Incentive-based Search for Efficient Equilibria of the Public Goods Game

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Abstract

The "best-shot" public goods game is a network game, defined on a social network. The simple version of the public goods game (PGG) has a fixed utility for a player who has at least a single neighbor buying the good. Players in the general version of PGG have additional utility when multiple neighbors purchase the good. The general version of the public goods game is shown to be a potential game, establishing the convergence to a stable state (i.e., a pure Nash equilibrium – PNE) by best-response dynamics. One can think of best-response dynamics as a distributed algorithm that runs in a fixed order of players/agents and is guaranteed to converge to a PNE.

A new distributed algorithm is proposed for finding PNEs with improved efficiency by the use of transfer of payoffs among players. For the simple version of PGG, it is shown that the proposed algorithm can stabilize an outcome that maximizes social welfare. For the general version of the game, the proposed procedure transforms any initial outcome into a stable solution at least as efficient as the initial outcome by using transfers. An extensive experimental evaluation on randomly generated PGGs demonstrates that whereas pure best-response dynamics converges on stable states that have lower efficiency than the initial outcome, the proposed procedure finds PNEs of higher efficiency.

Keywords: Public Goods game, Equilibria search, Payoff transfers

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1. Introduction

In graphical games [1], played on a social network, the utilities of players depend both on their own actions and on the actions taken by their neighbors in the network. A well known example of network games are “best-shot” public goods games (PGGs) [2, 3], in which players share common goods. Each player chooses whether to take some action or avoid it. The action is associated with an investment in some local public good. The action might be in the form of performing some computational effort, where computation results can be easily shared locally. It can also be the buying of a book or some other product that is easily lent from one player to another. Each player wants to have at least one player in her neighborhood taking the action, including herself. However, there is a cost associated with taking the action, so if some of the player’s neighbors take the action then the player would prefer to avoid it [4].

There are several versions to the public goods game. The simplest and most common one has a fixed price across all players for the buying of the good and a fixed utility for a player who has at least a single neighbor buying the good [2, 3]. More complex versions have different costs of buying the good for different players and an increasing utility for players with more than a single neighbor that buy the good [5]. One may interpret this increasing utility as a shorter waiting time for using the good by the player, when a larger number of its neighbors buy the good (cf. [3]). Most former studies address the simple version of the PGG, where finding efficient equilibria is equivalent to finding a maximal independent set of vertices in the network [6]. Consequently, the finding of a Pure-strategy Nash Equilibrium (PNE) in the simple version is well studied.

However, when one moves to the general version of PGG, which is at the focus of the present study, equilibria are no longer correlated with the independent sets of the network. The multiplicity of PNEs in a PGG drove the study of Galeoti et al. [7] to consider Bayesian NEs, instead of PNEs. Alternatively, the present paper focuses on distributed search algorithms for improved-efficiency PNEs of the general version of the PGG.
The present article has evolved from a paper that was published in the WI-IAT conference [8], which focused only on the simple version of the PGG. For the simple version it was shown that the game is an ordinal potential game, and that every PNE is Pareto optimal [8]. On a different theme, it has been shown in [8] that the optimal state of the simple version of the PGG (e.g., with the largest social welfare) can be stabilized by the use of transfer of payoffs. The present article deals with the general version of the PGG, and among other contributions, provides a distributed algorithm for finding improved-efficiency PNEs that did not appear in [8].

The present study proves that all versions of the public goods game are potential games, by constructing a non-trivial potential function for the general version of the PGG (see Section 3.1). It follows that the best-response dynamics is guaranteed to converge to a PNE (cf. [9, 10]). For the general version of the PGG it is shown that convergence of best-response dynamics is guaranteed in up to $2 \cdot K \cdot n^2$ steps, where $n$ is the number of agents in the PGG network and $K$ is the largest number of neighbors from which a player can benefit in the general version of PGG (see Sections 2 and 3). This forms a non trivial extension to the results for the simple version of PGG in [8].

Typically, there are many PNEs in a public goods game [5, 11]. This has lead former studies to consider networks with a very specific utility function that entails a unique PNE [5]. Or, on a different theme, consider probabilistic stable states (i.e., Bayesian NE) on networks with partial information for players [7]. Equilibria of PGG with a certain form of achieving higher social welfare have been also found to be stable to perturbations [12]. Interestingly, experimental studies demonstrate that people playing these games seem to favor equilibria of high social welfare [11].

