  $U$ :

$$\Gamma_H = \left\{ A \subseteq U : \text{There exists } i \in [m] \text{ such that } \left| A \cap \bigcup_{j=i}^m L_j \right| \geq k_i \right\}.$$

That is, a set  $A \subseteq U$  is in  $\Gamma_H$  if and only if it contains at least  $k_i$  users from the  $i$ th level and above, for some  $i \in [m]$ .

The family of HTASs was introduced by Simmons in [26] and further studied by Brickell who proved their ideality [6]. An explicit ideal scheme for these access structures was constructed in [32].

**Remark 3.2** Without loss of generality, we assume that  $|L_i| > k_i - k_{i+1}$  for every  $i \in [m-1]$ , and  $|L_m| \geq k_m$ . Indeed, if  $|L_i| \leq k_i - k_{i+1}$  for some  $i \in [m-1]$ , then the  $i$ th threshold condition in the HTAS definition implies the  $(i+1)$ th threshold condition and, consequently, the  $i$ th condition is redundant.

**Definition 3.3 (Tripartite Access Structure – TPAS)** Let  $U$  be a set of  $n$  users, such that  $U = A \cup B \cup C$ , where  $A$ ,  $B$ , and  $C$  are disjoint, and  $A$  and  $C$  are nonempty. Let  $m, d, t$  be positive integers such that  $m > t$ . Then the following is a tripartite access structure (TPAS) on  $U$ :

$$\Delta_1 = \{X \subseteq U : (|X| \geq m \text{ and } |X \cap (B \cup C)| \geq m - d) \text{ or } |X \cap C| \geq t\},$$

Namely, a set  $X$  is in  $\Delta_1$  if either it has at least  $m$  users,  $(m-d)$  of which are from  $B \cup C$ , or it has at least  $t$  users from  $C$ . If  $|B| \leq d + t - m$ , then the following is another type of a tripartite access structure:

$$\Delta_2 = \{X \subseteq U : (|X| \geq m \text{ and } |X \cap C| \geq m - d) \text{ or } |X \cap (B \cup C)| \geq t\}.$$

That is, a set  $X$  is in  $\Delta_2$  if either it has at least  $m$  users,  $(m-d)$  of which are from  $C$ , or it has at least  $t$  users from  $B \cup C$ .

TPASs, introduced herein, generalize the concept of bipartite access structure that was presented in [22]. We show that TPASs are ideal by constructing a linear ideal secret sharing scheme that realizes them. Our scheme is a generalization of a scheme from [22] for bipartite access structures.

**Definition 3.4 (Composition of Access Structures)** Let  $U_1$  and  $U_2$  be disjoint sets of users and let  $\Gamma_1$  and  $\Gamma_2$  be access structures over  $U_1$  and  $U_2$  respectively. Let  $u_1 \in U_1$ , and set  $U = U_1 \cup U_2 \setminus \{u_1\}$ . Then the composition of  $\Gamma_1$  and  $\Gamma_2$  via  $u_1$  is

$$\Gamma = \{X \subseteq U : X_1 \in \Gamma_1 \text{ or } (X_2 \in \Gamma_2 \text{ and } X_1 \cup \{u_1\} \in \Gamma_1) \text{ where } X_1 = X \cap U_1 \text{ and } X_2 = X \cap U_2\};$$

namely,  $X \subseteq U$  is authorized in this access structure if either  $X_1 = X \cap U_1$  is authorized in  $\Gamma_1$ , or  $X_1 \cup \{u_1\}$  is authorized in  $\Gamma_1$  and  $X_2 = X \cap U_2$  is authorized in  $\Gamma_2$ .

The motivation behind this definition is as follows. We start with an access structure  $\Gamma_1$  over  $U_1$ . Then, we allow the replacement of one the users in  $U_1$ , denoted  $u_1$ , by substitute users of a disjoint set  $U_2$ , where  $\Gamma_2$  specifies the subsets of  $U_2$  that are legitimate substitutes for  $u_1$ . Hence, in the set  $U = U_1 \cup U_2 \setminus \{u_1\}$  that is obtained from  $U_1$  after replacing  $u_1$  with  $U_2$ , a subset is authorized if it is either a subset in  $\Gamma_1$  that does not include  $u_1$  or it was obtained from a subset in  $\Gamma_1$  that did include  $u_1$ , by replacing  $u_1$  with an authorized subset of substitutes from  $\Gamma_2$ . In Lemma 8.1 we show that  $\Gamma$  is ideal if and only if both  $\Gamma_1$  and  $\Gamma_2$  are ideal.

### 3.2 The Characterization

Following the notation convention in Section 2.3, the set  $U$  is viewed as a sequence which is ordered in a monotonic non-decreasing order according to the weights. Let  $M$  be the lexicographically minimal minterm of  $\Gamma$  (that is,  $M \in \Gamma$  is a minterm and  $M \prec M'$  for all other minterms  $M' \in \Gamma$ ). It turns out that the form of  $M$  plays a significant role in the characterization of  $\Gamma$ .

If  $M$  is a prefix of  $U$ , namely  $M = U_{1,k} = \{u_1, \dots, u_k\}$  for some  $k \in [n]$ , then, as we prove in Section 6.3, the access structure is a HTAS of at most three levels. If  $M$  is a lacunary prefix, in the sense that  $M = U_{1,k} \setminus \{u_\ell\}$  for  $1 \leq \ell < k \leq n$ , then, as we prove in Section 7, the access structure is a TPAS. Otherwise, if  $M$  is neither a prefix nor a lacunary prefix, the access structure is a composition of two weighted threshold access structures over smaller sets. More specifically, we identify a prefix  $U_{1,k}$ , where  $1 < k < n$ , that could be replaced by a single substitute user  $u$ , and then show that  $\Gamma$  is a composition of a WTAS on  $U_{1,k}$  and another WTAS on  $U_{k+1,n} \cup \{u\}$ . Since  $\Gamma$  is ideal, so are the two smaller WTASs, as implied by Lemma 8.1. Hence, this result, which we prove in Section 8, completes the characterization of ideal WTASs in a recursive manner.

Hence, our main result in this paper is as follows.

**Theorem 3.5 (The Characterization Theorem)** *Let  $U$  be a set of users,  $w : U \rightarrow \mathbb{N}$  be a weight function,  $T$  be a threshold, and  $\Gamma$  be the corresponding WTAS. Then  $\Gamma$  is ideal if and only if one of the following three conditions holds:*

- *The access structure  $\Gamma$  is a HTAS.*
- *The access structure  $\Gamma$  is a TPAS.*
- *The access structure  $\Gamma$  is a composition of  $\Gamma_1$  and  $\Gamma_2$ , where  $\Gamma_1$  and  $\Gamma_2$  are ideal WTASs defined over sets of users smaller than  $U$ .*

*In particular, if  $\Gamma$  is an ideal WTAS then there exists a linear ideal secret sharing scheme that realizes it.*

## 4 Matroids and Ideal Secret Sharing Schemes

Ideal secret sharing schemes and matroids are strongly related. If an access structure is ideal, then there is a matroid that reflects its structure. On the other hand, every matroid that is representable over some finite field is the reflection of some ideal access structure. In this section we review some basic results from the theory of matroids and describe their relation to ideal secret sharing schemes. For more background on matroid theory the reader is referred to [21].

Matroids are a combinatorial structure that generalizes both linear spaces and the set of circuits in an undirected graph. They are a useful tool in several fields of theoretical computer science, e.g., optimization algorithms. A matroid  $\mathcal{M} = (V, \mathcal{I})$  is a finite set  $V$  and a collection  $\mathcal{I}$  of subsets of  $V$  that satisfy the following three axioms:

- (I1)  $\emptyset \in \mathcal{I}$ .
- (I2) If  $X \in \mathcal{I}$  and  $Y \subseteq X$  then  $Y \in \mathcal{I}$ .
- (I3) If  $X$  and  $Y$  are members of  $\mathcal{I}$  with  $|X| = |Y| + 1$  then there exists an element  $x \in X \setminus Y$  such that  $Y \cup \{x\} \in \mathcal{I}$ .

The elements of  $V$  are called the *points* of the matroid and the sets in  $\mathcal{I}$  are called the *independent* sets of the matroid. Axiom (I1) assures that there is at least one independent set in  $\mathcal{I}$ . Axiom (I2) asserts that the collection  $\mathcal{I}$  is closed under containment. Finally, Axiom (I3) enables the expansion of every small independent set in  $\mathcal{I}$ . A *dependent set* of the matroid is any subset of  $V$  that is not independent. The minimal dependent sets are called *circuits*. A matroid is said to be *connected* if for any two points there exists a circuit that contains both of them.

We now discuss the relations between ideal secret sharing schemes and matroids. Let  $\Gamma$  be an access structure over a set of users  $U = \{u_1, \dots, u_n\}$ . If  $\Gamma$  is ideal, then, by the results of [7, 18], there exists a matroid  $\mathcal{M}$  corresponding to  $\Gamma$ . The points of  $\mathcal{M}$  are the users in  $U$  together with an additional point, denoted  $u_0$ , that could be thought of as representing the dealer. We denote hereinafter by  $\mathcal{C}_0 = \{X \cup \{u_0\} : X \text{ is a minterm of } \Gamma\}$  the set of all  $\Gamma$ -minterms, supplemented by  $u_0$ .

**Theorem 4.1 ([7, 18])** *Let  $\Gamma$  be a connected ideal access structure. Then there exists a connected matroid  $\mathcal{M}$  such that  $\mathcal{C}_0$  is exactly the set of circuits of  $\mathcal{M}$  containing  $u_0$ .*

The next result implies the uniqueness of the matroid  $\mathcal{M}$  that corresponds to a given connected ideal access structure, as discussed in Theorem 4.1, and, additionally, it provides means to identify *all* the circuits of that matroid.

**Lemma 4.2 ([21, Theorem 4.3.2])** *Let  $e$  be an element of a connected matroid  $\mathcal{M}$  and let  $\mathcal{C}_e$  be the set of circuits of  $\mathcal{M}$  that contain  $e$ . Then all of the circuits of  $\mathcal{M}$  that do not contain  $e$  are the minimal sets of the form*

$$(C_1 \cup C_2) \setminus \bigcap \{C_3 : C_3 \in \mathcal{C}_e, C_3 \subseteq C_1 \cup C_2\}$$

where  $C_1$  and  $C_2$  are distinct circuits in  $\mathcal{C}_e$ .

The unique matroid whose existence and uniqueness are guaranteed by Theorem 4.1 and Lemma 4.2 is referred to as *the matroid corresponding to  $\Gamma$* .

This next definition will enable us to explicitly define the matroid corresponding to  $\Gamma$  using the authorized sets in  $\Gamma$ .

**Definition 4.3 (Critical User)** *Let  $M_1$  and  $M_2$  be distinct minterms of  $\Gamma$ . A user  $x \in M_1 \cup M_2$  is critical for  $M_1 \cup M_2$  if the set  $M_1 \cup M_2 \setminus \{x\}$  is unauthorized. In addition, we define*

$$D(M_1, M_2) = (M_1 \cup M_2) \setminus \{x \in M_1 \cup M_2 : x \text{ is critical for } M_1 \cup M_2\}.$$

With this definition, the following corollary is a straightforward result of Theorem 4.1 and Lemma 4.2.

**Corollary 4.4** *Let  $\Gamma$  be a connected ideal access structure. Then there exists a unique connected matroid  $\mathcal{M}$  such that  $\mathcal{C}_0$  is exactly the set of circuits of  $\mathcal{M}$  containing  $u_0$ . Furthermore, the minimal sets in  $\{D(M_1, M_2) : M_1, M_2 \text{ are minterm of } \Gamma\}$  are the circuits of the matroid that do not contain  $u_0$ .*

In the sequel we use the following simple consequence of Corollary 4.4.

**Corollary 4.5** *Let  $M_1$  and  $M_2$  be two distinct minterms of  $\Gamma$ . Then  $D(M_1, M_2)$  is a dependent set of  $\mathcal{M}$ .*

Note that  $D(M_1, M_2)$  is a dependent set of  $\mathcal{M}$ , but is not necessarily a circuit of  $\mathcal{M}$ . The following basic result from matroid theory provides another way of establishing the dependency of a set in a matroid.

**Lemma 4.6 ([21, Page 16])** *Let  $C_1$  and  $C_2$  be two distinct circuits in a matroid,  $e \in C_1 \cap C_2$ , and  $d \in C_1 \setminus C_2$  such that  $d \neq e$ . Then there exists a circuit  $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$ , where  $d \in C_3$ .*

In most applications of the lemma we will ignore  $d$  and the fact that  $d \in C_3$ . Since every minterm of  $\Gamma$  is properly contained in a circuit of  $\mathcal{M}$ , it forms an independent set of  $\mathcal{M}$ . However, these are not necessarily the only independent sets in  $\mathcal{M}$ . The following claim is a simple corollary of Axiom (I3) in the matroid definition.

**Claim 4.7** *Let  $\mathcal{M}$  be a matroid and  $I$  be an independent set of size  $m$ . Then any independent set of size  $\ell < m$  can be extended to an independent set of size  $m$ .*

Finally, the next lemma is applicable when adding an element to an independent set results in a dependent set.

**Lemma 4.8** *Let  $I$  be an independent set in a matroid  $\mathcal{M}$  and let  $e$  be an element of  $\mathcal{M}$  such that  $I \cup \{e\}$  is dependent. Then  $\mathcal{M}$  has a unique circuit contained in  $I \cup \{e\}$  and that circuit contains  $e$ .*

To prove that an access structure is not ideal we use the above results that define the dependent sets in the matroid corresponding to  $\Gamma$ . In particular, we use the basic fact that an independent set of  $\mathcal{M}$  cannot contain a dependent set (see Axiom (I2)). By Theorem 4.1, for every minterm  $M \in \Gamma$ , the set  $M \cup \{u_0\}$  is a circuit of  $\mathcal{M}$ . Thus, the set  $M$  is an independent set of  $\mathcal{M}$ . Therefore, if  $D$  is a dependent set of  $\mathcal{M}$ , it cannot be contained in any minterm of  $\Gamma$ . The next example shows how to use the above statements in order to demonstrate that a given access structure is not ideal.

**Example 4.9** Consider the WTAS  $\Gamma$  on the set  $U = \{u_1, u_2, u_3, u_4\}$  with weights  $w(u_1) = w(u_2) = w(u_3) = 1$  and  $w(u_4) = 2$  and threshold  $T = 3$ . The minterms of  $\Gamma$  are  $\{u_1, u_2, u_3\}$ ,  $\{u_1, u_4\}$ ,  $\{u_2, u_4\}$ , and  $\{u_3, u_4\}$ . It follows from Benaloh and Leichter [4] that this access structure is not ideal.<sup>2</sup> We first give a simple proof, based on

<sup>2</sup>In [9] it was shown that if the domain of secrets is  $S$ , the size of the domain of shares of at least one user in that access structure must be at least  $|S|^{1.5}$ . That result improved upon previous bounds that were derived in [8].

Corollary 4.5, showing that  $\Gamma$  is not ideal. Assume that it is ideal and consider the minterms  $M_1 = \{u_1, u_4\}$  and  $M_2 = \{u_2, u_4\}$ . The set  $\{u_1, u_2\}$  is unauthorized and thus  $u_4$  is critical for  $M_1 \cup M_2$ . On the other hand, the users  $u_1$  and  $u_2$  are not critical for  $M_1 \cup M_2$ . Therefore, by Corollary 4.5, the set  $D(M_1, M_2) = \{u_1, u_2\}$  is a dependent set of  $\mathcal{M}$ , the matroid corresponding to  $\Gamma$ . However, the set  $\{u_1, u_2, u_3\}$  is a minterm of  $\Gamma$  and, consequently, it is independent in  $\mathcal{M}$ . Since  $\{u_1, u_2\} \subset \{u_1, u_2, u_3\}$ , we arrive at the absurd conclusion that a dependent set is contained in an independent set. This contradiction implies that  $\Gamma$  is not ideal.

We proceed to sketch an alternative proof for the non-ideality of  $\Gamma$ , similar to the proof that was given in [4]. We present this proof for two reasons. First, we want to show that using matroids simplifies and shortens the proofs substantially. Matroid-based proofs, however, tend to be technical and, consequently, not very intuitive. The alternative proof that we present herein reveals some of the intuition that is sometimes obfuscated in matroid-based proofs.

Assume towards contradiction that  $\Gamma$  has an ideal secret sharing scheme over some domain of secrets  $S$ . Letting  $s_i$  denote the share of user  $u_i$  for  $1 \leq i \leq 4$ , we will show that  $s_2$  may be computed from  $s_1$  (this statement is equivalent to the statement in the first proof that the set  $\{u_1, u_2\}$  is dependent). Having shown that, we will conclude that  $\{u_1, u_3\}$  can reconstruct the secret (since  $\{u_1, u_2, u_3\}$  is an authorized set and  $u_2$  is unnecessary in the presence of  $u_1$ ), in contradiction with the definition of  $\Gamma$ .

We need to show that  $s_2$  is uniquely determined by  $s_1$ . Assume, by contradiction, that there exists  $s_1 \in S_1$  and  $s_2, s'_2 \in S_2$ , where  $s_2 \neq s'_2$ , such that both assignments of shares to  $\{u_1, u_2\}$ , namely  $\langle s_1, s_2 \rangle$  and  $\langle s_1, s'_2 \rangle$ , are possible. Since  $\{u_1, u_2\}$  is unauthorized, that set cannot learn any information on the secret from its shares. Hence, given the pair of shares  $\langle s_1, s_2 \rangle$ , every secret is possible. Hence, for every  $s \in S$ , there exists at least one share  $s_4(s)$  for  $u_4$  such that  $\langle s_1, s_2, s_4(s) \rangle$  is a possible assignment of shares to  $\{u_1, u_2, u_4\}$  given the secret  $s$ . On the other hand, since the shares of  $\{u_1, u_2, u_4\}$  determine the secret,  $s_4(s)$  must be unique, since otherwise the domain of shares of  $u_4$  would be larger than the domain of secrets, in contradiction with ideality. Similarly, for every  $s \in S$  there exists a unique share  $s'_4(s)$  such that  $\langle s_1, s'_2, s'_4(s) \rangle$  is another possible assignment of shares to  $\{u_1, u_2, u_4\}$  given the secret  $s$ .

In view of the above, the sets  $\{s_4(s) : s \in S\}$  and  $\{s'_4(s) : s \in S\}$  are of size  $|S|$ . However, since they are both subsets of  $S_4$  and, by ideality,  $|S_4| = |S|$ , they both equal  $S_4$ . One conclusion is that each secret  $s \in S$  is consistent with each share  $s_4 \in S_4$ . Another conclusion is that there are secrets  $a, b \in S$  such that  $s_4(a) = s'_4(b) = s_4$ . If  $a \neq b$ , then the pair of shares  $\langle s_1, s_4 \rangle$  is insufficient to determine whether the secret is  $a$  or  $b$ , in contradiction with the correctness requirement that  $\{u_1, u_4\} \in \Gamma$  should be able to reconstruct the secret. Thus,  $a = b$ , whence both pairs of shares  $\langle s_2, s_4 \rangle$  and  $\langle s'_2, s_4 \rangle$  are possible assignments of shares to  $\{u_2, u_4\}$  given that the secret is  $a$ . But this implies that the size of  $S_2$  is at least  $|S| + 1$  (by going over all secrets  $s \in S$  and looking for a corresponding assignment of shares where the share of  $u_4$  is  $s_4$ ) thus contradicting ideality.

## 4.1 Restrictions

**Definition 4.10 (Restriction)** *Let  $Y, X \subseteq U$  be two disjoint subsets of users. The restriction of  $\Gamma$  that is induced by  $Y$  on  $X$  is defined as the following access structure:*

$$\Gamma_{Y,X} = \{Z \subseteq X : Z \cup Y \in \Gamma\}.$$

In other words,  $\Gamma_{Y,X}$  consists of all subsets of  $X$  that complete  $Y$  to an authorized set in  $\Gamma$ . Since  $\Gamma_{Y,X}$  is defined over a smaller set of users, restrictions can be helpful in recursively characterizing the structure of  $\Gamma$ . The following known result assures us that if  $\Gamma$  is ideal,  $\Gamma_{Y,X}$  is ideal as well. For completeness, we give a proof herein.

