Expected maximum of subgaussian variables

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Devroye+Lugosi, Combinatorial Methods in Density Estimation

LEMMA 2.2. *Let $\sigma > 0$, $n \geq 2$, and let $Y_1, \ldots, Y_n$ be real-valued random variables such that for all $s > 0$ and $1 \leq i \leq n$, $\mathbf{E}\{e^{sY_i}\} \leq e^{s^2\sigma^2/2}$. Then*

$$\mathbf{E}\left\{\max_{i\leq n} Y_i\right\} \leq \sigma\sqrt{2\ln n}.$$

If, in addition, $\mathbf{E}\{e^{s(-Y_i)}\} \leq e^{s^2\sigma^2/2}$ for every $s > 0$ and $1 \leq i \leq n$, then for any $n \geq 1$,

$$\mathbf{E}\left\{\max_{i\leq n} |Y_i|\right\} \leq \sigma\sqrt{2\ln(2n)}.$$

PROOF. By Jensen's inequality, for all $s > 0$:

$$\begin{aligned} e^{s\mathbf{E}\{\max_{i\leq n} Y_i\}} &\leq \mathbf{E}\{e^{s\max_{i\leq n} Y_i}\} \\ &= \mathbf{E}\left\{\max_{i\leq n} e^{sY_i}\right\} \\ &\leq \sum_{i=1}^{n} \mathbf{E}\{e^{sY_i}\} \\ &\leq ne^{s^2\sigma^2/2}. \end{aligned}$$

Thus,

$$\mathbf{E}\left\{\max_{i\leq n} Y_i\right\} \leq \frac{\ln n}{s} + \frac{s\sigma^2}{2},$$

and taking $s = \sqrt{2\ln n/\sigma^2}$ yields the first inequality. Finally, note that $\max_{i\leq n} |Y_i| = \max(Y_1, -Y_1, \ldots, Y_n, -Y_n)$ and apply the first inequality to prove the second. □