On the theme of real world versions of the public goods game on a network, Boncinelli et al. [12] propose a typical problem of car pool grouping on a social network. Similarly to many PGGs, the game incorporates multiple Nash equilibria that correspond to different stable car-pools on the social network. True to a real world example, the efficiencies of the different equilibria differ widely and the resulting pollution depends on the structure and graph of the social network. The present paper shows (in Section 3) that by using a simple distributed search algorithm like best response, the car pool grouping seekers on a social network can within a bounded number
of rounds find a stable solution to the problem. The proposed new algorithm is shown 
(in Section 6) to guarantee an equilibrium of higher social welfare to the problem of 
car pool grouping on a social network (that, say, minimizes pollution) by using the 
mechanism of side-payments.

PNEs with high social welfare are commonly considered to be efficient, but finding 
a PNE with a maximal social welfare (SW) is computationally intensive [3]. Since 
general PGGs are potential games, one can achieve stability by using best-response 
dynamics as a distributed search algorithm performed by all agents in a fixed order, 
which is guaranteed to converge to a stable outcome. The natural next step is at the 
focus of the present study – to design a distributed algorithm for PGG players that 
converges to an efficient PNE (of high social welfare). Since efficient outcomes of 
PGG are not necessarily stable, the tradeoff between efficiency and stability motivates 
the use of an incentive mechanism to promote stability in efficient states (see Section 4).

A possible approach for incentivizing unsatisfied players to agree on some preferred outcomes is to use transfers of payoffs among the players. Transfers of payoffs 
(e.g., money) take place between players, such that players who gain from some outcome may want to pay others that are unsatisfied by it. Transfers of payoffs can help promote efficiency by providing incentives to some players for seeing more fully the impact of their actions [13]. This is an important result of Jackson and Wilkie [13] who studied the efficiency of equilibria that are achieved by using side payments among the players. The present study continues this line of research by exploiting the machinery of transfer functions (side payments) and combining it with a search algorithm, in order to find efficient equilibria for the public goods game.

For the simple version of PGG, where a single copy of the public good produces 
the maximal utility for all neighbors, a distributed iterative procedure is proposed for 
finding efficient PNEs. The proposed procedure works on any initial arbitrary outcome, 
not necessarily stable. Transfers of payoffs are used in order to ensure stability. The 
resulting outcome is ensured to be at least as efficient as the original outcome and it is shown that the proposed procedure can stabilize the optimal outcome of a PGG 
(i.e., of highest social welfare). This is in contrast to the work of DallAsta et al. [14], 
that use simulated annealing to search for an optimal outcome but does not guarantee
finding it in finite time. In the second stage of the proposed procedure the new outcome is transformed into a PNE by applying the transfers of payoffs among the players. All transfers are contracted by the agents during the run of the iterative improvement procedure in the first stage of the algorithm. In this respect the proposed distributed algorithms represent a multi-agents (multi-players) procedure that can be embarked upon in a distributed manner and produce a not-worse outcome for all participants.

For the general version of PGG an analogous distributed iterative procedure is proposed in Section 5. It uses transfers of payoffs to ensure stability and is guaranteed to converge to a PNE, but not necessarily of improved efficiency. The proposed algorithm is guaranteed to converge to a PNE within $2 \cdot K \cdot n^2$ steps at most, where $n$ is the number of agents and $K$ is the maximal number of accessible public goods from which an agent can benefit (see Section 2). An extensive empirical evaluation on randomly generated general PGGs demonstrates that while best-response dynamics converge to PNEs of lower efficiency than the original outcome, the proposed iterative improvement algorithm, which uses transfers of payoffs, converges to PNEs of improved efficiency.