**Lemma 4.11** *Let  $\Gamma$  be an access structure over a set of users  $U$ . Let  $Y, X \subseteq U$  be sets such that  $Y \notin \Gamma$  and  $X \cap Y = \emptyset$ . If  $\Gamma$  is ideal, then  $\Gamma_{Y,X}$  is ideal.*

**Proof:** Let  $Y = \{u_{i_1}, \dots, u_{i_{|Y|}}\}$ . We construct an ideal secret sharing scheme  $\Pi'$  for  $\Gamma_{Y,X}$  using an ideal scheme  $\Pi$  for  $\Gamma$ . Fix an arbitrary secret  $s' \in S$ , and let  $s_Y = \langle s'_{i_1}, \dots, s'_{i_{|Y|}} \rangle = \Pi_Y(s')$  be a possible vector of shares for the users of  $Y$ . Let  $s \in S$  be a secret to be distributed by  $\Pi'$ . The scheme  $\Pi'$  randomly chooses an  $n$ -tuple  $\Pi(s) = \langle s_1, \dots, s_n \rangle$  such that  $s_{i_j} = s'_{i_j}$  for all  $1 \leq j \leq |Y|$  (such a selection is possible because  $Y$  is unauthorized, whence every assignment of shares to  $Y$  is consistent with every secret). Every user  $u_j \in X$  is then assigned by  $\Pi'$  the share  $s_j$ . (The shares of  $Y$ , namely  $\langle s'_{i_1}, \dots, s'_{i_{|Y|}} \rangle$ , are considered “publicly known”.)

The scheme is correct since if  $Z \in \Gamma_{Y,X}$  then  $Y \cup Z \in \Gamma$ , and thus the members of  $Z$ , who know the shares  $s_Y$  of  $Y$ , can reconstruct the secret as in the scheme  $\Pi$  for  $\Gamma$ . Similarly, the scheme is private, since if  $Z \notin \Gamma_{Y,X}$ , then



$Y \cup Z \notin \Gamma$ , and thus the users of  $Z$ , who know the shares of  $Y$ , will not have any information regarding the secret. The scheme is ideal since the share-domain of all the users is the same as in the original ideal scheme.  $\square$

The following result gives a relation between the matroids corresponding to  $\Gamma$  and  $\Gamma_{Y,X}$ .

**Lemma 4.12** *Let  $\Gamma$  be an ideal access structure over a set of users  $U$ . Let  $Y, X \subseteq U$  be sets such that  $X \cap Y = \emptyset$ , and  $Y$  is independent in the matroid corresponding to  $\Gamma$ . If a set  $I \subseteq X$  is independent in the matroid corresponding to  $\Gamma_{Y,X}$ , then it is independent in the matroid corresponding to  $\Gamma$ .*

**Proof:** Let  $\mathcal{M} = \langle U \cup \{u_0\}, \mathcal{I} \rangle$  be the matroid associated with  $\Gamma$ . We define the matroid  $\mathcal{M}' = \langle X \cup \{u_0\}, \mathcal{I}' \rangle$  as follows: A set  $Z \subseteq X \cup \{u_0\}$  is in  $\mathcal{I}'$  if and only if  $Y \cup Z \in \mathcal{I}$ . This is indeed a matroid: Axiom (I1) holds because  $Y \in \mathcal{I}$  so that  $\emptyset \in \mathcal{I}'$ . As for Axiom (I2),  $Y \cup B \in \mathcal{I}$  for every  $B \in \mathcal{I}'$ ; therefore, if  $A \subseteq B$  then  $Y \cup A \in \mathcal{I}$  whence  $A \in \mathcal{I}'$ . Axiom (I3) holds as well. Let  $A, B \in \mathcal{I}'$  be such that  $|A| > |B|$ . Since  $Y \cup A \in \mathcal{I}$  and  $Y \cup B \in \mathcal{I}$  while  $|Y \cup A| > |Y \cup B|$ , there must be an element  $x \in A \setminus B$  such that  $Y \cup (B \cup \{x\}) \in \mathcal{I}$ . Therefore,  $B \cup \{x\} \in \mathcal{I}'$ , as required.

We claim that  $\mathcal{M}'$  is the unique matroid corresponding to  $\Gamma_{Y,X}$ . That is, for every  $M \subseteq X$ , the set  $M \cup \{u_0\}$  is a circuit of  $\mathcal{M}'$  if and only if  $M$  is a minterm of  $\Gamma_{Y,X}$ . Proving that, we get that if  $I \in \mathcal{I}'$  then  $Y \cup I \in \mathcal{I}$  and, therefore,  $I$  itself is independent in  $\mathcal{M}$ .

Suppose  $M \cup \{u_0\}$  is a circuit of  $\mathcal{M}'$ . Then  $Y \cup M \cup \{u_0\}$  is dependent in  $\mathcal{M}$ , and, consequently must contain a circuit of  $\mathcal{M}$ . That circuit must contain  $M \cup \{u_0\}$  because otherwise there would be a proper subset  $B \subset M \cup \{u_0\}$  such that  $Y \cup B$  is dependent in  $\mathcal{M}$ , thus implying that  $B$  is dependent also in  $\mathcal{M}'$ , in contradiction with our assumption that  $M \cup \{u_0\}$  is a circuit of  $\mathcal{M}'$ . Let  $Y' \cup M \cup \{u_0\}$  be that circuit of  $\mathcal{M}$ , where  $Y' \subseteq Y$ . Hence, by the characterization of the matroid that corresponds to a given access structure,  $Y' \cup M$  is a minterm of  $\Gamma$ . Hence,  $Y' \cup M$  is in  $\Gamma$ , and, consequently,  $M \in \Gamma_{Y,X}$ . It is left to show that  $M$  is a minterm of  $\Gamma_{Y,X}$ . Let  $M'$  be a proper subset of  $M$ . Since  $M \cup \{u_0\}$  is a circuit of  $\mathcal{M}'$ , the set  $M' \cup \{u_0\}$  is independent in  $\mathcal{M}'$ . This implies that  $Y \cup M' \cup \{u_0\}$  is independent in  $\mathcal{M}$ . Therefore  $Y \cup M' \notin \Gamma$ , and thus  $M' \notin \Gamma_{Y,X}$ . This completes the proof of the first direction.

For the second direction, assume that  $M$  is a minterm of  $\Gamma_{Y,X}$ . We next prove that  $M \cup \{u_0\}$  is dependent in  $\mathcal{M}'$ . The set  $Y \cup M$  is authorized in  $\Gamma$ . Let  $Y' \subseteq Y$  be such that  $Y' \cup M \in \Gamma$  and  $|Y'|$  is minimal (it is possible that  $Y' = \emptyset$ ). We claim that  $Y' \cup M$  is a minterm of  $\Gamma$ : indeed, every user in  $Y'$  is critical in  $Y' \cup M$ , as implied by the minimality of  $|Y'|$ ; on the other hand, every user in  $M$  is critical, for otherwise  $M$  would have a proper subset  $M' \subset M$  such that  $Y \cup M' \in \Gamma$  thus contradicting our assumption that  $M$  is a minterm of  $\Gamma_{Y,X}$ . It follows that  $Y' \cup M \cup \{u_0\}$  is a circuit of  $\mathcal{M}$ . Therefore, the set  $Y \cup M \cup \{u_0\}$  is dependent in  $\mathcal{M}$ . This implies that  $M \cup \{u_0\}$  is dependent in  $\mathcal{M}'$ .

To complete the proof we need to prove that  $M \cup \{u_0\}$  is a circuit in  $\mathcal{M}'$ . Assume towards contradiction that  $M \cup \{u_0\}$  properly contains a circuit of  $\mathcal{M}'$ . If this circuit is of the form  $M' \cup \{u_0\}$ , for some  $M' \subset M$ , then by the proof of the first direction,  $M'$  is a minterm of  $\Gamma_{Y,X}$ , contradicting the fact that  $M$  is a minterm of  $\Gamma_{Y,X}$ . Therefore, the circuit is some  $M' \subseteq M$ . This implies that  $Y \cup M'$  is dependent in  $\mathcal{M}$ . Hence, there exists  $Y'' \subseteq Y$ , such that  $Y'' \cup M'$  is a circuit of  $\mathcal{M}$ . However, as shown earlier,  $Y' \cup M \cup \{u_0\}$  is a circuit of  $\mathcal{M}$  as well. Applying Lemma 4.6 for these two circuits, with  $e$  being an arbitrary user in  $M'$ , and  $d = u_0$ , we get that there is a circuit contained in  $(M \setminus \{e\}) \cup Y \cup \{u_0\}$ , that contains  $u_0$ . This implies that  $(M \setminus \{e\}) \cup Y \in \Gamma$ , whence  $(M \setminus \{e\}) \in \Gamma_{Y,X}$ , in contradiction to our assumption that  $M$  is a minterm of  $\Gamma_{Y,X}$ .  $\square$

A special family of restrictions of  $\Gamma$  are restrictions in which  $Y = \emptyset$ , i.e.  $\Gamma_{\emptyset, X}$ . Such a restriction consists of all subsets of  $X$  that are authorized in  $\Gamma$ . We shall refer to such a restriction simply as *the restriction of  $\Gamma$  to  $X$* . We conclude with a trivial claim regarding restrictions of WTASs.

**Claim 4.13** *Let  $X, Y \in U$  such that  $X \cap Y = \emptyset$ . If  $\Gamma$  is a WTAS, then  $\Gamma_{Y,X}$  is a WTAS.*

## 5 WTASs and Matroids

In this section we prove several properties of matroids that are associated with ideal WTASs. These properties will serve us later in characterizing ideal WTASs. Let  $\Gamma$  be an ideal WTAS on  $U = \{u_1, \dots, u_n\}$  corresponding to a weight function  $w : U \rightarrow \mathbb{N}$  and a threshold  $T$ . Let  $\mathcal{M}$  be the matroid corresponding to  $\Gamma$ .

An authorized set of  $\Gamma$  may contain many different minterms. The following simple lemma shows that one of these minterms is a suffix of the set.

**Lemma 5.1** *If  $X = \{x_1, \dots, x_k\} \in \Gamma$ , it contains a suffix minterm, namely, there exists  $i \in [k]$  such that  $X_{i,k} = \{x_i, \dots, x_k\}$  is a minterm.*

**Proof:** Let  $s \in [k]$  be the largest index such that  $X_{s,k}$  is authorized. We claim that  $X_{s,k}$  is a minterm. Indeed,  $x_s$  is critical for  $X_{s,k}$  since removing it from  $X_{s,k}$  we get  $X_{s+1,k}$ , which is unauthorized by the choice of  $s$ . Consequently, every user  $x_i \in X_{s,k}$  is critical because  $x_s$  has the smallest weight in  $X_{s,k}$ . Therefore,  $X_{s,k}$  is a minterm.  $\square$

We saw that minterms of  $\Gamma$  are independent sets of  $\mathcal{M}$ . Different minterms of  $\Gamma$  may be of different sizes. By Claim 4.7, if  $M$  and  $M'$  are minterms and  $|M| < |M'|$ , we can add (at least  $|M'| - |M|$ ) users to  $M$  such that the resulting set is still an independent set of  $\mathcal{M}$ . The next lemma asserts that the users that may be added to  $M$  must have a smaller weight than the weight of every member of  $M$ :

**Lemma 5.2** *Let  $M$  be a minterm of  $\Gamma$ . Let  $y \in U \setminus M$  be a user such that  $w(M_{\min}) \leq w(y)$ . Then  $M \cup \{y\}$  is a dependent set of  $\mathcal{M}$ .*

**Proof:** Let  $X = (M \setminus \{M_{\min}\}) \cup \{y\}$  be the set that is obtained by replacing the minimal element of  $M$  with  $y$ . Since  $w(X) \geq w(M)$ , the set  $X$  is authorized and, thus, it contains a minterm  $M'$ . Moreover,  $M \neq M'$  since  $M_{\min} \in M \setminus M'$ . Therefore, by Corollary 4.5, the set  $M \cup M' = M \cup \{y\}$  is dependent in  $\mathcal{M}$ .  $\square$

While  $\Gamma$  may have minterms of different cardinalities, we show in the next lemma that whenever two minterms have the same minimal member they must be of the same size.

**Lemma 5.3** *Let  $X$  and  $Y$  be minterms of the access structure  $\Gamma$  such that  $X_{\min} = Y_{\min}$ . Then  $|X| = |Y|$ .*

**Proof:** As  $X$  and  $Y$  are minterms, they are independent sets of the matroid  $\mathcal{M}$ . Assume that  $|X| < |Y|$ . Then, by Axiom (I3) of the matroid definition, there exists  $y \in Y \setminus X$  such that the set  $X \cup \{y\}$  is independent. However, as  $X_{\min} = Y_{\min}$  is a user with the minimal weight in both  $X$  and  $Y$ , we have that  $w(X_{\min}) \leq w(y)$ . Consequently, in view of Lemma 5.2, the set  $X \cup \{y\}$  is dependent. This contradiction implies that  $|X| \geq |Y|$ . Arguing along the same lines we conclude that also  $|X| \leq |Y|$ , whence both minterms are of the same size.  $\square$

In the matroid  $\mathcal{M}$  corresponding to an access structure  $\Gamma$ , each minterm  $M$  of  $\Gamma$  defined a circuit  $M \cup \{u_0\}$  in  $\mathcal{M}$ . This may be viewed as a *local* relation between the minterms of  $\Gamma$  and the dependent sets of  $\mathcal{M}$ . However, when dealing with WTASs, the *global* structure of  $\Gamma$  enables us to prove the dependency of various sets in  $\mathcal{M}$ . It turns out that the lexicographic order on the minterms of  $\Gamma$ , with respect to the relation  $\prec$ , is strongly related to dependence in  $\mathcal{M}$ . This is demonstrated through the following definition and lemmas.

**Definition 5.4 (Canonical Complement)** *Let  $P$  be a prefix of some minterm of  $\Gamma$ . Let  $Y \subseteq U$  be the lexicographically minimal set such that:*

- $P_{\max} \prec Y_{\min}$ , and
- The set  $P \cup Y$  is a minterm of  $\Gamma$ .

*Then the set  $Y$  is called the canonical complement of  $P$ .*

The following lemma shows that replacing the canonical complement by a user that precedes the first element of the canonical complement results in a dependent set.

**Lemma 5.5** *Let  $P$  be a prefix of some minterm of  $\Gamma$ . Let  $Y = \{y_1, \dots, y_t\}$  be the canonical complement of  $P$ , and  $b$  be a user such  $P_{\max} \prec b \prec y_1$ . Then  $P \cup \{b\}$  is dependent. Furthermore, the set  $P \cup \{b\}$  includes a unique circuit that contains  $b$ .*

**Proof:** If  $P = \emptyset$ , then, since  $\Gamma$  is connected, there exists a minterm that starts with  $u_1$ , whence  $y_1 = u_1$ . Therefore, it cannot be that  $b \prec y_1$  and thus the claim is trivially true. Otherwise, if  $P \neq \emptyset$ , denote by  $M_1$  the minterm  $M_1 = P \cup Y$ . Let  $X_2 = (M_1 \setminus \{P_{\max}\}) \cup \{b\}$  be the set resulting from replacing  $P_{\max}$  with  $b$  in  $M_1$ . Since  $w(P_{\max}) \leq w(b)$ , the set  $X_2$  is authorized (though not necessarily a minterm). Let  $M_2$  be the suffix minterm contained in  $X_2$  (such a minterm exists in view of Lemma 5.1). It must be that  $b \in M_2$ , since otherwise  $M_2 \subseteq Y$ , where  $Y$  is a proper subset of a minterm and thus is unauthorized.

Let  $A = M_1 \cup M_2 = P \cup \{b\} \cup Y$ . We proceed to show that every user in  $Y$  is critical for  $A$ . This will show that  $D(M_1, M_2) \subseteq (M_1 \cup M_2) \setminus Y = P \cup \{b\}$ . But since, by Corollary 4.5, the set  $D(M_1, M_2)$  is dependent, this will imply that also  $P \cup \{b\}$  is dependent. We also observe that it is sufficient to show that  $Y_{\min} = y_1$  is critical for  $A$ ; this will imply that also all other members of  $Y$ , having weight that is no smaller than  $w(y_1)$ , are also critical for  $A$ .

In view of the above, it suffices to show that  $y_1$  is critical for  $A$ . Suppose this is not the case, namely, the set  $A \setminus \{y_1\}$  is authorized. Since  $A \setminus \{y_1\}$  results from  $M_1$  by replacing  $y_1$  by  $b$  where  $w(b) \leq w(y_1)$ , and since  $M_1$  is a minterm, it must be that  $A \setminus y_1$  is also a minterm. But this is a contradiction to the choice of  $y_1$  as the first element in the canonical complement of  $P$ . Hence, all the elements of  $Y$  are critical for  $A$ , and, consequently,  $P \cup \{b\}$  is dependent.

Since  $P$  is part of a minterm, it must be that  $P$  is independent. Thus, by Claim 4.8, the set  $P \cup \{b\}$  must contain a unique circuit that contains  $b$ .  $\square$

The next lemma is a generalization of Lemma 5.5. Given a prefix  $P$  of some minterm, it gives a sufficient condition for a set  $B \subseteq U$  so that  $P \cup B$  is dependent in  $\mathcal{M}$ . It plays an essential role in our study of the characterization of ideal WTASs.

**Lemma 5.6** *Let  $P$  be a prefix of some minterm of  $\Gamma$ . Let  $Y = \{y_1, \dots, y_t\}$  be the canonical complement of  $P$ , and  $B = \{b_1, \dots, b_j\}$  be a set such that  $P_{\max} \prec B_{\min}$  and  $b_j \prec y_j$ . Then, the set  $P \cup B$  is dependent.*

**Proof:** The proof is by induction on  $j$ . The case  $j = 1$  is handled by Lemma 5.5. We assume that the lemma holds for all  $i \in [j - 1]$  and proceed to prove it for  $i = j$ . If  $b_i \prec y_i$  for some  $i \in [j - 1]$ , then the statement of the lemma holds by induction (in fact, in that case we conclude that  $P \cup B_{1,i}$  is dependent, which is a stronger claim than what we need to show). Therefore, we may assume hereinafter that  $y_i = b_i$  or  $y_i \prec b_i$  for all  $i \in [j - 1]$ .

Assume towards contradiction that  $P \cup B$  is independent. Denote  $D = P \cup \{y_1\} \cup B$  and  $P' = P \cup \{y_1\}$ . The canonical complement of  $P'$  is  $Y_{2,t}$ . On the other hand, since  $P'_{\max} = y_1 \preceq b_1 \prec b_2$ , and  $b_j \prec y_j$ , the induction hypothesis, applied to  $P'$  and  $B_{2,j}$ , implies that  $P' \cup B_{2,j}$  is dependent. Since  $P' \cup B_{2,j} \subseteq D$ , the set  $D$  is also dependent. Therefore, since we assumed that  $P \cup B$  is independent, Claim 4.8 implies that  $D$  contains a unique circuit  $C$  that contains  $y_1$ . The uniqueness of the circuit  $C$  in  $D$  and the fact  $P' \cup B_{2,j}$  is a dependent subset of  $D$ , implies that  $C \subseteq P' \cup B_{2,j}$ . This, in turn, implies that  $b_1 \notin C$ . This will allow us to derive a contradiction.

There are three possible cases regarding the position of  $b_1$  with respect to the members of  $Y$ :  $y_1 = b_1$ , or  $y_1 \prec b_1 \prec y_2$ , or  $y_2 \preceq b_1$ . The first case is trivial: if  $y_1 = b_1$  then  $P \cup B = P' \cup B_{2,j}$  and, thus, it is dependent, in contradiction to our assumption.

If  $y_1 \prec b_1 \prec y_2$  then by Lemma 5.5, applied to  $P'$  and  $b_1$ , the set  $P' \cup \{b_1\}$  is dependent and contains a circuit  $C'$  that contains  $b_1$ . As  $b_1 \in C'$  and  $b_1 \notin C$ , we conclude that  $C \neq C'$ . However, as  $C' \subseteq D$ , this contradicts the uniqueness of the circuit  $C$  in  $D$ .

The only case left is  $y_2 \preceq b_1$ . As  $P'$  is independent, there must be an index  $s \in \{2, \dots, j\}$  such that  $b_s \in C$  (for, otherwise, as  $b_1 \notin C$ ,  $C \cap B = \emptyset$ , and, consequently,  $C \subseteq P'$ ). Consider the set  $B^s = B \setminus \{b_s\}$ . Since  $|B^s| = j - 1$ ,  $P'_{\max} = y_1 \prec b_1 = B_{\min}^s$ , and the  $(j - 1)$ th element of  $B^s$  (which is either  $b_j$ , if  $s < j$ , or  $b_{j-1}$ ) precedes the  $(j - 1)$ th element of the canonical complement of  $P'$  (which is  $y_j$ ), we conclude, by the induction hypothesis, that  $P' \cup B^s$  is dependent. Therefore, it contains a circuit  $C''$  that does not contain  $b_s$ . Since  $b_s \in C$  but  $b_s \notin C''$ , we infer that  $C \neq C''$ . However, as  $C'' \subseteq D$ , this contradicts the uniqueness of  $C$  in  $D$ . This completes the proof that the set  $P \cup B$  is dependent.  $\square$

## 6 WTASs and HTASs

In this section we discuss the family of hierarchical threshold access structures (HTASs), from Definition 3.1, and their relation to WTASs. In Section 6.1 we prove certain properties of the matroids associated with such access structures. In Section 6.2 we characterize the intersection between HTASs and WTASs. Finally, in Section 6.3, we show that if an ideal WTAS has a minterm in the form of a prefix of  $U$ , then it is in fact an HTAS.

When discussing a HTAS over some set  $U = \{u_1, \dots, u_m\}$ , we shall assume that the users in  $U$  are ordered according to their position in the hierarchy, from the lowest level to the highest. Namely, that

$$L_i = U_{\ell_i, \ell_{i+1}-1} = \{u_{\ell_i}, \dots, u_{\ell_{i+1}-1}\} \quad \forall i \in [m] \quad (1)$$

for some sequence  $\ell_1 = 1 < \ell_2 < \dots < \ell_m < \ell_{m+1} = n + 1$ . Given a nonempty subset  $A \subseteq U$ , if  $A_{\min} \in L_i$ , then  $A$  is said to be of level  $i$  and it is denoted by  $L(A) = i$ .

## 6.1 The Matroid Associated with HTASs

Since any HTAS  $\Gamma_H$  is ideal [6, 32], Theorem 4.1 implies that there exists a matroid  $\mathcal{M}$  that is associated with it. We derive herein some properties of such matroids that will be used later in our study of the structure of matroids corresponding to ideal WTASs.

**Claim 6.1** *Let  $\Gamma_H$  and  $\mathcal{M}$  be an HTAS and its associated matroid. Then every circuit of  $\mathcal{M}$  not containing  $u_0$  is a union of two distinct minterms.*

**Proof:** By Lemma 4.2, the circuits of the matroid  $\mathcal{M}$  that do not contain  $u_0$  are the union of two minterms of  $\Gamma_H$ , excluding users that are critical to that union. All that needs to be shown is that if  $M_1, M_2$  are two minterms in  $\Gamma_H$ , no user in their union,  $u \in M_1 \cup M_2$ , is critical. Assume, without loss of generality, that  $L(M_1) = i \leq L(M_2)$ . Then, as  $M_1$  is a minterm, we conclude that it satisfies the  $i$ th threshold condition with an equality, namely,  $M_1$  has  $k_i$  users from  $\bigcup_{j=i}^m L_j$ . Hence, since  $M_2 \subseteq \bigcup_{j=i}^m L_j$  and  $M_2 \neq \emptyset$ , the union  $M_1 \cup M_2$  has at least  $k_i + 1$  users from  $\bigcup_{j=i}^m L_j$ . Therefore, we can remove any user from  $M_1 \cup M_2$  and still have an authorized set. Hence, no user  $u \in M_1 \cup M_2$  is critical for the union.  $\square$

**Claim 6.2** *Let  $\Gamma_H$  and  $\mathcal{M}$  be an HTAS and its associated matroid. Then  $U_{1,k_1+1}$  is a circuit of  $\mathcal{M}$ .*

**Proof:** Consider the two sets  $U_{1,k_1}$  and  $U_{2,k_1+1}$ . The first one is a minterm of  $\Gamma_H$ . Regarding the second, there are two options: either  $U_{2,k_1+1}$  is a minterm (satisfying the  $k_1$ -threshold condition), or it satisfies already the  $k_2$ -threshold condition (this may happen only if  $|L_1| = \ell_1 = k_1 - k_2 + 1$ , which is the minimal possible value for the size of that level). In both cases,  $U_{2,k_1+1}$  contains a minterm that includes  $u_{k_1+1}$ . Hence,  $U_{1,k_1+1}$  is a union of two distinct minterms and, therefore, by Corollary 4.5, it is dependent. We claim that it is actually a circuit. If not, then it must contain a circuit  $C$  of size smaller than  $k_1 + 1$ . By Claim 6.1, this circuit is a union of two minterms. Since its size is smaller than  $k_1 + 1$ , it must be a union of two minterms from levels  $L_2$  and above. However, by Remark 3.2, the set  $U_{1,k_1+1}$  contains at most  $k_2$  users from levels  $L_2$  and above, whence  $U_{1,k_1+1}$  contains at most one minterm from levels  $L_2$  and above. Therefore,  $C$  must be a circuit of  $\mathcal{M}$ .  $\square$

## 6.2 The Intersection of WTASs and HTASs

Let  $U$  be a set of users and let  $\Gamma$  be a monotone access structure over  $U$  that is both a WTAS and a HTAS. Namely, on one hand, there exist a weight function  $w : U \rightarrow \mathbb{N}$  and a threshold  $T \in \mathbb{N}$  such that  $\Gamma$  is the corresponding WTAS on  $U$  and, on the other hand, there exists a hierarchy in  $U$ , where  $U = \bigcup_{i=1}^m L_i$ , and thresholds  $k_1 > k_2 > \dots > k_m$  such that  $\Gamma$  is also the corresponding HTAS on  $U$ . Our goal herein is to characterize such access structures that belong to those two classes of access structures.

It should be noted that in the following discussion we can not assume that the levels of  $U$  in the HTAS are nicely organized in a monotonic order as in (1). Let  $A_i$ , where  $i \in [m - 1]$ , denote a subset of  $k_i - k_{i+1}$  users of largest weights in  $L_i$ , and let  $A_m$  be a subset of  $k_m - 1$  users of largest weights in  $L_m$ . We also let  $a_i$  denote a user of minimal weight in  $L_i$ , for all  $i \in [m]$ . Note that, by Remark 3.2, we have that  $a_i \in L_i \setminus A_i$  for every  $i \in [m]$ .

We set  $A = \bigcup_{i=1}^m A_i$  and  $B = A_m \cup \{a_m\}$ . The set  $A$  is unauthorized since it fails to satisfy each of the threshold conditions by exactly one user, while  $B$  is authorized since it is composed of  $k_m$  users of level  $L_m$ . Therefore  $w(A) < T$  and  $w(B) \geq T$ . But as  $B = A \cup \{a_m\} \setminus (A \setminus A_m)$ , we infer that

$$w(A \setminus A_m) < w(a_m). \quad (2)$$

We refer to the last level  $L_m$  in an HTAS as trivial if  $k_m = 1$  (namely, all users of such a level are self-sufficient). We proceed to characterize access structures that are both WTAS and HTAS.

**Lemma 6.3** *If the last level is not trivial, then the number of levels  $m$  in the HTAS is at most 2.*

**Proof:** Assume that  $m \geq 3$  and that the last level is not trivial, i.e.,  $k_m \geq 2$ . Let  $Z_m$  be a subset of  $k_m - 2 \geq 0$  users of largest weight in  $L_m$ . Define  $Z_{m-1} = A_{m-1} \cup \{a_{m-1}\}$  and  $Z_{m-2} = A_{m-2} \cup \{a_{m-2}\}$ . Their union  $Z = Z_{m-2} \cup Z_{m-1} \cup Z_m$  is authorized since its cardinality is  $|Z| = |A_{m-2}| + |A_{m-1}| + k_m = k_{m-2}$ , and consequently, it satisfies the  $(m-2)$ th threshold condition.

Next, consider the set  $B = Z \cup \{a_m\} \setminus (A_{m-1} \cup A_{m-2}) = Z_m \cup \{a_{m-2}, a_{m-1}, a_m\}$ . Since, by (2),  $w(A_{m-1} \cup A_{m-2}) < w(a_m)$ , the definition of  $B$  implies that  $w(B) > w(Z)$ . Therefore, as  $Z$  is authorized, so is  $B$ . However, while  $B$  is authorized in the WTAS, it is unauthorized in the HTAS, as implied by the following inequalities:  $|B \cap L_m| = |Z_m \cup \{a_m\}| = k_m - 1 < k_m$ ,  $|B \cap (L_{m-1} \cup L_m)| = k_m < k_{m-1}$ , and  $|B| = k_m + 1 < k_{m-2}$ . This contradiction implies that either  $m \leq 2$  or the last level is trivial.  $\square$

According to Lemma 6.3, access structures that are both WTAS and HTAS and have no self-sufficient users have a very simple hierarchical structure: either there is one level (in which case the access structure is a simple threshold access structure) or two levels. In the latter case, we show below that if the difference between  $k_1$  and  $k_2$  is larger than 1, the size of the first level is minimal (see Remark 3.2).

**Lemma 6.4** *If the last level is not trivial,  $m = 2$ , and  $k_1 - k_2 > 1$ , then  $|L_1| = k_1 - k_2 + 1$ .*

**Proof:** Assume towards contradiction that  $|L_1| > k_1 - k_2 + 1$ . Let  $Z_2$  be a subset with  $k_2 - 2$  users of largest weight in  $L_2$  and  $Z_1 = A_1 \cup \{b_1, b_2\}$ , where  $b_1$  and  $b_2$  are two users in  $L_1 \setminus A_1$  (note that our assumptions imply that  $Z_1$  and  $Z_2$  are well defined). Their union  $Z = Z_1 \cup Z_2$  is authorized since  $|Z| = k_1$ . Next, consider the set  $B = Z \cup \{a_2\} \setminus A_1 = Z_2 \cup \{a_2, b_1, b_2\}$ . Since, by (2),  $w(A_1) < w(a_2)$ , we conclude that  $B$  is also authorized. Hence, it must be also authorized in the HTAS. However, as  $|B \cap L_2| = k_2 - 1$  and  $|B| = k_2 + 1 < k_1$ , the set  $B$  is unauthorized in the HTAS.  $\square$

Lemmas 6.3 and 6.4 imply that if  $\Gamma$  is an access structure on  $U$  that is both a WTAS and an HTAS, and  $U$  has no self-sufficient users (i.e., the last level in the HTAS is not trivial), then either  $m = 1$  or  $m = 2$ , where in the latter case either  $k_1 = k_2 + 1$  or  $|L_1| = k_1 - k_2 + 1$ . If, on the other hand, the access structure has self-sufficient users, i.e., the last level in the HTAS is trivial, then its restriction to the first  $m - 1$  levels is still a WTAS as well as an HTAS, and, in addition, that restriction has no self-sufficient users. This implies that the structure of that restriction is as described above. This is summarized as follows:

**Claim 6.5** *Let  $\Gamma$  be both a WTAS and an HTAS. Then the HTAS-parameters of  $\Gamma$  satisfy one of the following conditions:*

1.  $m = 1$ .
2.  $m = 2$  and  $k_1 = k_2 + 1$ .
3.  $m = 2$  and  $|L_1| = k_1 - k_2 + 1$ .
4.  $m \in \{2, 3\}$ , the level  $L_m$  is trivial, and the restriction of  $\Gamma_H$  to the first  $m - 1$  levels is of the form that is described in cases (1)-(3).

To complete the characterization we proceed to prove the converse. Namely, that any HTAS with parameters as described in Claim 6.5 is also a WTAS. This is shown by constructing the appropriate weight function and threshold in each case.

**Claim 6.6** *Let  $\Gamma_H$  be an HTAS that satisfies one of the conditions in Claim 6.5. Then there exist a weight function and a threshold such that  $\Gamma_H$  coincides with the corresponding WTAS.*

**Proof:** Case (1) is a simple threshold, so we can assign each user the weight 1, and set the threshold to  $k_1$ . In case (2), the weight assigned to every user in  $L_1$  is  $k_2$ , the weight assigned to users in  $L_2$  is  $k_1$ , and the threshold is  $k_1 \cdot k_2$ . In case (3), the weight assigned to users in  $L_1$  and  $L_2$  is 1 and  $k_1 - k_2 + 1$ , respectively, while the threshold is  $k_2(k_1 - k_2 + 1)$ . In case (4), we assign weights to the users in the first  $m - 1$  levels and set an appropriate threshold according to the condition that is satisfied by the restriction of  $\Gamma_H$  to the first  $m - 1$  levels and the rules that were described above for the first three cases; in addition, we assign to every user in the trivial level a weight that equals the threshold. In all four cases it is a straightforward matter to verify that the induced WTAS coincides with the original HTAS.  $\square$

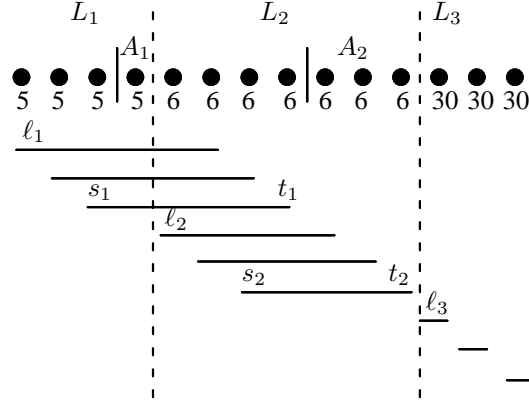


Figure 1: A WTAS that is also an HTAS.

### 6.3 Ideal WTASs with a Prefix Minterm are HTASs

In this section we make the first step towards proving our main result, Theorem 3.5. Let  $\Gamma$  be an ideal WTAS over a set  $U$  of  $n$  users, corresponding to a weight function  $w : U \rightarrow \mathbb{N}$  and a threshold  $T$ . Assume that  $U$  possesses a prefix minterm  $U_{1,k}$  for some  $k \in [n]$  (namely, there exists  $k \in [n]$  such that the  $k$  users of smallest weights form a minterm). We claim that  $\Gamma$  is an HTAS. We first describe the partition of  $U$  into levels and determine the corresponding thresholds. Denoting the resulting HTAS by  $\Gamma_H$ , we proceed to prove that  $\Gamma = \Gamma_H$ .

The decomposition of  $U$  to levels will respect the order of users according to their weights. Namely, each level will be a run of  $U$  and our goal is to determine the transition points between one level and the subsequent one. Since  $U_{1,k}$  is a minterm,  $U_{1,i}$  is authorized for every  $i \in \{k, \dots, n\}$ . By Lemma 5.1, for every such  $i$  there exists a run-minterm ending at  $u_i$ . Let us denote the length of that run-minterm by  $\mu_i$ . By the non-decreasing monotonicity of the weights, we infer that the sequence of lengths  $\boldsymbol{\mu} = (\mu_i)_{k \leq i \leq n}$  is monotonically non-increasing. Denote by  $m$  the number of distinct values assumed by the sequence  $\boldsymbol{\mu}$ , and let us denote those values by  $k_1 > \dots > k_m$ . Then the HTAS  $\Gamma_H$  is defined as follows:  $m$  is the number of levels and  $k_i$  is the  $i$ th threshold. As for the levels, we denote by  $\ell_i$ , where  $i \in [m]$ , the index of the first user in the first run-minterm of length  $k_i$  (e.g.,  $\ell_1 = 1$  since  $U_{1,k}$  is the first run-minterm of length  $k = k_1$  and its first user is  $u_1$ ); then the  $i$ th level in the hierarchy is  $L_i = U_{\ell_i, \ell_{i+1}-1}$ , where  $\ell_{m+1} = n + 1$ .

In addition, we denote by  $U_{s_i, t_i}$  the right-most run-minterm whose length is  $k_i$ , where  $i \in [m]$ , and consider the set  $A_i = U_{s_i+1, \ell_{i+1}-1}$ . As  $U_{s_i, t_i}$  is the last minterm that contains  $k_i$  users and  $U_{\ell_{i+1}, t_i+1}$  is the first minterm that contains  $k_{i+1}$  users, the set  $A_i$  consists of the last  $k_i - k_{i+1}$  users in  $L_i$  (where  $k_{m+1} = 1$ ). An important observation regarding those subsets is that given  $i \in [m]$  and  $u_h \in A_i$ , there is no run-minterm of the WTAS  $\Gamma$  that starts with  $u_h$ ; indeed, if  $U_{h,j}$  was a run-minterm then it would be a proper subset of the minterm  $U_{s_i, t_i}$  if  $j \leq t_i$ , or a proper superset of the minterm  $U_{\ell_{i+1}, t_i+1}$  if  $j \geq t_i + 1$ .

An illustration of the construction of the levels of the HTAS appears in Figure 1. The example in the illustration is of a WTAS with 14 users of weights 5, 5, 5, 5, 6, 6, 6, 6, 6, 6, 30, 30, 30 and threshold  $T = 30$ . The vertical dashed lines indicate the three levels in the corresponding HTAS (the third one being a trivial one) and the horizontal lines indicate all of the run-minterms in that access structure.

Next, we prove that the WTAS  $\Gamma$  coincides with the HTAS  $\Gamma_H$  described above.

**Theorem 6.7** *Let  $\Gamma$  be an ideal WTAS over  $U$  that has a prefix minterm. Then  $\Gamma$  is an HTAS.*

**Proof:** Let  $\Gamma_H$  be the HTAS as described above. We will prove that  $\Gamma = \Gamma_H$ , thus showing that  $\Gamma$  is an HTAS. We start with proving that  $\Gamma_H \subseteq \Gamma$ . Let  $X \in \Gamma_H$ . Then for some  $i \in [m]$  the set  $X$  has at least  $k_i$  users from  $\bigcup_{j=i}^m L_j$ . Letting  $B_i = U_{\ell_i, \ell_i+k_i-1}$  denote the set of the first  $k_i$  users from  $\bigcup_{j=i}^m L_j$ , the non-decreasing monotonicity of the weights implies that  $w(X) \geq w(B_i)$ . By the construction of levels in  $\Gamma_H$ , the set  $B_i$  is a minterm of  $\Gamma$ , whence  $w(B_i) \geq T$ . Therefore,  $w(X) \geq T$  and, consequently,  $X \in \Gamma$ .

Conversely, assume that  $X \notin \Gamma_H$ . Then  $X$  has at most  $k_i - 1$  users from  $\bigcup_{j=i}^m L_j$ , for every  $i \in [m]$ . Consider the set  $A = \bigcup_{i=1}^m A_i$ . By the definition of  $A$ , it has exactly  $k_i - 1$  users from  $\bigcup_{j=i}^m L_j$ , for every  $i \in [m]$ . Moreover,

$A$  is the set with the maximal weight among the sets that are unauthorized in the HTAS and thus  $w(X) \leq w(A)$ . Therefore, it suffices to show that  $A \notin \Gamma$  in order to conclude that  $X \notin \Gamma$  and, thus, complete the proof.

To this end, assume that  $A \in \Gamma$ . Then  $A$  contains some minterm  $M \in \Gamma$ . Assume that  $M$  is of level  $i$ ,  $L(M) = i$ , namely,  $i$  is the lowest level for which  $M \cap L_i \neq \emptyset$ . Then  $M \cap A_i$  is a prefix of  $M$ . In order to arrive at a contradiction, we proceed to show that there can be no minterm that has a prefix which is a subset of  $A_i$ .

Assume, by contradiction, that there are such minterms, and let  $M'$  be the lexicographically minimal minterm of that sort. Let  $u_h = M'_{\min}$  and let  $j$  be the maximal index such that  $M' = U_{h,j} \cup Z$  for some  $Z \subset U$ . Since  $u_h \in A_i$  and we observed earlier that no run-minterm starts in  $A_i$ , we conclude that  $Z \neq \emptyset$  (and  $u_{j+1} \prec Z_{\min}$  because of the maximality of  $j$ ). We claim that  $j < t_i$ . Indeed, if  $j \geq t_i$ , then  $M'$  is a proper superset of  $\hat{M} = U_{\ell_{i+1}, t_i} \cup \{Z_{\min}\}$ . Since  $w(\hat{M}) \geq w(U_{\ell_{i+1}, t_i+1}) \geq T$ , we get a contradiction since a minterm  $M'$  cannot be a proper superset of an authorized set  $\hat{M}$ .