The plan of the paper is as follows. In Section 2 all versions of the public goods game are defined formally. Section 3 proves the first main contribution of the paper - that the general version of the PGG is a potential game, implying the convergence of best-response dynamics to a PNE. The efficiency of PNEs of public goods games is presented and discussed in Section 4. In order to find efficient PNEs for the public goods game, Section 5 presents the mechanism of transfer functions and Section 6 proposes the second main contribution of the paper – an innovative iterative improvement algorithm for finding PNEs of improved efficiency, which uses transfers of payoffs. The algorithm for the simple version of PGGs is proven to secure improved-efficiency equilibria. An extensive experimental evaluation and discussion are given in Section 7 and demonstrate the improved efficiency of PNEs resulting from the proposed algorithm, over those achieved by best-response dynamics. Section 8 summarizes our conclusions.
2. The Game Model

A common and compact representation for network games is the graphical model proposed by Kearns et al. [1]. This model assumes that each player interacts with only a limited number of players (her neighbors). It is natural to think that a network game is based on some underlying graph describing the players’ interactions network.

The present study focuses on the “best-shot” public goods game [7, 14, 15], which is a well known example of a network game. A PGG is composed of a finite set of players \( N = \{1, \ldots, n\} \) that are connected by a network \( G = \{N, E\} \). Each vertex in \( G \) represents a player, and the edges \( E \) represent the interaction structure of the game. Given a player \( i \in N \), the set of \( i \)’s neighbors is denoted by \( N_i \); these are the players whose actions impact \( i \)’s payoff. The neighborhood of \( i \) is the set \( \{i\} \cup N_i \).

In the “best-shot” public goods game each player chooses an action \( v_i \in V_i \) where the set of possible actions is \( V_i = \{T, F\} \). The choice \( v_i = T \) denotes taking the action (i.e., investment in some local public good), whereas the choice \( v_i = F \) represents avoiding action. An outcome, \( v = (v_1, \ldots, v_n) \in V_1 \times \ldots \times V_n \), is a collection of choices, one for each player.

Taking the action by player \( i \) comes with a cost \( 0 < C_i < 1 \). Therefore, player \( i \) will try to avoid taking the action if possible. Obtaining the use of the local public good is essential for each player in the “best-shot” public goods game. Specifically, each player wants that either one of her neighbors will invest in a local public good or that she will take the action herself. In both cases the player obtains access to the public good.

It is convenient to define distinct states of a player \( i \in N \) (at outcome \( v \)). These states are defined as follows:

\[
\text{state}_i(v) := \begin{cases} 
F_l, & \text{if } v_i = F \\
T_l, & \text{if } v_i = T 
\end{cases}
\] (1)

where \( l \) denotes the number of neighbors that choose to invest in the local public good (i.e., the number of \( k \in N_i, v_k = T \)). The state \( F_l \) (of player \( i \)) denotes that the player avoids taking the action \( (v_i = F) \) and has exactly \( l \) neighbors that take the action. If
the player is taking the action and has exactly \( l \) neighbors that also invest in the local public good, then the player’s state is \( T_l \).

Given the definition of players’ states, the utility of player \( i \) for outcome \( v \) is defined as follows:

\[
  u_i(v) := \begin{cases} 
    \min(\mathbb{K}, l + 1) - C_i, & \text{if } \text{state}_i(v) = T_l \\
    \min(\mathbb{K}, l), & \text{if } \text{state}_i(v) = F_l
  \end{cases}
\]  

(2)

Here, \( \mathbb{K} \) defines the maximal number of players in \( i \)’s neighborhood which can affect player \( i \)’s utility. In other words, the player’s utility grows with the number of players in her neighborhood that take the action, up to a limit of \( \mathbb{K} \). Consequently, the tuple \( \langle G, \mathbb{K}, C_1, \ldots, C_n \rangle \) defines a “best-shot” public goods game where \( G \) is an underlying interactions network, \( \mathbb{K} \) is the maximal number of public goods that can affect \( i \)’s utility and \( C_1, \ldots, C_n \) are the costs for each player for taking the action.

Most previous studies of equilibria of the PGG \([7, 14, 15, 8]\) dealt with the simplified version of the “best-shot” public goods game. In the simplified version the costs for taking the action are equal for all players (i.e., \( \forall i \in N, C_i = C \)) and the number of public goods that each player can benefit from accessing to is limited to 1 (i.e., \( \mathbb{K} = 1 \)). In such a case, the utility of player \( i \) for some outcome \( v \in V_1 \times \ldots \times V_n \) can be simplified to:

\[
  u_i(v) := \begin{cases} 
    1 - C, & \text{if } v_i = T \\
    1, & \text{if } v_i = F, \exists j \in N_i : v_j = T \\
    0, & \text{if } v_i = F, \forall j \in N_i : v_j = F
  \end{cases}
\]  