Next, define  $Q = M' \cup \{u_{j+1}\} \setminus \{u_j\}$ . The set  $Q$  is authorized and, by Lemma 5.1, it contains a suffix minterm  $M''$  that must contain  $u_{j+1}$ , for otherwise it would be a proper subset of  $M'$ . Therefore,  $M' \cup M'' = M' \cup \{u_{j+1}\} = U_{h,j} \cup \{u_{j+1}\} \cup Z$ . We claim that all members of  $Z$  are critical for this union. Assume, by contradiction, that  $M^* = M' \cup \{u_{j+1}\} \setminus \{z\}$  is authorized, for some  $z \in Z$ . Since  $w(u_{j+1}) \leq w(z)$  and  $M'$  was a minterm, also  $M^*$  is a minterm. But  $M^*$  is a minterm that starts within  $A_i$  and  $M^* \prec M'$ , thus contradicting our choice of  $M'$ . Hence, by Corollary 4.5, the set  $U_{h,j+1}$  is dependent in  $\mathcal{M}$ . However, since  $s_i < h$  and  $j+1 \leq t_i$ , this dependent set is properly contained in the minterm  $U_{s_i, t_i}$ , leading to a contradiction. Hence,  $A \notin \Gamma$ , and the proof is thus complete.  $\square$

## 7 Ideal WTASs and TPASs

In the previous section we dealt with the case where the lexicographically minimal minterm of  $\Gamma$  is a prefix of  $U$ . Here, we handle the case where the lexicographically minimal minterm of  $\Gamma$  is a lacunary prefix, namely, it takes the form  $M = U_{1,d} \cup U_{d+2,k}$  for some  $1 \leq d \leq k-2$  and  $k \leq n$ . Throughout this section we assume that there is at least one minterm starting with the user  $u_2$ , and that there are no self-sufficient users. If this is not the case, then  $\Gamma$  is a simple composition of access structures as shown in Section 8.2. The section is organized as follows: first we show that  $U_{2,k}$  must be a minterm of  $\Gamma$ . Consequently, we show that  $\Gamma$  is one of the two tripartite access structure (TPASs) that were defined in Definition 3.3. Finally, in Section 7.1, we design linear ideal secret sharing schemes that realize TPASs.

Consider the set  $U_{2,k}$ . As  $U_{2,k} = M \cup \{u_{d+1}\} \setminus \{u_1\}$  and  $w(u_{d+1}) \geq w(u_1)$ , the set  $U_{2,k}$  is authorized. We proceed to show that it is in fact a minterm of  $\Gamma$ .

**Claim 7.1** *Let  $\Gamma$  be an ideal WTAS with  $M = U_{1,d} \cup U_{d+2,k}$  being its lexicographical minimal minterm for some  $1 \leq d \leq k-2$  and  $k \leq n$ . Then if there exists a minterm that has  $u_2$  as its minimal user,  $U_{2,k}$  is a minterm of  $\Gamma$ .*

**Proof:** Let  $M'$  be the lexicographically minimal minterm starting with  $u_2$ . Assume towards contradiction that the authorized set  $U_{2,k}$  is not a minterm. The minterm  $M'$  cannot be a run-minterm since  $U_{2,k-1}$  is unauthorized (as its weight does not exceed that of  $M \setminus \{u_1\}$  which is a proper subset of the minterm  $M$ ) and  $U_{2,k}$  is already an authorized set that is not a minterm. Let  $s$  be the maximal index such that  $U_{2,s} \subset M'$ . In that case, as  $M'$  is not a run-minterm,  $M' = U_{2,s} \cup Z$  where  $Z \neq \emptyset$  and  $u_{s+1} \prec Z_{\min}$ . We claim that  $s \leq k-2$ : Indeed, if  $s \geq k-1$  then  $w(M') \geq w(U_{2,k-1}) + w(Z) \geq w(U_{2,k})$ ; however, this is impossible since  $U_{2,k}$  is an authorized set that is not a minterm while  $M'$  is assumed to be a minterm, and both sets include the user  $u_2$  as the user of minimal weight.

As  $Z$  is the canonical complement of  $U_{2,s}$ , Lemma 5.5 implies that the set  $U_{2,s+1}$  is dependent. Hence, it contains a circuit  $C_1$  of the corresponding matroid  $\mathcal{M}$ . The circuit  $C_1$  must include the user  $u_{d+1}$  (the user that is missing in the minterm  $M$ ), since, otherwise, as  $s+1 \leq k-1$ , the circuit  $C_1$  would be a subset of the minterm  $M = U_{1,d} \cup U_{d+2,k}$ . We claim, and prove in the next paragraph, that  $U_{1,d+1}$  is also a circuit of  $\mathcal{M}$ . Since  $u_1 \in U_{1,d+1}$  but  $u_1 \notin C_1$ , the sets  $C_1$  and  $U_{1,d+1}$  are two distinct circuits. In addition, both circuits include  $u_{d+1}$ . Therefore, by Lemma 4.6, the set  $C_1 \cup U_{1,d+1} \setminus \{u_{d+1}\}$  is dependent. However, this set is contained in the minterm  $M = U_{1,d} \cup U_{d+2,k}$ . This contradiction implies that  $U_{2,k}$  is a minterm in  $\Gamma$ .

Hence, we need only to show that  $U_{1,d+1}$  is a circuit of  $\mathcal{M}$ . Let  $Y = U_{d+2,k}$ ,  $X = U_{1,d+1}$  and consider  $\Gamma_{Y,X}$ , namely, the restriction of  $\Gamma$  induced by  $Y$  on  $X$ . By Lemma 4.11, it is an ideal WTAS, and  $U_{1,d}$  is a prefix minterm in that access structure. Therefore, by Theorem 6.7, the access structure  $\Gamma_{Y,X}$  is an HTAS, and, by Claim 6.2, the set

$U_{1,d+1}$  is a circuit in the matroid corresponding to  $\Gamma_{Y,X}$ . By Lemma 5.5, applied with  $P = U_{1,d}$ , the set  $U_{d+2,k}$  as the canonical complement of  $P$ , and  $b = u_{d+1}$ , the set  $P \cup b = U_{1,d+1}$  is dependent in  $\mathcal{M}$ . Assume, by contradiction that it is not a circuit. Therefore, it possesses a proper subset  $C' \subset U_{1,d+1}$  that is a circuit. By Lemma 4.12, the set  $C'$  would have to be dependent in the matroid corresponding to  $\Gamma_{Y,X}$ . But as  $U_{1,d+1}$  is a circuit in that matroid, this is impossible. Hence,  $U_{1,d+1}$  is also a circuit in  $\mathcal{M}$ . This completes the proof.  $\square$

In view of the above, as we assume herein that there are minterms that start with  $u_2$ , we conclude that  $U_{2,k}$  is a minterm. This is a prefix minterm in the restriction of  $\Gamma$  to  $U_{2,n}$ , denoted hereinafter  $\Gamma'$ . Since  $\Gamma'$  is ideal by Lemma 4.11, and a WTAS by Claim 4.13, and it has a prefix minterm, Theorem 6.7 implies that it is also an HTAS. Since we assumed that  $\Gamma$  has no self-sufficient users, then by Claim 6.5, the HTAS  $\Gamma'$  has at most two levels.

We separate our discussion to two cases, according to the number of levels in  $\Gamma'$ . First, we consider the case where  $\Gamma'$  is a simple threshold access structure.

**Claim 7.2** *If  $\Gamma'$  is an HTAS of one level (i.e., it is a threshold access structure), then a set  $X \subseteq U$  is in  $\Gamma$  if and only if  $|X| \geq k - 1$  and  $|X \cap U_{d+2,n}| \geq k - 1 - d$ .*

**Proof:** Let us begin with sets  $X$  that do not include the user  $u_1$ , namely,  $X \subseteq U_{2,n}$ . For such sets, the second threshold condition ( $|X \cap U_{d+2,n}| \geq k - 1 - d$ ) is implied by the first one ( $|X| \geq k - 1$ ). Since we assumed that  $\Gamma'$ , which is the restriction of  $\Gamma$  to  $U_{2,n}$ , is a simple threshold access structure, and the value of the threshold equals  $|U_{2,k}| = k - 1$ , the statement is straightforward in this case.

Next, consider sets  $X$  such that  $u_1 \in X$ . If  $X$  satisfies the two conditions in the claim, then its weight is no less than the weight of  $M = U_{1,d} \cup U_{d+2,k}$ . As  $w(M) \geq T$ , we conclude that  $X \in \Gamma$ . Conversely, assume that  $X \in \Gamma$  but it fails to satisfy one of the two conditions in the claim. Without loss of generality, we may assume that  $X$  is a minterm of  $\Gamma$ . Thus, by Lemma 5.3,  $|X| = |M| = k - 1$ . Therefore,  $X$  may violate only the second condition in the claim, namely,  $|X \cap U_{d+2,n}| < k - 1 - d$ . But then  $U_{1,d+1} \subset X$  and that cannot be since then  $X \prec M = U_{1,d} \cup U_{d+2,k}$  and  $M$  was assumed to be the lexicographically minimal minterm in  $\Gamma$ .  $\square$

The access structure that is described in Claim 7.2 is a tripartite access structure with  $B = \emptyset$ . This special case of tripartite access structures is referred to as *bipartite*. In a bipartite access structure there are two classes of users, and users in the same class play an equivalent role in the access structure. Namely, in order to determine whether a given subset is authorized, it suffices to examine how many users it has in each class. The two classes we have here are  $U_{1,d+1}$  and  $U_{d+2,n}$ . In [22], all ideal bipartite access structures were characterized and realized by suitable linear secret sharing schemes. (The access structure in Claim 7.2 falls also under the class of access structures that were introduced and realized by explicit linear secret sharing schemes in [32].)

We are left with the case where  $\Gamma'$  is an HTAS of exactly two levels. Assume that  $X \in \Gamma'$  if it has at least  $k_1$  users from  $U_{2,n}$  or at least  $k_2$  users from  $U_{r+1,n}$  for some  $3 \leq r \leq n - 2$ . As  $U_{2,k}$  is a minterm,  $k_1 = k - 1$ .

**Claim 7.3** *Under the above assumptions, a set  $X \subseteq U$  is in  $\Gamma$  if and only if  $|X \cap U_{r+1,n}| \geq k_2$  or both  $|X| \geq k_1$  and  $|X \cap U_{d+2,n}| \geq k_1 - d$ .*

**Proof:** Assume first that  $X \subseteq U$  satisfies one of the above two threshold conditions. If  $|X \cap U_{r+1,n}| \geq k_2$  then  $X \in \Gamma'$  and thus  $X \in \Gamma$ . If  $X$  satisfies the second condition then it also belongs to  $\Gamma$  since the set of minimal weight  $X$  that satisfies both  $|X| \geq k_1$  and  $|X \cap U_{d+2,n}| \geq k_1 - d$  is  $X = M = U_{1,d} \cup U_{d+2,k}$ , and  $M \in \Gamma$ .

Assume next that  $X \in \Gamma$ . Without loss of generality, we may assume that  $X$  is a minterm and prove that it satisfies one of the two threshold conditions in the claim. We separate the discussion into two cases. If  $u_1 \notin X$  then  $X \in \Gamma'$ . Thus either  $|X \cap U_{r+1,n}| \geq k_2$  or  $|X| \geq k_1$ . If  $|X| \geq k_1$  then, as  $u_1 \notin X$ , it must be that  $|X \cap U_{d+2,n}| \geq k_1 - d$ . If, on the other hand,  $u_1 \in X$ , then by Lemma 5.3, we get that  $|X| = k_1$ . But then, since no minterm contains  $U_{1,d+1}$ , it must be that  $|X \cap U_{d+2,n}| \geq k_1 - d$ .  $\square$

The access structure  $\Gamma$ , as appears in Claim 7.3, coincides with one of the two TPASs in Definition 3.3. In case  $r \geq d + 1$ , we set  $A = U_{1,d+1}$ ,  $B = U_{d+2,r}$ ,  $C = U_{r+1,n}$  and then  $\Gamma$  is of type  $\Delta_1$ ,

$$\Gamma = \{X \subseteq U : (|X| \geq k_1 \text{ and } |X \cap (B \cup C)| \geq k_1 - d) \text{ or } |X \cap C| \geq k_2\}.$$



If, on the other hand,  $r < d + 1$ , we claim that  $\Gamma$  is of type  $\Delta_2$ . To that end, we set  $A = U_{1,r}$ ,  $B = U_{r+1,d+1}$ , and  $C = U_{d+2,n}$ . By Claim 7.3, the structure of  $\Gamma$  is

$$\Gamma = \{X \subseteq U : (|X| \geq k_1 \text{ and } |X \cap C| \geq k_1 - d) \text{ or } |X \cap (B \cup C)| \geq k_2\}.$$

It remains to show that the size of the set  $B$  is bounded, in accord with Definition 3.3.

**Claim 7.4** *If  $r < d + 1$ , then  $|B| < d + k_2 - k_1$ .*

**Proof:** The access structure  $\Gamma'$ , the restriction of  $\Gamma$  to  $U_{2,n}$ , is an HTAS of two levels. A set  $X$  is in  $\Gamma'$  if it has at least  $k_1$  users from  $U_{2,n}$  or at least  $k_2$  users from  $U_{r+1,n}$ . Therefore,  $U_{2,k} = U_{2,k_1+1}$  is a minterm and so is  $U_{r+1,r+k_2}$ . Consequently,  $r + k_2 > k_1 + 1$  (since otherwise the second minterm would have been a proper subset of the first one). As  $|B| = |U_{r+1,d+1}| = d + 1 - r$ , we infer that  $|B| < d + k_2 - k_1$ .  $\square$

**Example 7.5** Consider the set  $U = \{u_1, \dots, u_9\}$ , and let  $\Gamma$  be a WTAS where the weights are 16, 16, 17, 18, 19, 24, 24, 24, and 24 and the threshold is 92. First note that there is no prefix minterm as  $w(U_{1,5}) = 16 + 16 + 17 + 18 + 19 = 86 < 92$  and  $w(U_{2,6}) = 16 + 17 + 18 + 19 + 24 = 94 \geq 92$ , and thus  $U_{1,6}$  is not a minterm. The lexicographically minimal minterm is  $U_{1,3} \cup U_{5,6}$ , and so  $d = 3$  and  $k = 6$ . The HTAS structure resulting from the restriction of  $\Gamma$  to  $U_{2,9}$  is composed of two levels  $U_{2,5}$  and  $U_{6,9}$ , that is  $r = 5$ , with thresholds  $k_1 = 5$  and  $k_2 = 4$ . Since  $r > d + 1$ , the access structure is of type  $\Delta_1$ . Setting  $A = U_{1,4}$ ,  $B = \{u_5\}$ , and  $C = U_{6,9}$ , a set is authorized if it has at least 4 users from  $C$ , or if it has at least 5 users, where at least 2 of whom are from  $B \cup C$ .

We summarize this section in the next theorem.

**Theorem 7.6** *Let  $\Gamma$  be an ideal WTAS with  $M = U_{1,d} \cup U_{d+2,k}$  being its lexicographical minimal minterm for some  $1 \leq d \leq k - 2$  and  $k \leq n$ . If there exists a minterm in  $\Gamma$  that has  $u_2$  as its minimal member and  $\Gamma$  has no self-sufficient users, then  $\Gamma$  is a TPAS.*

## 7.1 Ideal Linear Secret Sharing Schemes for TPASs

We now turn to describe ideal linear secret sharing schemes that implement the TPASs  $\Delta_1$  and  $\Delta_2$  in Definition 3.3. Those schemes extend the ideas of the schemes that were presented in [22] for bipartite access structures. In fact, if we take  $B = \emptyset$  then our schemes coincide with the schemes of [22].

Let us first briefly recall the definition of those access structures. Assume that  $|U| = n$  and  $U = A \cup B \cup C$ , where  $A$ ,  $B$ , and  $C$  are disjoint, and  $A, C \neq \emptyset$ . Let  $m, d, t \in \mathbb{N}$  be such that  $m > t$ . Then

$$\Delta_1 = \{X \subseteq U : (|X| \geq m \text{ and } |X \cap (B \cup C)| \geq m - d) \text{ or } |X \cap C| \geq t\},$$

and

$$\Delta_2 = \{X \subseteq U : (|X| \geq m \text{ and } |X \cap C| \geq m - d) \text{ or } |X \cap (B \cup C)| \geq t\},$$

where in  $\Delta_2$  the size of  $B$  is restricted to  $|B| \leq d + t - m$ .

Let  $\mathbb{F}$  be a sufficiency large finite field. For a  $\lambda \in \mathbb{F}$  and an integer  $s$  define the Vandermonde vector  $\mathbf{V}_s(\lambda) := (1, \lambda, \lambda^2, \dots, \lambda^{s-1})$  (for clarity, we denote hereinafter all vectors with bold-faced letters). Recall that for any set  $A \subseteq \mathbb{F}$  if  $|A| \leq s$  then the vectors  $\{\mathbf{V}_s(\lambda) : \lambda \in A\}$  are linearly independent. Thus, for any set  $A \subseteq \mathbb{F}$  if  $|A| \geq s$ , the vectors  $\{\mathbf{V}_s(\lambda) : \lambda \in A\}$  span the entire space  $\mathbb{F}^s$ . Let  $E = \mathbb{F}^m$  and  $E_1$  and  $E_2$  be subspaces of  $E$  of dimensions  $d$  and  $t$ , respectively, such that if  $d + t \geq m$  then  $E = E_1 + E_2$ , otherwise  $E_1 \cap E_2 = \{\vec{0}\}$ . Denote  $r = \dim(E_1 \cap E_2) = \max(0, d + t - m)$ . If  $r > 0$ , let  $\Lambda = \{\lambda_1, \dots, \lambda_r\}$  be a set of elements from  $\mathbb{F}$ , otherwise  $\Lambda = \emptyset$ . Let

$$\varphi_1 : \mathbb{F}^d \rightarrow E_1 \text{ and } \varphi_2 : \mathbb{F}^t \rightarrow E_2$$

be two isomorphisms such that

$$\varphi_1(\mathbf{V}_d(\lambda_i)) = \varphi_2(\mathbf{V}_t(\lambda_i)) \quad \forall \lambda_i \in \Lambda. \quad (3)$$

Finally, we define the functions  $\mathbf{a} : \mathbb{F} \rightarrow E_1$  and  $\mathbf{b} : \mathbb{F} \rightarrow E_2$  as follows:

$$\mathbf{a}(\lambda) = \varphi_1(\mathbf{V}_d(\lambda)) \text{ and } \mathbf{b}(\lambda) = \varphi_2(\mathbf{V}_t(\lambda)) \quad \forall \lambda \in \mathbb{F}.$$

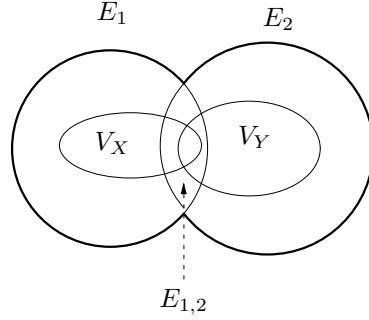


Figure 2: The linear spaces in the proof of Lemma 7.7.

The target vector in the ideal linear secret sharing schemes that we construct below is  $\mathbf{b}(\lambda_T)$  for some  $\lambda_T \in \mathbb{F} \setminus \Lambda$ . Note that the target vector is in  $E_2 \setminus E_1$ .

Next, we describe the function  $\eta : U \rightarrow E$  that assigns vectors from  $E$  to users in  $U$ , so that the vectors of authorized subsets span the target vector, while the vectors of unauthorized subsets do not, i.e.,  $\mathbf{b}(\lambda_T) \in \text{span}\{\eta(S)\}$  iff  $S \in \Gamma$ . By Theorem 2.5, this implies that the scheme realizes  $\Gamma$ . The technique we use is similar for both types of access structures,  $\Gamma = \Delta_1$  and  $\Gamma = \Delta_2$ . We assign the vectors to users one at a time and prove correctness by induction. Assume that we already assigned vectors to all users in  $U' \subset U$  and that we wish to find a proper assignment of a vector to an additional user  $u \in U \setminus U'$ . We shall denote this stage in the assignment by  $\langle U', u \rangle$ . The goal is to find an assignment  $\eta(u)$  of a vector, so that the following conditions hold:

- C1. If  $S \subseteq U'$ ,  $S \notin \Gamma$  but  $S' := S \cup \{u\} \in \Gamma$ , then  $\eta(u)$  should be such that  $\mathbf{b}(\lambda_T) \in \text{span}\{\eta(S')\}$ .
- C2. If  $S \subseteq U'$ ,  $S \notin \Gamma$  and  $S' := S \cup \{u\} \notin \Gamma$ , then  $\eta(u)$  should be such that  $\mathbf{b}(\lambda_T) \notin \text{span}\{\eta(S')\}$ .

We shall always look for the vector  $\eta(u)$  in a single-parameterized family of vectors  $\{\zeta(\lambda) \in E : \lambda \in \mathbb{F}\}$  (in the constructions that we present here,  $\zeta(\lambda)$  is one of  $\mathbf{a}(\lambda)$ ,  $\mathbf{b}(\lambda)$ , and  $\mathbf{V}_m(\lambda)$ ). Using Lemma 7.7 below, we shall show that at each stage of the assignment,  $\langle U', u \rangle$ , and for every  $S \subseteq U'$ , where  $S \notin \Gamma$ , there are finitely many bad choices of the parameter  $\lambda$  that give rise to vectors  $\eta(u) = \zeta(\lambda)$  that would violate the relevant condition above. Hence, since  $|2^{U'} \setminus \Gamma|$  is bounded, there exists a good choice of  $\lambda$  such that  $\eta(u) = \zeta(\lambda)$  satisfies all necessary conditions, provided that the field is sufficiently large. Thus, in the rest of the proof we bound the number of bad values for a specific set  $S$ .

Since we are interested only in proving that both TPASSs,  $\Delta_1$  and  $\Delta_2$ , are ideal, we do not address herein the question of how large  $\mathbb{F}$  should be with respect to the parameters of the access structure, nor we are interested in the efficiency of the proposed schemes. However, similarly to [22], it can be verified that the size of the field that our schemes require, which is the size of the domain of secrets and shares, is  $2^{O(n)}$ .

**Lemma 7.7** *Let  $X \subseteq \mathbb{F}$  and  $Y \subseteq \mathbb{F} \setminus \Lambda$  be such that  $|X| < d$  and  $|X| + |Y| < m$ . Let  $V = \text{span}\{\mathbf{a}(x), \mathbf{b}(y) : x \in X, y \in Y\}$ . Then  $E_1 \setminus V \neq \emptyset$ .*

**Proof:** Hereinafter, let  $V_X = \text{span}\{\mathbf{a}(x) : x \in X\}$ ,  $V_Y = \text{span}\{\mathbf{b}(y) : y \in Y\}$ , and  $E_{1,2} = E_1 \cap E_2$  (see Figure 2 for an illustration of those linear spaces). Let us begin with the fairly easy case when  $m \geq d + t$ . In that case, since  $E_1 \cap E_2 = \{\vec{0}\}$ , we claim that  $V \cap E_1 = V_X$ , whence  $\dim(E_1 \setminus V) = \dim E_1 - \dim V_X = d - |X| > 0$ . Indeed, let  $\mathbf{u}$  be an arbitrary vector from  $V \cap E_1$ . Then, on one hand,  $\mathbf{u} = \mathbf{v}_x + \mathbf{v}_y$  for some  $\mathbf{v}_x \in V_X$  and  $\mathbf{v}_y \in V_Y$ , and on the other hand  $\mathbf{u} \in E_1$ . Therefore,  $\mathbf{v}_y = \mathbf{u} - \mathbf{v}_x \in E_1 - V_X = E_1$ . But since  $\mathbf{v}_y \in V_Y \subseteq E_2$  and  $E_1 \cap E_2 = \{\vec{0}\}$ , we infer that  $\mathbf{v}_y = \vec{0}$ . This implies that  $\mathbf{u} = \mathbf{v}_x \in V_X$ .

Next, we prove the claim for the case where  $r = d + t - m > 0$ . Without loss of generality, we may assume that  $|Y| = t - k$  for some  $0 \leq k \leq r$ . Indeed, if  $|Y| > t$  it may be reduced to  $Y^- \subset Y$  of size  $|Y^-| = t$  without affecting  $V$ . If, on the other hand,  $|Y| < t - r$ , it may be extended to a superset  $Y^+ \supset Y$  of size  $|Y^+| = t - r$ . Since  $|X| + |Y^+| < d + t - r = m$ , we may prove the statement for  $X$  and  $Y^+$  and then infer that it also holds for  $X$  and  $Y$ .

Assume, towards contradiction, that  $E_1 \subseteq V$ , whence  $\dim(V \cap E_1) = \dim E_1 = d$ . We claim that this assumption implies the following two statements:

$$\dim(V_Y \cap E_{1,2}) = r - k \quad \text{and} \quad \dim(V_X \cap E_{1,2}) < k. \quad (4)$$

After proving these two statements, we may derive the sought-after contradiction as follows. Since we assumed that  $E_1 \subseteq V = V_X + V_Y$ , all vectors  $e \in E_{1,2}$  may be expressed as  $e = x + y$  where  $x \in V_X$  and  $y \in V_Y$ . But since  $\dim E_{1,2} = r > \dim(V_X \cap E_{1,2}) + \dim(V_Y \cap E_{1,2})$ , as implied by the two statements in (4), there exists at least one vector  $e \in E_{1,2}$ , such that  $e = x + y$ , for which either  $x \in E_1 \setminus E_2$  or  $y \in E_2 \setminus E_1$ . Without loss of generality, we assume that  $x \in E_1 \setminus E_2$ . But then  $x = e - y \in E_{1,2} + V_Y \subseteq E_2$ , as opposed to our assumption that  $x \notin E_2$ .

Next, we prove the statements in (4). Let us begin with the equality in (4). The dimension of  $E_{1,2} + V_Y$  is  $t$  since the  $t - k$  base vectors of  $V_Y$ , namely  $\{\mathbf{b}(y) : y \in Y\}$ , may be extended to a full basis of  $E_2$  by any  $k$  base vectors of  $E_{1,2}$  of the form  $\mathbf{b}(\lambda)$ ,  $\lambda \in \Lambda$  (recall that  $|\Lambda| = r \geq k$  and that  $Y \subset \mathbb{F} \setminus \Lambda$ ). Therefore,

$$\dim(V_Y \cap E_{1,2}) = \dim V_Y + \dim E_{1,2} - \dim(E_{1,2} + V_Y) = (t - k) + r - t = r - k.$$

In order to prove the inequality in (4), we first prove that

$$\dim(V_X + E_{1,2}) = d. \quad (5)$$

Since  $V_X + E_{1,2} \subseteq E_1$  and  $\dim E_1 = d$ , all we need to show is that  $E_1 \subseteq V_X + E_{1,2}$ . We assumed that  $E_1 \subseteq V = V_X + V_Y$ . Hence, for each  $e \in E_1$  there exist  $x \in V_X$  and  $y \in V_Y$  such that  $e = x + y$ . But then  $y = e - x \in E_1 + V_X = E_1$ . Since  $y \in V_Y \subseteq E_2$  we conclude that  $y \in E_{1,2}$ . This implies that  $e \in V_X + E_{1,2}$ , whence we arrive at the required conclusion that  $E_1 \subseteq V_X + E_{1,2}$ . Finally, the inequality in (4) is a straightforward consequence of (5). Indeed,

$$\dim(V_X \cap E_{1,2}) = \dim V_X + \dim E_{1,2} - \dim(V_X + E_{1,2}) = |X| + r - d;$$

recalling that  $|X| < m - |Y| = m - t + k$  and  $r = t + d - m$  we infer that  $\dim(V_X \cap E_{1,2}) < k$ . That concludes the proof.  $\square$

### 7.1.1 Realizing $\Delta_1$

Here,  $\Gamma$  is an access structure of type  $\Delta_1$ . We assign the vectors in three stages. We first describe the assignment  $\boldsymbol{\eta}(u)$  for all  $u \in C$ , then for  $u \in A$  and finally for  $u \in B$ .

**Step 1: Assigning vectors to users in  $C$ .** Let  $C = \{c_i\}_{1 \leq i \leq |C|}$ , and let  $\gamma_i$ , where  $1 \leq i \leq |C|$ , be distinct elements of  $\mathbb{F} \setminus \Lambda$ , all different from  $\lambda_T$ . Then

$$\boldsymbol{\eta}(c_i) = \mathbf{b}(\gamma_i) \quad 1 \leq i \leq |C|.$$

This is essentially the Shamir  $t$ -out-of- $|C|$  scheme. Therefore, every  $t$  users in  $C$  can reconstruct  $\mathbf{b}(\lambda_T)$  while no  $t - 1$  users can.

**Step 2: Assigning vectors to users in  $A$ .** Let  $A = \{a_i\}_{1 \leq i \leq |A|}$ . Our claim is that, assuming  $\mathbb{F}$  is large enough, we may find  $\alpha_i \in \mathbb{F}$ , where  $1 \leq i \leq |A|$ , so that the assignment  $\boldsymbol{\eta}(a_i) = \mathbf{a}(\alpha_i)$  is consistent with  $\Gamma$ , restricted to  $A \cup C$ , and, in addition, for any  $S \subseteq A \cup C$  where  $S \notin \Gamma$ , the following equality holds,

$$\dim \text{span}\{\boldsymbol{\eta}(S)\} = |S \cap (B \cup C)| + \min(|S \cap A|, d). \quad (6)$$

(Herein, we could have replaced  $S \cap (B \cup C)$  with  $S \cap C$  since  $S$  does not include users from  $B$ .) Namely, whenever we augment an unauthorized set  $S \subseteq A \cup C$  with an additional user from  $A$ , the dimension of the corresponding subspace that is spanned by the vectors owned by  $S$  increases by 1, up to a limit of  $d$  users from  $A$  (i.e., all users from  $A$  beyond the first  $d$  ones are redundant). Using (6) we guarantee that if  $S \in \Gamma$  and  $|S \cap C| < t$  then the vectors assigned to  $S$  span  $E$ , and in particular, span the target vector  $\mathbf{b}(\lambda_T)$ . We shall prove this claim by induction. Assume that we already assigned vectors to  $A_{1,j} := \{a_1, \dots, a_j\}$  for some  $0 \leq j < |A|$ , so that  $\Gamma$  is respected and (6) holds for all  $S \in 2^{U'} \setminus \Gamma$ , where  $U' = A_{1,j} \cup C$ . This assumption is obviously true for  $j = 0$ . We proceed to look for an assignment  $\boldsymbol{\eta}(a_{j+1}) = \mathbf{a}(\alpha_{j+1})$  for the next user from  $A$  so that conditions C1 and C2 hold for all  $S \in 2^{U'} \setminus \Gamma$

(in order to keep respecting  $\Gamma$ ) and equality (6) holds for all  $S \in 2^{U' \cup \{a_{j+1}\}} \setminus \Gamma$ . As explained earlier, it suffices to concentrate on a single set  $S \in 2^{U'} \setminus \Gamma$  and bound the number of bad choices of  $\alpha_{j+1} \in \mathbb{F}$  for that specific set.

Let  $S \in 2^{U'} \setminus \Gamma$  be such that  $S' := S \cup \{a_{j+1}\} \in \Gamma$ . Since  $|S'| = |S| + 1$  but  $|S' \cap (B \cup C)| = |S \cap (B \cup C)|$ , the fact that  $S \notin \Gamma$  and  $S' \in \Gamma$  implies that  $|S| = m - 1$  and  $|S \cap (B \cup C)| \geq m - d$ . This, in turn, implies that  $|S \cap A_{1,j}| < d$ . Therefore, by Lemma 7.7, applied to  $X = \{\alpha_i\}_{1 \leq i \leq j}$  and  $Y = \{\gamma_i\}_{1 \leq i \leq |C|}$ , the vectors owned by  $S$  span a subspace of  $E_1$  of dimension  $d - 1$  at the most. Hence, for all but at most  $d - 1$  values of  $\alpha_{j+1} \in \mathbb{F}$ , we have  $\dim \text{span}\{\boldsymbol{\eta}(S')\} = \dim \text{span}\{\boldsymbol{\eta}(S)\} + 1$ . Now, by the induction hypothesis, the set  $S$  satisfies equality (6). Therefore, since  $|S \cap A| = |S \cap A_{1,j}| < d$ , we get that

$$\dim \text{span}\{\boldsymbol{\eta}(S)\} = |S \cap C| + \min(|S \cap A|, d) = |S \cap C| + |S \cap A| = |S| = m - 1.$$

This implies that  $\dim \text{span}\{\boldsymbol{\eta}(S')\} = m = \dim E$  so that  $\mathbf{b}(\lambda_T) \in \text{span}\{\boldsymbol{\eta}(S')\}$  and  $CI$  holds.

Let  $S \in 2^{U'} \setminus \Gamma$  be such that  $S' := S \cup \{a_{j+1}\} \notin \Gamma$ . If  $|S \cap A_{1,j}| \geq d$ , then  $\text{span}\{\boldsymbol{\eta}(S')\} = \text{span}\{\boldsymbol{\eta}(S)\}$  for any choice of  $\alpha_{j+1}$ ; hence, no matter how we chose  $\alpha_{j+1}$  in this case, the augmented set  $S'$  will remain incapable of spanning the target vector. Also, since any  $d$  vectors of the form  $\mathbf{a}(\alpha_i)$  span the same subspace  $E_1$ , any choice of  $\alpha_{j+1}$  will result with an assignment that keeps respecting equality (6).

Hence, we restrict our attention to subsets  $S \in 2^{U'} \setminus \Gamma$  for which  $|S \cap A_{1,j}| < d$ . We claim that in this case  $|S| \leq m - 2$ . Assume, by contradiction, that  $|S| \geq m - 1$ . As  $|S \cap A_{1,j}| < d$  we infer that  $|S \cap C| \geq m - d$ . Since  $|S' \cap C| = |S \cap C| \geq m - d$  and  $|S'| = |S| + 1 \geq m$ , we infer that  $S' \in \Gamma$ , as opposed to our assumption. Hence,  $|S| \leq m - 2$ . Our goal is to find  $\alpha_{j+1}$  so that the target vector is not in  $\text{span}\{\boldsymbol{\eta}(S')\}$  (in order to keep respecting  $\Gamma$ ) and that equality (6) still holds for all  $S \in 2^{U'} \setminus \Gamma$ . Concentrating on the first part of our mission, we want to find  $\alpha_{j+1}$  so that  $\mathbf{a}(\alpha_{j+1}) \notin \text{span}\{\boldsymbol{\eta}(S) \cup \{\mathbf{b}(\lambda_T)\}\}$ . By Lemma 7.7, for  $X = \{\alpha_i\}_{1 \leq i \leq j}$  and  $Y = \{\gamma_i\}_{1 \leq i \leq |C|} \cup \{\lambda_T\}$ , the space  $\text{span}\{\boldsymbol{\eta}(S) \cup \{\mathbf{b}(\lambda_T)\}\} \cap E_1$  is a subspace of  $E_1$  of dimension  $d - 1$  at the most. We infer that for all but possibly  $d - 1$  values of  $\alpha_{j+1} \in \mathbb{F}$ , the vector  $\mathbf{a}(\alpha_{j+1})$  increases the dimension of  $\text{span}\{\boldsymbol{\eta}(S) \cup \{\mathbf{b}(\lambda_T)\}\}$  by 1. It is easy to see that for all such selections, we get an assignment that will also respect equality (6). This completes the proof that a proper assignment of vectors exists for all users in  $A \cup C$ . We are ready to proceed to the third and final stage in the assignment.

**Step 3: Assigning vectors to users in  $B$ .** Here we assign vectors to the users in  $B$  (note that  $B$  may be empty). Let  $B = \{b_i\}_{1 \leq i \leq |B|}$ . As before, we claim that, provided  $\mathbb{F}$  is large enough, there exist  $\beta_i \in \mathbb{F}$ , where  $1 \leq i \leq |B|$ , so that the assignment  $\boldsymbol{\eta}(b_i) = \mathbf{V}_m(\beta_i)$  is consistent with  $\Gamma$ , and, in addition, for any  $S \notin \Gamma$ , equality (6) holds. Namely, whenever we augment an unauthorized set  $S$  with an additional user from  $B$ , the corresponding subspace that is spanned by the vectors owned by  $S$  increases by 1. We shall prove this claim by induction. Assume that we already assigned vectors to  $B_{1,j} := \{b_1, \dots, b_j\}$  for some  $0 \leq j < |B|$ , so that  $\Gamma$  is respected and (6) holds for all  $S \in 2^{U'} \setminus \Gamma$ , where  $U' = B_{1,j} \cup A \cup C$ . This assumption is obviously true for  $j = 0$ . Next, we look for an assignment for the next user from  $B$ , that is  $\boldsymbol{\eta}(b_{j+1}) = \mathbf{V}_m(\beta_{j+1})$ , so that conditions  $CI$  and  $C2$  hold for all  $S \in 2^{U'} \setminus \Gamma$  (in order to keep respecting  $\Gamma$ ) and equality (6) holds for all  $S \in 2^{U' \cup \{b_{j+1}\}} \setminus \Gamma$ . Again, we concentrate on a single set  $S \in 2^{U'} \setminus \Gamma$  and bound the number of bad choices of  $\beta_{j+1} \in \mathbb{F}$  for that specific set.

Let  $S \in 2^{U'} \setminus \Gamma$  be such that  $S' := S \cup \{b_{j+1}\} \in \Gamma$ . Since  $S \notin \Gamma$  but  $S' \in \Gamma$ , and the added user  $b_{j+1}$  is from  $B$ , we conclude that while  $S$  fails to satisfy the first of the two threshold conditions in the definition of  $\Delta_1$ , i.e.,  $|S| < m$  or  $|S \cap (B \cup C)| < m - d$ , the set  $S'$  does satisfy that condition, namely,  $|S'| \geq m$  and  $|S' \cap (B \cup C)| \geq m - d$ . There are two cases to consider here, according to which of the two threshold conditions were violated by  $S$ :

1.  $|S| = m - 1$ .
2.  $|S| \geq m$  but  $|S \cap (B \cup C)| = m - d - 1$ .

In the first case, we claim that  $|S \cap A| \leq d$ . Indeed, since  $|S' \cap (B \cup C)| \geq m - d$  and  $|S' \cap (B \cup C)| = |S \cap (B \cup C)| + 1$ , we have  $|S \cap (B \cup C)| \geq m - d - 1$ , whence  $|S \cap A| = |S| - |S \cap (B \cup C)| \leq (m - 1) - (m - d - 1) = d$ . Applying the induction hypothesis to  $S$ , we get by equality (6) that

$$\dim \text{span}\{\boldsymbol{\eta}(S)\} = |S \cap (B \cup C)| + \min(|S \cap A|, d) = |S \cap (B \cup C)| + |S \cap A| = |S| = m - 1.$$

Therefore, for all but  $m - 1$  values of  $\beta_{j+1}$ , we get that  $\dim \text{span}\{\boldsymbol{\eta}(S')\} = \dim \text{span}\{\boldsymbol{\eta}(S)\} + 1 = m$ , whence  $\mathbf{b}(\lambda_T) \in \text{span}\{\boldsymbol{\eta}(S')\}$ .

Next, we show how to reduce the second case to the first case. In the second case  $|S \cap A| = |S| - |S \cap (B \cup C)| \geq m - (m - d - 1) = d + 1$ . Hence,  $S$  includes too many users from  $A$ , since for all  $a \in A$ , the vector  $\boldsymbol{\eta}(a)$  is chosen from the  $d$ -dimensional space  $E_1$ . Let  $a \in S \cap A$  and define  $S_1 = S \setminus \{a\}$  and  $S'_1 = S_1 \cup \{b_{j+1}\}$ . Obviously,  $S_1 \notin \Gamma$  and, as can be easily verified,  $S'_1 \in \Gamma$ . Since  $\text{span}\{\boldsymbol{\eta}(S)\} = \text{span}\{\boldsymbol{\eta}(S_1)\}$  and  $\text{span}\{\boldsymbol{\eta}(S')\} = \text{span}\{\boldsymbol{\eta}(S'_1)\}$ , if  $\beta_{j+1}$  is chosen properly for  $S_1$ , it will be also an appropriate choice for  $S$ . Hence, we may repeat this reduction stage until we reach a set  $S_1$  of size  $|S_1| = m - 1$ , and then apply the arguments of the first case to conclude that all but  $m - 1$  values of  $\beta_{j+1}$  are appropriate choices.