(3)

The next section shows that the general version of the public goods game is an ordinal potential game. This implies the use of better-response dynamic for reaching a stable (equilibrium) state (Section \([3]\)). When addressing the efficiency of equilibria, an innovative procedure for finding efficient pure Nash equilibria (PNE) in the public goods game is described in Section \([5]\).
3. Stability of the Public Goods Game

The Pure Nash Equilibrium (PNE) is a central concept in game theory. An outcome is a PNE, if every player does not prefer to change her strategy, given the strategies of all other players in this outcome. Formally, an outcome \( v \) is a PNE if the following holds:

\[
\forall i \in N, \exists v'_i \in V_i \text{ s.t., } u_i(v_i, v_{-i}) < u_i(v'_i, v_{-i}) \tag{4}
\]

where \( v_{-i} \) is the standard notation for the combined strategies of all players except \( i \). It is easy to see that the states \( T_{l<K} \) and \( F_{l\geq K} \) are stable. A player in one of these states has no incentive to change her strategy unilaterally. The states \( T_{l\geq K} \) and \( F_{l<K} \) are not stable. A player in state \( F_{l<K} \) would prefer to change her strategy to \( T \), whereas a player in state \( T_{l\geq K} \) would prefer to change her strategy to \( F \).

In the next subsection it is shown that the general version of the public goods game is an ordinal potential game \([9, 10]\). This implies that a best-response procedure converges to a PNE.

3.1. A Potential Game Perspective

The concept of a potential game is used to define games in which the change in players’ utilities corresponds to the change in some global potential function \([9, 10]\). Since the incentives of all players are mapped into a global potential function, it can be a useful tool to analyze the attributes of equilibria states of the game, as will be demonstrated below.

A game is an ordinal potential game if there exists a (potential) function \( \Phi : V_1 \times \ldots \times V_n \rightarrow \mathbb{R} \) such that \( \forall v \in V_1 \times \ldots \times V_n, \forall v'_i \in V_i \) it holds that

\[
u_i(v'_i, v_{-i}) - u_i(v_i, v_{-i}) > 0 \iff \Phi(v'_i, v_{-i}) - \Phi(v_i, v_{-i}) > 0 \tag{5}\]

Namely, in (ordinal) potential games the sign of the difference of individual payoffs for each player that arises from individually changing one’s strategy (ceteris paribus) has the same sign as the difference in values of these states in the potential function.

In order to show that PGG is an ordinal potential game, one must provide a global potential function and then prove that the properties of the global potential function (as
given in Equation 5) hold. We term such a potential function as viable. Consider a

\[ \Phi(v) := \sum_{i \in N} \phi_i(v) \]  

(6)

where \( \phi_i(v) \) denotes the share of player \( i \) in the global potential function in outcome \( v \). An intuitive and simple version of \( \phi_i(v) \) was proposed for the simplified version of the “best-shot” public goods game in a former paper [8]:

\[ \phi_i(v) := \begin{cases} 
0 & \text{if } \text{state}_{i}(v) = T_{l \geq 1} \\
C & \text{if } \text{state}_{i}(v) = F_l \\
1 & \text{if } \text{state}_{i}(v) = T_{l=0} 
\end{cases} \]  

(7)

Equations 6 and 7 have been proven to form a potential function for the case of \( K = 1 \) [8]. Consider a naive generalization of Equation 7 for \( K \geq 2 \):

\[ \phi_i(v) := \begin{cases} 
0 & \text{if } \text{state}_{i}(v) = T_{l \geq K} \\
C & \text{if } \text{state}_{i}(v) = F_l \\
1 & \text{if } \text{state}_{i}(v) = T_{l < K} 
\end{cases} \]  

(8)

Let us see why Equations 6 and 7 no longer form a viable potential function, even for the case of \( K = 2 \). Consider the case were agent \( i \) is in state \( \text{state}_{i}(v) = F_l \), having a single neighbor \( j \) in state \( \text{state}_{j}(v) = T_1 \). Agent \( i \)‘s strategy change from \( v_i = F \) to \( v_i = T \) results in an increment of \( 1 - C \) in its utility, but has a negative effect on the global potential. The change in agent \( i \)’s potential share is \( 1 - C \), while the change in \( j \)’s potential share is \(-1\). Consequently, the total change in the global potential equals \(-C\) which contradicts Equation 5.