Let  $S \in 2^{U'} \setminus \Gamma$  be such that  $S' := S \cup \{b_{j+1}\} \notin \Gamma$ . Arguing along the same lines as above, we may assume, without loss of generality, that  $|S \cap A| \leq d$ , since any  $d$  members of  $A$  span  $E_1$ . We claim that  $|S| \leq m - 2$ . Indeed, if  $|S| \geq m - 1$  then  $|S \cap (B \cup C)| \geq m - 1 - d$ , as  $|S \cap A| \leq d$ . This implies that  $|S' \cap (B \cup C)| \geq m - d$  and  $|S'| \geq m$ , whence  $S' \in \Gamma$ , in contradiction to our assumption. Therefore, the dimension of  $\text{span}\{\boldsymbol{\eta}(S) \cup \{\mathbf{b}(\lambda_T)\}\}$  is at most  $m - 2 + 1$ . Consequently, for all but possibly  $m - 1$  values of  $\beta_{j+1} \in \mathbb{F}$ , we have  $\mathbf{V}_m(\beta_{j+1}) \notin \text{span}\{\boldsymbol{\eta}(S) \cup \{\mathbf{b}(\lambda_T)\}\}$ . Hence, for all but possibly  $m - 1$  values of  $\beta_{j+1} \in \mathbb{F}$ , we get that  $\mathbf{b}(\lambda_T) \notin \text{span}\{\boldsymbol{\eta}(S) \cup \{\mathbf{V}_m(\beta_{j+1})\}\}$ . Clearly, all such selections yield assignments that respect equality (6).

This concludes the description of the assignment  $\boldsymbol{\eta} : U \rightarrow E$  and the proof that it yields an ideal linear sharing scheme that realizes a given access structure  $\Gamma$  of type  $\Delta_1$ .

### 7.1.2 Realizing $\Delta_2$

Here we propose an ideal linear secret sharing scheme for access structures  $\Gamma$  that are a TPAS of type  $\Delta_2$ . Recall that, by Definition 3.3, in such access structures  $|B| \leq |\Lambda| = r = d + t - m$ . We concentrate on cases where  $r > 0$  (if  $r = 0$  then  $B = \emptyset$  and then we get a bipartite access structure).

**Step 1: Assigning vectors to users in  $B \cup C$ .** The users in  $B$  and  $C$  are assigned distinct vectors of the form  $\mathbf{b}(\beta_i)$  and  $\mathbf{b}(\gamma_i)$ , respectively, where  $\{\beta_i\}_{1 \leq i \leq |B|} \subset \Lambda$  and  $\{\gamma_i\}_{1 \leq i \leq |C|} \subset \mathbb{F} \setminus (\Lambda \cup \{\lambda_T\})$ . Clearly, any  $t$  users from  $B \cup C$  may reconstruct  $\mathbf{b}(\lambda_T)$  while no  $t - 1$  users can.

**Step 2: Assigning vectors to users in  $A$ .** Let  $A = \{a_i\}_{1 \leq i \leq |A|}$ . Our claim is that, assuming  $\mathbb{F}$  is large enough, we may find  $\alpha_i \in \mathbb{F}$ , where  $1 \leq i \leq |A|$ , so that the assignment  $\boldsymbol{\eta}(a_i) = \mathbf{a}(\alpha_i)$  is consistent with  $\Gamma$ , and, in addition, for any  $S \subseteq U$  where  $S \notin \Gamma$ , the following equality holds,

$$\dim \text{span}\{\boldsymbol{\eta}(S)\} = |S \cap C| + \min(|S \cap (A \cup B)|, d). \quad (7)$$

We shall prove this claim by induction. Assume that we already assigned vectors to  $A_{1,j} := \{a_1, \dots, a_j\}$  for some  $0 \leq j < |A|$ , so that  $\Gamma$  is respected and (7) holds for all  $S \in 2^{U'} \setminus \Gamma$ , where  $U' = A_{1,j} \cup B \cup C$ . This assumption is true for  $j = 0$  since then, on one hand,  $\dim \text{span}\{\boldsymbol{\eta}(S)\} = |S|$ , while on the other hand  $S \cap A = \emptyset$  and  $|S \cap B| \leq |B| \leq d + t - m \leq d$ . We proceed to look for an assignment  $\boldsymbol{\eta}(a_{j+1}) = \mathbf{a}(\alpha_{j+1})$  for the next user from  $A$ , so that conditions C1 and C2 hold for all  $S \in 2^{U'} \setminus \Gamma$  and equality (7) holds for all  $S \in 2^{U' \cup \{a_{j+1}\}} \setminus \Gamma$ .

Let  $S \in 2^{U'} \setminus \Gamma$  be such that  $S' := S \cup \{a_{j+1}\} \in \Gamma$ . We infer that  $|S| = m - 1$  and  $|S \cap C| \geq m - d$ . This implies that  $|S \cap (A \cup B)| = |S| - |S \cap C| \leq d - 1$ . By Lemma 7.7, applied to  $X = \{\alpha_i\}_{1 \leq i \leq j} \cup \{\beta_i\}_{1 \leq i \leq |B|}$  and  $Y = \{\gamma_i\}_{1 \leq i \leq |C|}$  (here we use the fact that  $\beta_i \in \Lambda$  so that, in view of (3),  $\mathbf{a}(\beta_i) = \mathbf{b}(\beta_i)$ ), the vectors owned by  $S$  span a subspace of  $E_1$  of dimension  $d - 1$  at the most. Hence, for all but  $d - 1$  values of  $\alpha_{j+1} \in \mathbb{F}$ ,  $\dim \text{span}\{\boldsymbol{\eta}(S')\} = \dim \text{span}\{\boldsymbol{\eta}(S)\} + 1$ . Now, by the induction hypothesis, the set  $S$  satisfies equality (7). Therefore, since  $|S \cap (A \cup B)| < d$ , we get that

$$\begin{aligned} \dim \text{span}\{\boldsymbol{\eta}(S)\} &= |S \cap C| + \min(|S \cap (A \cup B)|, d) \\ &= |S \cap C| + |S \cap (A \cup B)| = |S| = m - 1. \end{aligned}$$

This implies that  $\dim \text{span}\{\boldsymbol{\eta}(S')\} = m$  so that  $\mathbf{b}(\lambda_T) \in \text{span}\{\boldsymbol{\eta}(S')\}$ .

Let  $S \in 2^{U'} \setminus \Gamma$  be such that  $S' := S \cup \{a_{j+1}\} \notin \Gamma$ . There are two cases to consider here. If  $|S \cap (A \cup B)| \geq d$ , then any choice of  $\alpha_{j+1}$  is good. This is because, in view of (3), the vectors given to users in  $B$  are  $\mathbf{b}(\beta_i) = \mathbf{a}(\beta_i)$  where  $\beta_i \in \Lambda$ , and, consequently, if  $|S \cap (A \cup B)| \geq d$ , the vectors held by the users of  $S$  already span all of  $E_1$ . Hence, for any choice of  $\alpha_{j+1}$ ,  $\text{span}\{\boldsymbol{\eta}(S')\} = \text{span}\{\boldsymbol{\eta}(S)\}$  and  $S'$  will still satisfy equality (7).

This leaves us with the case  $|S \cap (A \cup B)| < d$ . In that case  $|S| \leq m - 2$  (since if  $|S| \geq m - 1$  then  $|S \cap C| = |S| - |S \cap (A \cup B)| \geq m - d$  and  $|S'| = |S| + 1 \geq m$ , and consequently,  $S'$  would have been authorized). Our goal is to find  $\alpha_{j+1}$  so that the target vector is not in  $\text{span}\{\boldsymbol{\eta}(S')\}$  (in order to keep respecting  $\Gamma$ ) and that equality (7) still holds. Concentrating on the first part of our mission, we want to find  $\alpha_{j+1}$  so that  $\mathbf{a}(\alpha_{j+1}) \notin \text{span}\{\boldsymbol{\eta}(S) \cup \{\mathbf{b}(\lambda_T)\}\}$ . By Lemma 7.7, for  $X = \{\alpha_i\}_{1 \leq i \leq j} \cup \{\beta_i\}_{1 \leq i \leq |B|}$  and  $Y = \{\gamma_i\}_{1 \leq i \leq |C|} \cup \{\lambda_T\}$ , the space  $\text{span}\{\boldsymbol{\eta}(S) \cup \{\mathbf{b}(\lambda_T)\}\} \cap E_1$  is a subspace of  $E_1$  of dimension  $d - 1$  at the most. Hence, for all but possibly  $d - 1$  values of  $\alpha_{j+1} \in \mathbb{F}$ , the vector  $\mathbf{a}(\alpha_{j+1})$  increases the dimension of  $\text{span}\{\boldsymbol{\eta}(S) \cup \{\mathbf{b}(\lambda_T)\}\}$  by 1. It is easy to see that for all such selections, we get an assignment that will also respect equality (7).

## 8 A Recursive Characterization of Ideal WTASs by Means of Composition

In this section we show that if  $\Gamma$  is an ideal WTAS that is not one of the structures that we identified in Sections 6.3 and 7, namely, an HTAS or a TPAS, then it is a composition of two ideal WTASs that are defined over smaller sets of users. By doing so, we obtain a recursive characterization of ideal WTASs.

### 8.1 WTASs and Composition of Access Structures

We begin with the following fundamental lemma that asserts that a composition of two access structures is ideal if and only if those two access structures are ideal.

**Lemma 8.1** *Let  $U_1$  and  $U_2$  be disjoint sets. Let  $u_1 \in U_1$ , and define  $U = U_1 \cup U_2 \setminus \{u_1\}$ . Suppose  $\Gamma_1$  and  $\Gamma_2$  are access structures over  $U_1$  and  $U_2$  respectively such that  $u_1$  is not redundant in  $\Gamma_1$  and  $\Gamma_2 \neq \emptyset$ . Furthermore, let  $\Gamma$  be the composition of  $\Gamma_1$  and  $\Gamma_2$  via  $u_1$ . Then  $\Gamma$  is ideal if and only if both  $\Gamma_1$  and  $\Gamma_2$  are ideal. Moreover, if both  $\Gamma_1$  and  $\Gamma_2$  have an ideal linear secret sharing schemes, then  $\Gamma$  has an ideal linear secret sharing scheme.*

**Proof:** Assume that  $\Gamma_1$  and  $\Gamma_2$  are ideal. We show that  $\Gamma$  is ideal by describing an ideal secret sharing scheme that realizes it. Given a secret  $s$ , we share it among the users of  $U_1$  using an ideal secret sharing scheme for  $\Gamma_1$ . Then, if  $s_1$  is the share of  $u_1$  in that scheme, we share it among the users of  $U_2$  using an ideal secret sharing scheme for  $\Gamma_2$ . The resulting scheme is clearly ideal, and its correctness and perfect security stem from the correctness and perfect security of the two schemes for  $\Gamma_1$  and  $\Gamma_2$ . It is easy to verify that if both secret sharing schemes for  $\Gamma_1$  and for  $\Gamma_2$  are linear, then so is the resulting scheme for  $\Gamma$ .

Conversely, suppose that  $\Gamma$  is ideal. Let  $M \subset U$  be a minterm of  $\Gamma$  and let  $M = M_1 \cup M_2$  where  $M_1 = M \cap (U_1 \setminus \{u_1\})$  and  $M_2 = M \cap U_2$ . We choose  $M$  for which  $M_2 \neq \emptyset$  (since  $u_1$  is not redundant in  $\Gamma_1$  and  $\Gamma_2 \neq \emptyset$  such a minterm exists). By Definition 3.4,  $M \in \Gamma$  if and only if  $M_1 \cup \{u_1\} \in \Gamma_1$  and  $M_2 \in \Gamma_2$ . Furthermore,

$$M_1 \cup M'_2 \in \Gamma \text{ iff } M'_2 \in \Gamma_2 \quad \text{and} \quad M'_1 \cup M_2 \in \Gamma \text{ iff } M'_1 \cup \{u_1\} \in \Gamma_1. \quad (8)$$

From the first observation above we conclude that  $\Gamma_2$  coincides with  $\Gamma_{M_1, U_2}$  – the restriction of  $\Gamma$  that  $M_1$  induces on  $U_2$ . Since by Lemma 4.11 any restriction of an ideal access structure is ideal, we infer that  $\Gamma_2$  is ideal. As for  $\Gamma_1$ , let  $x$  be an arbitrary element of  $M_2$ , let  $U' = U_1 \setminus \{u_1\} \cup \{x\}$ , and consider the restriction

$$\Gamma' := \Gamma_{M_2 \setminus \{x\}, U'} = \{Q \subseteq U' : (M_2 \setminus \{x\}) \cup Q \in \Gamma\}.$$

We claim, and prove below, that  $\Gamma'$  is isomorphic to  $\Gamma_1$  under the natural bijection from  $2^{U_1}$  to  $2^{U'}$  (i.e.,  $A \subseteq U_1$  is mapped to  $A' := A \setminus \{u_1\} \cup \{x\} \in U'$  if  $u_1 \in A$ , and to  $A' = A$  otherwise). Therefore, since  $\Gamma'$  is a restriction of  $\Gamma$ , it is ideal, as implied by Lemma 4.11, and, consequently, so is  $\Gamma_1$ .

We conclude with a proof of the isomorphism. Assume that  $A \in \Gamma_1$ . If  $u_1 \notin A$  then  $A \in \Gamma$ . Hence, also  $A' = A \in \Gamma$  and, in particular,  $A' \in \Gamma'$ . If  $u_1 \in A$  then  $(A \setminus \{u_1\}) \cup M_2 \in \Gamma$  (this is implied by the second observation in (8)); hence,  $A' \cup (M_2 \setminus \{x\}) \in \Gamma$  and, therefore,  $A' \in \Gamma'$ . Conversely, assume that  $A' \in \Gamma'$ , namely,  $(M_2 \setminus \{x\}) \cup A' \in \Gamma$ . If  $x \notin A'$  then  $A' \subseteq U_1 \setminus \{u_1\}$ . As  $M_2$  is a minterm of  $\Gamma_2$  (this is implied by the first observation in (8)),  $M_2 \setminus \{x\} \notin \Gamma_2$ . Hence, by Definition 3.4, we conclude that  $A' \in \Gamma_1$ . But in that case  $A = A'$  so that  $A \in \Gamma_1$ . If  $x \in A'$  then  $(A' \setminus \{x\}) \cup M_2 \in \Gamma$  from which it follows that  $A = (A' \setminus \{x\}) \cup \{u_1\} \in \Gamma_1$ .  $\square$

The recursive characterization of ideal WTASs will be obtained by distinguishing between two types of users. Specifically, we shall identify a subset of so-called strong users that takes the form of a suffix,  $S = U_{k,n}$ , where  $k \geq 3$ ,

and then the complement subset will be thought of as the subset of weak users,  $W = U_{1,k-1}$ . A subset of strong users will be called  $S$ -cooperative if it is unauthorized, but it may become authorized if we add to it some weak users.

**Definition 8.2 (Cooperative Set)** *Given  $Y \subseteq S$ , if  $Y \notin \Gamma$  but  $W \cup Y \in \Gamma$ , then  $Y$  is called an  $S$ -cooperative set.*

By Claim 4.13, the access structure  $\Gamma_{Y,W}$ , the restriction of  $\Gamma$  induced by  $Y$  on  $W$ , is a WTAS for any partition  $U = W \cup S$  and  $Y \subseteq S$ . We proceed to define a condition on the set  $S$ , such that if it is satisfied for some suffix  $S = U_{k,n}$ , where  $k \geq 3$ , the access structure  $\Gamma$  is a composition of two ideal WTASs that are defined on sets smaller than  $U$ .

**Definition 8.3 (Strong Set)** *If for any two  $S$ -cooperative sets  $Y_1, Y_2 \subseteq S$ , the corresponding restrictions of  $\Gamma$  to  $W$  coincide, i.e.  $\Gamma_{Y_1,W} = \Gamma_{Y_2,W}$ , the set  $S$  is called a strong set of users.*

If  $S$  is a strong set of users, there exists an access structure on  $W$ , denoted  $\Gamma_W$ , such that  $\Gamma_W = \Gamma_{Y,W}$  for all cooperative subsets  $Y \subseteq S$ . In that case, every minterm  $M \in \Gamma$  is either contained in  $S$  or  $M \cap W \in \Gamma_W$ . The following theorem shows that if  $S$  is a strong set of users,  $\Gamma$  is a composition of two ideal WTASs.

**Theorem 8.4** *Let  $\Gamma$  be an ideal WTAS over  $U$ . Suppose  $S = U_{k,n}$ , for some  $k \geq 3$ , is a strong set of users. Then  $\Gamma$  is a composition of two ideal WTASs, where each access structure is defined on a set smaller than  $U$ .*

**Proof:** Let  $Y$  be an arbitrary  $S$ -cooperative set,  $Q \subseteq W$  be a minterm of  $\Gamma_{Y,W}$ , and  $x$  be an arbitrary element of  $Q$ . Both  $\Gamma_{Y,W}$  and  $\Gamma' = \Gamma_{Q \setminus \{x\}, S \cup \{x\}}$  are restrictions of  $\Gamma$ . Thus, by Lemma 4.11, they are both ideal, and by Claim 4.13 they are both WTASs. Let  $\Delta$  be the composition of  $\Gamma_{Y,W}$  and  $\Gamma'$  via  $x$  (since  $x$  belongs to the domains of both access structures,  $W$  and  $S \cup \{x\}$ , we may define the composition by first replacing the user  $x$  in  $\Gamma'$  with a copy  $x'$  and then compose  $\Gamma_{Y,W}$  and  $\Gamma'$  via  $x'$ ). We note that  $\Delta$ , like  $\Gamma$ , is defined on  $U = W \cup S$ . We proceed to show that  $\Gamma = \Delta$ , thus proving the lemma (note that  $\Gamma_{Y,W}$  is defined on  $W$ , whose size is  $k-1 < n$ , and  $\Gamma'$  is defined on  $S \cup \{x\}$  whose size is  $n-k+2 < n$ ).

Let  $M$  be a minterm of  $\Gamma$ . Then either  $M \subseteq S$ , or  $M \cap W \neq \emptyset$ . If  $M \subseteq S$ , then  $M \in \Gamma'$ . Since  $x \notin M$  we conclude, by Definition 3.4, that  $M \in \Delta$ . If, on the other hand,  $M \cap W \neq \emptyset$ , define  $M_W = M \cap W$  and  $M_S = M \cap S$ . Therefore,  $M_S$  is an  $S$ -cooperative set. Since  $S$  is a strong set of users, we infer that  $\Gamma_{M_S,W} = \Gamma_{Y,W}$ , whence  $M_W \in \Gamma_{Y,W}$ . In addition,  $Q \in \Gamma_{Y,W} = \Gamma_{M_S,W}$ . This implies that  $M_S \cup Q \in \Gamma$  and, consequently, that  $M_S \cup \{x\} \in \Gamma'$ . Therefore, by Definition 3.4,  $M = M_W \cup M_S \in \Delta$ .

For the other direction, let  $M$  be a minterm of  $\Delta$ , and, as before,  $M_S = M \cap S$  and  $M_W = M \cap W$ . Then either  $M_S \in \Gamma'$  or both  $M_W \in \Gamma_{Y,W}$  and  $M_S \cup \{x\} \in \Gamma'$ . We separate the discussion to these two cases:

If  $M_S \in \Gamma'$ , then  $M_S \cup (Q \setminus \{x\}) \in \Gamma$ . That means that  $Q \setminus \{x\} \in \Gamma_{M_S,W}$ . On the other hand, as  $Q$  is a minterm of  $\Gamma_{Y,W}$ , it must hold that  $Q \setminus \{x\} \notin \Gamma_{Y,W}$ . Hence,  $\Gamma_{M_S,W} \neq \Gamma_{Y,W}$ . This implies that  $M_S$  is not  $S$ -cooperative. But since  $M_S \cup (Q \setminus \{x\}) \in \Gamma$ , the only way  $M_S$  is not  $S$ -cooperative is if  $M_S$  is authorized by itself in  $\Gamma$ . However,  $M = M_W \cup M_S$  was a minterm of  $\Delta$ . Therefore,  $M_W$  must be empty whence  $M = M_S$ . This brings us to the sought-after conclusion that  $M \in \Gamma$ .

Now suppose that  $M_S \notin \Gamma'$  but  $M_W \in \Gamma_{Y,W}$  and  $M_S \cup \{x\} \in \Gamma'$ . Since  $M_S \cup \{x\} \in \Gamma'$  we get that  $Q \cup M_S \in \Gamma$ . On the other hand,  $M_S \notin \Gamma'$ , and thus  $M_S \notin \Gamma$ . Therefore,  $M_S$  is  $S$ -cooperative. Since  $M_W \in \Gamma_{Y,W}$ , and  $S$  is a strong set of users, this implies that  $M_W \in \Gamma_{M_S,W}$ . Therefore  $M_W \cup M_S \in \Gamma$ , and thus  $M \in \Gamma$ .  $\square$

## 8.2 Simple Compositions

In this section we show that in two simple cases, an ideal WTAS is a composition of two ideal WTASs defined on sets smaller than  $U$ . The first case is when there are self-sufficient users and the second is where  $u_2$  starts no minterm of  $\Gamma$ .

Suppose  $u_n$  is a self-sufficient user. Let  $\Gamma' = \Gamma_{\emptyset, U_{1,n-1}}$  be the restriction of  $\Gamma$  to  $U_{1,n-1}$  (namely, all authorized sets that do not include  $u_n$ ). By Lemma 4.11, the access structure  $\Gamma'$  is ideal, and by Claim 4.13, it is an ideal WTAS. Let  $\Gamma_\vee$  be the simple 1-out-of-2 access structure on the set  $\{u', u_n\}$ , where  $u'$  is an additional dummy user.  $\Gamma_\vee$  is clearly an ideal WTAS. It is easy to see that  $\Gamma$  is the composition of  $\Gamma'$  and  $\Gamma_\vee$  via  $u'$ .

If  $\Gamma$  has no minterm that starts with  $u_2$  then every minterm that contains  $u_1$  must contain also  $u_2$  (otherwise, we could have replaced  $u_1$  by  $u_2$  in order to get a minterm that starts with  $u_2$ ). Hence, for every  $U_{3,n}$ -cooperative set,  $Y \subseteq U_{3,n}$ , the access structure that  $Y$  induces on  $U_{1,2}$  is the same,  $\Gamma_{Y,U_{1,2}} = \{U_{1,2}\}$ . Therefore,  $U_{3,n}$  is a strong set

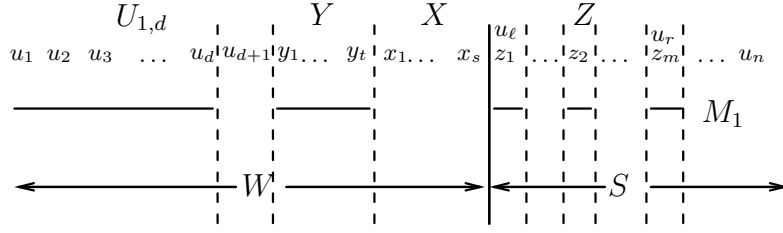


Figure 3: Notations for the composition.

of users in this case. Hence, by Theorem 8.4, the access structure  $\Gamma$  is a composition of two ideal WTASs that are defined on sets smaller than  $U$ .

To conclude, we proved the following lemma:

**Lemma 8.5** *Let  $\Gamma$  be an ideal WTAS over  $U$ . If  $\Gamma$  has self-sufficient users, or  $u_2$  starts no minterm of  $\Gamma$ , then  $\Gamma$  is a composition of two ideal WTASs that are defined on sets smaller than  $U$ .*

### 8.3 Identifying Composition Structures

In this section we show that if  $\Gamma$  is an ideal WTAS, but it is not one of the access structures that were characterized in Sections 6.3 and 7, then it is a composition of two ideal WTASs as described in Section 8.1. In view of Lemma 8.5, we assume hereinafter that  $u_2$  is the minimal user in some minterm of  $\Gamma$ .

Let  $M_1$  be the lexicographically minimal minterm in  $\Gamma$ . Let  $u_r$  be the maximal user in  $M_1$ . Then since  $\Gamma$  is neither an HTAS nor a TPAS, there must be at least two users in  $U_{1,r-1}$  that are not in  $M_1$ . Let  $u_\ell$  be the *minimal* user in  $M_1$  such that at least two users in  $U_{1,\ell-1}$  are missing from  $M_1$ , and let  $u_d$  be the *maximal* user in  $M_1$  such that  $U_{1,d} \subset M_1$ . We denote the users in  $M_1 \cap U_{d+1,\ell-1}$ , if there are any, by  $Y = \{y_1, \dots, y_t\}$ . Note that if  $Y$  is not empty then  $Y$  is a run of  $U$ , and  $y_1 = u_{d+2}$ . Next, if  $Y \neq \emptyset$  we denote the set of users of  $U \setminus M_1$  between  $y_t$  and  $u_\ell$  (excluding those two users) by  $X = \{x_1, \dots, x_s\}$ . Otherwise, we denote the set  $U_{d+2,\ell-1}$  as  $X = \{x_1, \dots, x_s\}$ . Finally, we denote the users of  $M_1 \cap U_{\ell,n}$  by  $Z = \{z_1, \dots, z_m\}$ . Note that the sets  $X$  and  $Z$  are never empty, and that  $z_1 = u_\ell$ . The above notations are depicted in Figure 3.

We claim that either  $U_{\ell,n}$  or  $U_{d+2,n}$  is a strong set of users. We start by partitioning  $U$  into  $W = U_{1,\ell-1}$  and  $S = U_{\ell,n}$ . We show that if all the  $S$ -cooperative sets are of the same size, then  $\Gamma_{Y_1,W} = \Gamma_{Y_2,W}$  for every two cooperative sets  $Y_1, Y_2 \subseteq S$ , namely,  $S$  is a set of strong users. If, however, that condition does not hold, we shall show that  $U_{d+2,n}$  is a strong set.

An important result in our study is that all  $S$ -cooperative sets are at least as large as  $Z$ , namely, their size is at least  $m$ .

**Claim 8.6** *Every minterm of the access structure  $\Gamma$  that intersects  $W$  contains at least  $m$  users from  $S$ .*

**Proof:** Assume towards contradiction that  $M$  is a minterm that intersects  $W$  such that  $|M \cap S| < m$ . The minterm  $M_1$  is an independent set of size  $d + t + m$ , and thus, by Claim 4.7, every independent set of  $\mathcal{M}$  that is smaller than  $d + t + m$  can be expanded to an independent set of size  $d + t + m$ . Therefore, if  $|M| < d + t + m$ , the minterm  $M$  can be expanded to an independent set  $I$  of size  $d + t + m$ ; otherwise, we set  $I = M$ . By Lemma 5.2, this expansion can only be done by adding to  $M$  users that precede  $M_{\min}$ . As  $M$  intersects  $W$ , the users in  $I \setminus M$  are all from  $W$ . Hence,  $|I \cap S| = |M \cap S| \leq m - 1$ . Therefore,  $|I \cap W| = |I| - |I \cap S| \geq d + t + m - (m - 1) = d + t + 1$ . Next, we view  $M_1$  as the canonical complement of the empty set (see Definition 5.4). Its  $(d + t + 1)$ th element is  $z_1$ . By Lemma 5.6 for  $P = \emptyset$ ,  $Y = M_1$ , and  $j = d + t + 1$ , any  $d + t + 1$  members of  $W$  form a dependent set. Hence  $I$ , which was assumed to be independent, contains a dependent set. This contradiction implies that  $|M \cap S| \geq m$ .  $\square$

#### 8.3.1 When All $S$ -Cooperative Sets are of the Same Size

Here we show that if all  $S$ -cooperative sets are of the same size, namely  $|Z| = m$ , the set  $S$  is a strong set. We accomplish this by showing that all the  $S$ -cooperative sets of size  $m$  induce the same access structure on  $W$ , which is



the access structure induced by the  $S$ -cooperative set  $Z$ . We begin by showing that the weight of every  $S$ -cooperative set is at least the weight of  $Z$ .

**Claim 8.7** *Let  $V \subseteq S$  be an  $S$ -cooperative set. Then  $w(V) \geq w(Z)$ .*

**Proof:** Assume towards contradiction that  $w(V) < w(Z)$  and consider the set  $W \cup Z$ . Since  $Z$  is  $S$ -cooperative, the set  $W \cup Z$  is authorized. Thus, by Lemma 5.1, it must contain a suffix minterm of the form  $B \cup Z$ , where  $B$  is a suffix of  $W$ . There are two possible cases: Either  $B \cup V$  is authorized, or not.

If  $B \cup V$  is authorized, then, since  $w(V) < w(Z)$ , the set  $B \cup V$  is a minterm. Hence, as  $B \cup V$  and  $B \cup Z$  are two minterms that have the same minimal user,  $B_{\min}$ , Lemma 5.3 implies that  $|V| = |Z| = m$ . The set  $B \cup V$  is independent in  $\mathcal{M}$ . If  $|B \cup V| < d + t + m$ , Claim 4.7 implies that  $B \cup V$  can be expanded to an independent set  $I$  of size  $d + t + m$ ; if  $|B \cup V| \geq d + t + m$ , we set  $I = B \cup V$ . By Lemma 5.2, all users in  $I \setminus (B \cup V)$  must be from  $W$ . Hence,  $I$  includes at least  $d + t$  users from  $W$ . On the other hand, since  $|V| = |Z| = m$  and  $w(V) < w(Z)$ , there must be an index  $j$  such that  $v_j \prec z_j$ . Since  $M_1$  is the canonical complement of the empty set  $\emptyset$ , we get from Lemma 5.6, applied to  $P = \emptyset$ ,  $Y = M_1$  and  $B = I_{1,d+t+j}$ , that the latter set is dependent. This is impossible since  $I$  is independent. Therefore,  $B \cup V$  cannot be authorized.

If  $B \cup V$  is unauthorized, we let  $Q$  be the canonical complement of  $B$ . Since  $B \cup Z$  is a minterm, we get from Lemma 5.3 that  $|Q| = |Z| = m$ . On the other hand, by Claim 8.6, since  $V$  is  $S$ -cooperative,  $|V| \geq m$ . Since  $B \cup V \notin \Gamma$  but  $B \cup Q \in \Gamma$ , we infer that  $w(V) < w(Q)$ . Therefore, there must be an index  $j \in [m]$  such that  $v_j \prec q_j$ . Thus, by Lemma 5.6, we get that  $B \cup V$  is dependent. On the other hand, since  $V$  is  $S$ -cooperative and  $B$  is a suffix of  $W$ , the set  $B \cup V$  may be expanded to an authorized superset by adding to it users that precede  $B_{\min}$ , one by one, until the first time that we get an authorized set. This construction, where in each stage we add a new user that is smaller than all current users in the set, guarantees that we end up with a minterm. But a minterm of  $\Gamma$  cannot contain a dependent set. Therefore, this case is not possible either. We are lead to the conclusion that  $w(V) \geq w(Z)$ .  $\square$

We are now ready to prove that all the cooperative sets of size  $m$  induce the same access structure on  $W$ .

**Claim 8.8** *Let  $V$  be an  $S$ -cooperative set of size  $m$ . Then  $\Gamma_{V,W} = \Gamma_{Z,W}$ .*

**Proof:** By Claim 8.7,  $w(V) \geq w(Z)$ . If  $w(V) = w(Z)$ , the claim is trivial, since  $\Gamma$  is a WTAS. Therefore, we assume that  $w(V) > w(Z)$ . We first show that  $U_{1,d} \cup Y \cup V$  is a minterm of  $\Gamma$ . Since  $M_1 = U_{1,d} \cup Y \cup Z$  is authorized, the set  $U_{1,d} \cup Y \cup V$  is authorized as well. Assume it is not a minterm. Then, by Lemma 5.1 it contains a suffix minterm of the form  $B \cup V$ , where  $B$  is a suffix of  $U_{2,d} \cup Y$ . Let  $Q$  be the canonical complement of  $B$ . Since  $B \cup V$  is a minterm, by Lemma 5.3, it must be that  $|Q| = |V| = m = |Z|$ . Since  $B \cup Q$  is authorized and  $B \cup Z$  is unauthorized,  $w(Z) < w(Q)$ . Therefore, there must be an index  $j \in [m]$  for which  $z_j \prec q_j$ . Thus, by Lemma 5.6, the set  $B \cup Z$  is dependent. However, this set is contained in  $U_{1,d} \cup Y \cup Z$ , which is a minterm. This contradiction implies that  $U_{1,d} \cup Y \cup V$  is a minterm of  $\Gamma$ . Consequently, since  $U_{1,d} \cup Y \cup Z$  is a minterm and  $U_{2,d} \cup Y \cup V$  is unauthorized (being a proper subset of a minterm), we infer that  $w(Z) + w(u_1) > w(V)$ .

We are now ready to prove that  $\Gamma_{Z,W} = \Gamma_{V,W}$ . Since we deal with the case where  $w(Z) < w(V)$ , the inclusion  $\Gamma_{Z,W} \subseteq \Gamma_{V,W}$  is obvious. For the opposite inclusion, it is sufficient to concentrate on minterms of  $\Gamma_{V,W}$ . Let  $M$  be a minterm of  $\Gamma_{V,W}$ . Thus,  $M \cup V \in \Gamma$ , and since  $M \cup V \setminus M_{\min} \notin \Gamma$ , the set  $M \cup V$  is a minterm in  $\Gamma$ . There are two possible cases: If  $u_1 \in M$ , the minterm  $M \cup V$  must be of the same size as  $M_1$  by Lemma 5.3. Since  $|M_1| = d + t + m$  and  $|V| = m$ , we get that  $|M| = d + t$ . As  $M_1 = U_{1,d} \cup Y \cup Z$  is the minimal minterm in  $\Gamma$  in terms of the precedence order  $\prec$ , the weight of  $U_{1,d} \cup Y$  is minimal among all sets of size  $d + t$  that are contained in a minterm. This implies that  $w(M) \geq w(U_{1,d} \cup Y)$ . This, in turn, implies that  $M \cup Z \in \Gamma$  and thus  $M \in \Gamma_{Z,W}$ .

The second case is when  $u_1 \notin M$ . Assume, towards contradiction, that  $M \cup Z \notin \Gamma$ . Let  $Q$  be the canonical complement of  $M$ . By Lemma 5.3,  $|Q| = |V| = m = |Z|$ . Since  $M \cup Z \notin \Gamma$  and  $M \cup Q \in \Gamma$ ,  $w(Z) < w(Q)$ . Therefore, there must be an index  $j \in [m]$  such that  $z_j \prec q_j$ . Thus, by Lemma 5.6,  $M \cup Z$  is dependent. However, since  $M \cup V \in \Gamma$  and  $w(Z) + w(u_1) > w(V)$ , we get that  $\{u_1\} \cup M \cup Z \in \Gamma$ . Moreover, it must be a minterm since any proper subset of  $\{u_1\} \cup M \cup Z$  is of weight that does not exceed that of the unauthorized set  $M \cup Z$ . We arrived at the absurd situation where the dependent set  $M \cup Z$  is contained in a minterm. This contradiction implies that  $M \cup Z$  is authorized, so  $M \in \Gamma_{Z,W}$ .  $\square$

Combining Claims 8.6 and 8.8 we arrive at the following corollary.

**Corollary 8.9** *If there are no  $S$ -cooperative sets of size larger than  $m$ , then  $S$  is a strong set.*

**Example 8.10** Consider the set  $U = \{u_1, \dots, u_8\}$ , and let  $\Gamma$  be a WTAS where the weights are 1, 1, 1, 1, 1, 3, 3, and 3 and the threshold is 6. The lexicographically minimal minterm is  $\{u_1, u_2, u_3, u_6\}$ , and so there is no prefix minterm and no lacunary minterm. In this example  $W = U_{1,5}$  and  $S = U_{6,8}$  and the access structure is a composition of a 3-out-of-5 threshold access structure on the week side  $W$  and a 2-out-of-4 threshold access structure on  $S \cup \{u'\}$ , where  $u'$  is an additional dummy user.

### 8.3.2 When Large $S$ -Cooperative Sets Exist

The conclusion from the previous section is that whenever all  $S$ -cooperative sets for  $S = U_{\ell,n}$  are of the same size (i.e.,  $|Z| = m$ ), the set  $S$  is a strong set and, hence, by Theorem 8.4, the access structure  $\Gamma$  is a composition of ideal WTASs that are defined over two smaller sets. Here, we continue to deal with the case where there are  $S$ -cooperative sets of size larger than  $m$ . In that case we identify another strong set. Specifically, we show that  $U_{d+2,n}$  is a strong set of users.

Recall that we assume that there are minterms that start with  $u_2$ . In order to prove that  $U_{d+2,n}$  is a strong set we show that for every  $U_{d+2,n}$ -cooperative set  $V$ , the access structure that it induces on  $U_{1,d+1}$ , namely  $\Gamma_{V,U_{1,d+1}}$ , is a  $d$ -out-of- $(d+1)$  threshold access structure. The proof has two stages. First we show that if  $M$  is a minterm of  $\Gamma$  such that  $|M \cap S| > m$  then  $M \cap U_{1,d+1} = \emptyset$ . Therefore, by Claim 8.6, every  $U_{d+2,n}$ -cooperative set  $V$  intersects  $S$  in exactly  $m$  users. In the second stage, we analyze the restricted access structure that  $Z$  induces on  $W$ , namely  $\Gamma_{Z,W}$ , and conclude that  $\Gamma_{V,U_{1,d+1}}$  is a  $d$ -out-of- $(d+1)$  access structure.

We start with two technical claims. The following claim shows that the weight of  $z_1$  is much larger than the weights of the users preceding it.

**Claim 8.11**  $w(u_{d+1}) + w(x_1) < w(z_1)$ .

**Proof:** Suppose the claim is false and  $w(u_{d+1}) + w(x_1) \geq w(z_1)$ . Then we can replace  $z_1$  by  $u_{d+1}$  and  $x_1$  in the minterm  $M_1$ , and get an authorized set  $M'_1$ . By Lemma 5.1, this set contains a suffix minterm. This minterm must intersect  $W$ , since  $Z$  by itself is unauthorized. But then  $M'_1$  would be a minterm that contains only  $m - 1$  elements of  $S$ , in contradiction to Claim 8.6.  $\square$

**Claim 8.12** *The sets  $U_{1,d} \cup Y \cup \{x_1\}$  and  $U_{1,d+1}$  are circuits of  $\mathcal{M}$  – the matroid that corresponds to  $\Gamma$ .*

**Proof:** We start with the set  $V := U_{1,d} \cup Y \cup \{x_1\}$ . Denote by  $\Gamma'$  the restriction  $\Gamma_{Z,W \setminus \{u_{d+1}\}}$ . By Lemma 4.11 and Claim 4.13,  $\Gamma'$  is an ideal WTAS. Moreover, the set  $U_{1,d} \cup Y$  is a prefix minterm of this access structure. Therefore, by Theorem 6.7,  $\Gamma'$  is a HTAS. Thus, by Claim 6.2,  $V$  is a circuit of the matroid that corresponds to  $\Gamma'$  and its size is  $|V| = d + t + 1$ . As  $M_1$  is the canonic complement of the empty set,  $z_1$  is its  $(d + t + 1)$ th element, and  $x_1 \prec z_1$ , we conclude by Lemma 5.6 that  $V$  is dependent in  $\mathcal{M}$ . It must be a circuit in  $\mathcal{M}$  for, otherwise, by Lemma 4.12, we would get a contradiction to the fact that it is a circuit in the matroid that corresponds to the restriction  $\Gamma'$ .

The proof for  $U_{1,d+1}$  is similar. Let  $\Gamma'' = \Gamma_{Y \cup Z, U_{1,d+1}}$  be the restriction that  $Y \cup Z$  induces on  $U_{1,d+1}$ . It is an ideal WTAS and  $U_{1,d}$  is a prefix minterm in it. Thus, by Claim 6.2,  $U_{1,d+1}$  is a circuit of the matroid that corresponds to  $\Gamma''$ . As  $u_{d+1}$ , which is the  $(d+1)$ th user in  $U_{1,d+1}$ , precedes the  $(d+1)$ th user in the canonical complement of the empty set (which is  $y_1$ , or  $z_1$  if  $Y = \emptyset$ ), we infer by Lemma 5.6 that  $U_{1,d+1}$  is dependent in  $\mathcal{M}$ . Finally, by Lemma 4.12, the set  $U_{1,d+1}$  must be a circuit of  $\mathcal{M}$ .  $\square$

We now turn to show that large  $S$ -cooperative sets are not contained in minterms that intersect  $U_{1,d+1}$ . This is done in two steps. First we show that if  $V$  is an  $S$ -cooperative set of size larger than  $m$ , there exists a suffix  $B$  of  $Y$  such that  $B \cup V$  is a minterm. Then, relying upon this result, we show that no minterm of  $\Gamma_{V,W}$  intersects  $U_{1,d+1}$ . These claims enable us to prove later on that  $U_{d+2,n}$  is a strong set.

**Claim 8.13** *If there exists an  $S$ -cooperative set  $V$  of size larger than  $m$ , there exists a suffix  $B$  of  $Y$  such that  $B \cup V$  is a minterm.*

**Proof:** Consider the set  $A := U_{1,d} \cup Y \cup V$ . Since  $|V| > |Z|$ , the set  $A$  cannot be a minterm, as implied by Lemma 5.3. By Claim 8.7,  $w(V) \geq w(Z)$ . Thus, the set  $A$  is authorized and, consequently, it contains a suffix minterm of the form  $B \cup V$  where  $B$  is a proper suffix of  $U_{1,d} \cup Y$ .

In order to prove the claim, we need to show that  $B \subseteq Y$ . Assume, towards contradiction, that it is not, namely, that  $B_{\min} \in U_{1,d}$ . Let  $Q = B \setminus \{B_{\min}\} \cup \{x_1\}$ . Since  $B_{\min} \prec x_1$ , the weight of  $Q$  is at least the weight of  $B$ , and

therefore,  $Q \cup V$  is an authorized set. Hence, it has a suffix minterm that must contain  $\{x_1\} \cup V$ . The union of this minterm and the minterm  $B \cup V$  is  $B \cup \{x_1\} \cup V$ . We claim that all the users in  $V$  are critical for  $B \cup \{x_1\} \cup V$ , in the sense of Definition 4.3. Let  $v_i \in V$  and assume that it is not critical, i.e.,  $B \cup \{x_1\} \cup V \setminus \{v_i\} \in \Gamma$ . Since  $B \cup V \setminus \{B_{\min}\} \notin \Gamma$  (because  $B \cup V$  is a minterm), we conclude that  $w(B \cup V \setminus \{B_{\min}\}) < w(B \cup \{x_1\} \cup V \setminus \{v_i\})$ , whence  $w(v_i) < w(B_{\min}) + w(x_1)$ . Therefore, as  $w(z_1) \leq w(v_i)$  and  $w(B_{\min}) \leq w(u_{d+1})$ , we arrive at the conclusion that  $w(z_1) < w(u_{d+1}) + w(x_1)$ , in contradiction with Claim 8.11. This implies that all of the users of  $V$  are critical for  $B \cup \{x_1\} \cup V$ . Thus, by Corollary 4.5, the set  $B \cup \{x_1\}$  is dependent. However,  $B \cup \{x_1\}$  is properly contained in  $U_{1,d} \cup Y \cup \{x_1\}$ , which, by Claim 8.12, is a circuit of  $\mathcal{M}$ . This contradiction implies that  $B$  does not intersect  $U_{1,d}$  and it is therefore a suffix of  $Y$ .  $\square$

**Remark 8.14** If there is an  $S$ -cooperative set  $V$  of size larger than  $m$ , the set  $Y$  is not empty. Otherwise, the set  $B$  in Claim 8.13 would be empty, whence  $V$  is a minterm in contradiction to our assumption that  $V$  is  $S$ -cooperative. Therefore, in the rest of this section we assume that  $Y$  is not empty.

**Claim 8.15** *If  $V$  is an  $S$ -cooperative set of size larger than  $m$  and  $A$  is a minterm of  $\Gamma_{V,W}$ , then  $A \cap U_{1,d+1} = \emptyset$ .*

**Proof:** Suppose there is a minterm  $A \in \Gamma_{V,W}$  such that  $A \cap U_{1,d+1} \neq \emptyset$ . We pick a minterm  $A$  of that sort having the property that  $A \cap (Y \cup X) \preceq M \cap (Y \cup X)$  for all minterms  $M \in \Gamma_{V,W}$  such that  $M \cap U_{1,d+1} \neq \emptyset$ . By Claim 8.13, there is a suffix  $B$  of  $Y$  such that  $B \in \Gamma_{V,W}$ . Therefore, as  $A$  is a minterm of  $\Gamma_{V,W}$ , it must be that  $Y \not\subseteq A$  (otherwise, if  $Y \subseteq A$ , the minterm  $A \cup V$  would have been a proper superset of the minterm  $B \cup V$  since  $A$  includes in addition users from  $U_{1,d+1}$ ). Let  $j$  be the smallest index such that  $y_j \notin A$  (recall that  $Y \neq \emptyset$ ) and consider the set  $Q = A \setminus \{A_{\min}\} \cup \{y_j\}$ . Since  $A_{\min} \in U_{1,d+1}$ , we conclude that  $w(A_{\min}) \leq w(y_j)$ . Thus  $Q \cup V$  is authorized in  $\Gamma$  and must contain a suffix minterm  $M'$  that contains  $y_j$ . The union of  $A \cup V$  and  $M'$  is  $A \cup \{y_j\} \cup V$ . The discussion now separated two cases: either  $A$  is a run of  $U$  or it is not.

If  $A$  is a run of  $U$ , then we first claim that every user of  $V$  is critical for  $A \cup \{y_j\} \cup V$ . Let  $v_i \in V$  and assume that it is not critical for  $A \cup \{y_j\} \cup V$ , i.e.,  $A \cup \{y_j\} \cup V \setminus \{v_i\} \in \Gamma$ . Since  $A \cup V \setminus \{A_{\min}\} \notin \Gamma$  (as  $A \cup V$  is a minterm), we conclude, by comparing the weights of the two latter sets, that  $w(v_i) < w(A_{\min}) + w(y_j)$ . However, since  $w(z_1) \leq w(v_i)$ ,  $w(y_j) \leq w(x_1)$ , and  $w(A_{\min}) \leq w(u_{d+1})$ , we get that  $w(z_1) < w(u_{d+1}) + w(x_1)$ , in contradiction with Claim 8.11. Therefore, all users in  $V$  are critical for  $A \cup \{y_j\} \cup V$ . Thus, by Corollary 4.5, the set  $A \cup \{y_j\}$  is dependent. Since we are now dealing with the case where  $A$  is a run of  $U$ , the set  $A \cup \{y_j\}$  must be contained in  $U_{1,d+1} \cup Y$ . Since  $A \cup \{y_j\}$  is dependent, it includes a circuit  $C$ . That circuit must include the user  $u_{d+1}$ , for, otherwise,  $C$  would have been a subset of the independent set  $U_{1,d} \cup Y$ . Moreover, the circuit  $C$  must also contain  $y_j$ , for otherwise  $C \subseteq A$  while  $A$  is contained in a minterm and thus is independent. On the other hand, by Claim 8.12, the set  $U_{1,d+1}$  is a circuit of  $\mathcal{M}$  that does not contain  $y_j$ . Therefore,  $u_{d+1}$  is in the intersection of the two distinct circuits  $U_{1,d+1}$  and  $C$ . Hence, by Lemma 4.6, the set  $U_{1,d+1} \cup C \setminus \{u_{d+1}\}$  is dependent. But that latter set is contained in the independent set  $U_{1,d} \cup Y$ . This contradiction settles the statement whenever  $A$  is a run of  $U$ .

If  $A$  is a not run of  $U$ , we claim that every user  $a_i \in A$  such that  $y_j \prec a_i$  is critical for  $A \cup \{y_j\} \cup V$ . Otherwise we could replace the user  $a_i$  in the minterm  $A \cup V$  with the user  $y_j$  and still get a minterm that intersects  $U_{1,d+1}$ . But this would contradict our choice of  $A$  as a minterm that is lexicographically minimal in  $Y \cup X$ . This implies that all users of  $A \cup \{y_j\} \cup V$  that have weight greater than or equal to  $y_j$  are critical for that set. In particular, the subset  $(A \cap X) \cup V$  consists of only critical users and, consequently, by Claim 4.5, the set  $A \setminus X \cup \{y_j\}$  is dependent. Since  $A$  is contained in a minterm, this dependent set must contain a circuit  $C$  that contains  $y_j$ . Since  $U_{1,d} \cup Y$  is an independent set of  $\mathcal{M}$ , it must be that  $u_{d+1} \in C$ , otherwise  $C$  is properly contained in an independent set. However, since  $U_{1,d+1}$  is a circuit of  $\mathcal{M}$ , we get that  $u_{d+1}$  is in the intersection of two distinct circuits,  $C$  and  $U_{1,d+1}$  (note that  $y_j \in C$  but  $y_j \notin U_{1,d+1}$ ). Therefore, by Lemma 4.6, the set  $C \cup U_{1,d+1} \setminus \{u_{d+1}\}$  is dependent. However, this set is properly contained in  $U_{1,d} \cup Y$  which is an independent set of  $\mathcal{M}$ , and thus we arrive at a similar contradiction as in the previous case.  $\square$

We now turn to the second stage of showing that  $U_{d+2,n}$  is a strong set. We will show that the existence of large  $S$ -cooperative sets and the existence of minterms that start with  $u_2$  imply that  $U_{d+2,n}$  is a strong set. By Claim 8.15, every  $U_{d+2,n}$ -cooperative set intersects  $S$  in exactly  $m$  users. By Claim 8.8, if  $V$  is an  $S$ -cooperative set of size  $m$  then  $\Gamma_{V,W} = \Gamma_{Z,W}$ . Therefore, it is enough to study the minterms in  $\Gamma_{Z,W}$  that intersect  $U_{1,d+1}$ . Somehow, surprisingly, the existence of a large  $S$ -cooperative set affects the structure of  $\Gamma_{Z,W}$ .

**Claim 8.16** *If there is an  $S$ -cooperative set  $V$  of size larger than  $m$ , there exists a suffix  $B$  of  $Y$  such that  $B \cup \{x_1\} \in \Gamma_{Z,W}$ .*

**Proof:** By Claim 8.13, there exists a suffix  $B$  of  $Y$  such that  $B \cup V$  is a minterm of  $\Gamma$ . We next prove that  $B \cup \{x_1\} \cup V \setminus \{v_1\} \in \Gamma$ . Replacing  $B_{\min}$  with  $x_1$  in  $B \cup V$ , we get an authorized set. Denote by  $M'$  the suffix minterm contained in that set and note that the union of  $M'$  and  $B \cup V$  is exactly  $B \cup \{x_1\} \cup V$ . We claim that  $v_1$  cannot be critical for  $B \cup \{x_1\} \cup V$ . If it was critical, then every user in  $V$  would also be critical for that set and thus, by Corollary 4.5, the set  $B \cup \{x_1\}$  would be dependent in  $\mathcal{M}$ . However, the set  $B \cup \{x_1\}$  is properly contained in  $U_{1,d} \cup Y \cup \{x_1\}$ , which is a circuit by Claim 8.12. Therefore,  $B \cup \{x_1\}$  is independent, and thus  $v_1$  cannot be critical for  $B \cup \{x_1\} \cup V$ . Hence, the set  $B \cup \{x_1\} \cup V \setminus \{v_1\}$  is authorized. But since it is obtained from the minterm  $B \cup V$  by replacing  $v_1$  with  $x_1 \prec v_1$ , it must be also a minterm.

We now prove the claim by induction on the size of  $V$ . If  $|V| = m + 1$ , then since  $B \cup \{x_1\} \cup V \setminus \{v_1\}$  is a minterm, we conclude that  $B \cup \{x_1\}$  is in  $\Gamma_{V',W}$ , where  $V' = V \setminus \{v_1\}$  is an  $S$ -cooperative set of size  $m$ . But since, in view of Claim 8.8,  $\Gamma_{V',W} = \Gamma_{Z,W}$ , we arrive at the sought-after conclusion that  $B \cup \{x_1\} \in \Gamma_{Z,W}$ . If  $|V| > m + 1$ , then  $V'$  is an  $S$ -cooperative set of size  $|V| - 1$ . Hence, we may apply the induction hypothesis to conclude that there exists a suffix  $B$  of  $Y$  such that  $B \cup \{x_1\} \in \Gamma_{Z,W}$ .  $\square$

The other assumption that affects the structure of  $\Gamma_{Z,W}$  is that there are minterms of  $\Gamma$  that start with  $u_2$ .

**Claim 8.17** *If  $u_2$  is the minimal user in some minterm of  $\Gamma$ , then  $U_{2,d+1} \cup Y$  is a minterm of  $\Gamma_{Z,W}$ .*

**Proof:** The access structure  $\Gamma_{Z,W}$  is an ideal WTAS and its lexicographically minimal minterm is  $U_{1,d} \cup Y$ . Let  $M$  be a minterm of  $\Gamma$  that starts with  $u_2$ . In particular,  $M$  intersects  $W$ . By Claims 8.6 and 8.15, the minterm  $M$  contains exactly  $m$  members of  $S$ . Thus, By Claim 8.8,  $\Gamma_{M \cap S, W} = \Gamma_{Z,W}$ . This implies that  $M \cap W$  is a minterm of  $\Gamma_{Z,W}$  that starts with  $u_2$ . Recall that, by Remark 8.14,  $Y \neq \emptyset$ . Hence, Claim 7.1 applies to the access structure  $\Gamma_{Z,W}$ , and the set  $U_{2,d+1} \cup Y$  is a minterm of  $\Gamma_{Z,W}$ .  $\square$

We may now describe the structure of  $\Gamma_{Z,W}$ . Since we assume that there exists a minterm of  $\Gamma$  that starts with  $u_2$ , the set  $U_{2,d+1} \cup Y$  is a minterm of  $\Gamma_{Z,W}$ , as implied by Claim 8.17. As  $U_{2,d+1} \cup Y$  is a run, and it is a prefix in  $W \setminus \{u_1\}$ , we get that the restriction  $\Gamma_{Z,W \setminus \{u_1\}}$  has a prefix minterm. Therefore, by Theorem 6.7, it is an HTAS. The threshold of the first level is  $|U_{2,d+1} \cup Y| = d + t$ . By Claim 8.16, the run minterm of this HTAS that ends with  $x_1$  starts with an element of  $Y$ . Therefore, the transition between the first and the second level in that HTAS (if exists) occurs within  $Y$ . This enables us to prove that  $U_{d+2,n}$  is a strong set of users.

**Claim 8.18** *Suppose there is an  $S$ -cooperative set of size larger than  $m$ , and that there is a minterm of  $\Gamma$  that starts with  $u_2$ . Then  $U_{d+2,n}$  is a strong set of users.*

**Proof:** In order to show that  $U_{d+2,n}$  is a strong set of users, we show that for any  $U_{d+2,n}$ -cooperative set  $V$ , the corresponding restriction induced by  $V$  on  $U_{1,d+1}$ , namely  $\Gamma_{V, U_{1,d+1}}$ , is a  $d$ -out-of- $(d+1)$  threshold access structure. Let  $V$  be a  $U_{d+2,n}$ -cooperative set. Hence, there exists  $A \subseteq U_{1,d+1}$  such that  $A \cup V$  is a minterm of  $\Gamma$ . Denote  $V' = V \cap W$  and  $V'' = V \cap S$ . Note that the set  $V''$  is  $S$ -cooperative. Moreover, since  $A \cup V$  is a minterm of  $\Gamma$ , there exists a minterm of  $\Gamma_{V'', W}$  that intersects  $U_{1,d+1}$ . Therefore, by Claim 8.15, we conclude that  $|V''| = m$ . Thus, by Claim 8.8,  $\Gamma_{V'', W} = \Gamma_{Z,W}$ . Consequently, since  $A \cup V'$  is a minterm of  $\Gamma_{V'', W}$ , it is also a minterm of  $\Gamma_{Z,W}$ .

We claim that the size of that minterm is  $|A \cup V'| = d + t$ . Indeed, if the minimal user in  $A \cup V'$  is  $u_1$  then, by Lemma 5.3,  $|A \cup V' \cup V''| = |M_1| = d + t + m$ , so, as  $|V''| = m$ , we get that  $|A \cup V'| = d + t$ . Otherwise,  $A \cup V' \subseteq W \setminus \{u_1\}$  and, consequently,  $A \cup V' \in \Gamma_{Z,W \setminus \{u_1\}}$ . But the latter access structure was shown above to be an HTAS; furthermore, we showed that the first level in that HTAS includes all of  $U_{1,d+1}$  and the corresponding threshold is  $d + t$ . Since  $A \cup V'$  starts within  $U_{1,d+1}$ , it is of size  $d + t$ .

We proceed to show that for any  $D \subseteq U_{1,d+1}$  of size  $d$ ,  $D \cup V \in \Gamma$ . Since  $w(D) \geq w(U_{1,d})$ , we need only to show that  $U_{1,d} \cup V \in \Gamma$ . We know that  $M_1 = U_{1,d} \cup Y \cup Z \in \Gamma$ . Since  $A \subseteq U_{1,d+1}$  it must be that  $|A| \leq d$  (otherwise  $A = U_{1,d+1}$  and then  $A \cup V$  would be a minterm that precedes  $M_1$  with respect to the lexicographical order). Hence,  $|V'| \geq t$ . Since  $Y$  consists of the first  $t$  users in  $U_{d+2,n}$ , we conclude that  $w(V') \geq w(Y)$ . Therefore  $V'$  can replace  $Y$  in  $M_1$  and so  $U_{1,d} \cup V' \in \Gamma_{Z,W} = \Gamma_{V'', W}$ . This implies that  $U_{1,d} \cup V' \cup V'' = U_{1,d} \cup V \in \Gamma$ , as required. Hence, every subset of  $d$  users from  $U_{1,d+1}$  completes  $V$  to an authorized set. We need to show now that no smaller subset of  $U_{1,d+1}$  completes  $V$  to an authorized set.

To that end, let  $B$  be a minterm of  $\Gamma_{V, U_{1,d+1}}$ . We proceed to show that  $|B| \geq d$ . Using the above notations,  $B \cup V' \cup V''$  is a minterm of  $\Gamma$ , or, equivalently,  $B \cup V'$  is a minterm of  $\Gamma_{V'', W} = \Gamma_{Z, W}$ . Arguing along the same lines as we did for  $A \cup V'$ , we conclude that  $|B \cup V'| = d + t$ . Assume towards contradiction that  $|B| < d$ . Then  $|V'| > t$  and, consequently, as  $V' \subset U_{d+2, n}$ , we get that  $w(V') \geq w(Y \cup \{x_1\})$ . But according to Claim 8.16,  $Y \cup \{x_1\} \in \Gamma_{Z, W}$ . Hence, since  $\Gamma_{Z, W} = \Gamma_{V'', W}$ , we infer that  $Y \cup \{x_1\} \cup V'' \in \Gamma$ . This implies that also  $V' \cup V'' = V$  is authorized, in contradiction to our assumption that  $V$  is a  $U_{d+2, n}$ -cooperative set. That completes the proof.  $\square$

## 8.4 Proof of Theorem 3.5 – The Characterization Theorem

Let  $\Gamma$  be an ideal WTAS defined on a set of users  $U$  and let  $M_1$  be its lexicographically minimal minterm. If either  $\Gamma$  has self-sufficient users or  $u_2$  starts no minterm of  $\Gamma$ , then, by Lemma 8.5, the access structure  $\Gamma$  is a composition of two ideal WTASs on smaller sets of users.

If  $M_1$  is a prefix then, by Theorem 6.7, the access structure  $\Gamma$  is an HTAS. If  $M_1$  is a lacunary prefix, namely,  $M_1 = U_{1,d} \cup U_{d+2,k}$  for some  $1 \leq d \leq k - 2$  and  $k \leq n$ , then, by Theorem 7.6, the access structure  $\Gamma$  is a TPAS. Otherwise, by Corollary 8.9 and Claim 8.18, there exists within  $U$  a subset of strong users. In this case, we conclude by Theorem 8.4 that the access structure  $\Gamma$  is a composition of two ideal WTASs that are defined on sets smaller than  $U$ .

As for the other direction, HTASs are ideal and may be realized by linear secret sharing schemes, as shown in [6, 32]. TPASs are also ideal and may be realized by linear secret sharing schemes, as shown herein in Section 7.1. Finally, given two ideal access structures, we showed in Lemma 8.1 how to construct an ideal secret sharing scheme for their composition. Hence, the composition is also ideal. Furthermore, by Lemma 8.1, if the secret sharing schemes for the two basic access structures are linear, so is the resulting scheme for the composition of the two access structures. This completes the proof of the characterization theorem.  $\square$

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