Since the potential function of Equation 7 cannot be naturally extended to fit the requirements for the general version of PGG, the new definition of the potential share is presented below together with an intuition regarding its correctness. The revised definition of the function \( \phi_i(v) \) is the following:

\[ \phi_i(v) := \begin{cases} 
a & \text{if } \text{state}_{i}(v) = F_l \\
b - j \cdot \Delta & \text{if } \text{state}_{i}(v) = T_l \end{cases} \]  

(9)
where \( a, b \) and \( \Delta \) are positive constants. Let us examine Figures 1a, 1b, and 1c that represent all possible state transitions of players in a PGG with \( K = 2 \). The purpose of these figures is to generate some intuition for the reasons of Equations 6 and 9 and the values of the constants \( a, b \) and \( \Delta \) to form a viable global potential function.

Recall that transitions of state occur when a player chooses to change her strategy, given the choices of her neighbors. The set of the possible state transitions corresponds to the maximal number of neighbors of the player. Figures 1a, 1b, and 1c present all possible transitions for a player in games with at most 1, 2 or 3 neighbors respectively (i.e., the maximal degree of nodes in the underlying graph is 1, 2, or 3).

The states in Figure 1 follow the notation in Equation 1. The edges (arrows) represent transitions among the states. Green arrows represent the change in the state of a player whose utility improved due to a change of her own strategy. All other edges represent state changes of a player due to a change of strategy by one of their neighbors. Only state transitions that change the share of a player in the global potential function (i.e., green/red/blue edges) are considered.

Consider Figure 1a which consists of only two possible strategy changes made by a player \( i \) that improve the player’s own utility (i.e., from \( F_0 \) to \( T_0 \) and from \( F_1 \) to \( T_1 \)). The change in the global potential due to the transition from state \( F_0 \) to \( T_0 \) equals \( b - a \). This stems from the fact that \( i \)’s share of potential at state \( T_0 \) equals \( b \) and the share of the potential at state \( F_0 \) equals \( a \). Since player \( i \) (which performs the change) has no other player in her neighborhood performing actions, this is the only change to the global potential. Therefore, in order to fulfill the requirements of the global potential function (Equation 5) it must have \( b - a > 0 \), i.e., \( b > a \).

The other state transition that improves the utility of a player, from \( F_1 \) to \( T_1 \); results in a change of \( b - \Delta - a \) to the potential share of player \( i \) that changed her strategy. Player \( i \) has exactly one neighbor that performs the action, therefore player \( i \)’s change of strategy changes the neighbor’s share in the global potential by exactly \(-\Delta\). Combined with the incurred change of \( b - \Delta - a \) to the share of player \( i \), the total change

\[ 1 \] Since all \( F_i \) states have exactly the same share of potential, the grey/vertical transitions between \( F_i \) states do not impact the global potential and are thus omitted from the following discussion.
to the global potential is exactly $b - \Delta - a - \Delta$, which in turn must be greater than 0 in order to satisfy the requirements of the global potential function. To summarize, it must hold that $b > a$ and $\Delta < \frac{b-a}{2}$.

By observing the state transitions for graphs with a maximal vertex degree of 2 (Figure 1b), it is easy to see that both of the previous constraints (i.e., $b > a$ and $\Delta < \frac{b-a}{2}$) must hold, for the same reasons. An additional strategy change that improves the player’s utility is from $T_2$ to $F_2$. The dashed line represents the border above which improving strategy changes are from $T$ to $F$. Such a transition results in a change to the global potential of $a - b + 2 \cdot \Delta + 2 \cdot \Delta$. This follows from the fact that the change to the potential share the player $i$ that changed her strategy is $a - b + 2 \cdot \Delta$ and the player has exactly two neighbors performing action, each of which adds $\Delta$ to her share in the global potential. Therefore, in order to satisfy the requirements of the global potential function, the inequality $a - b + 4 \cdot \Delta > 0$ (i.e., $\Delta > \frac{b-a}{2}$) must hold.

Figure 1c adds a single strategy change that improves the player’s utility, to those of Figure 1b. As a result, all previously defined constraints must still hold. The additional strategy change, from $T_3$ to $F_3$, leads to an additional inequality $a - b + 6 \cdot \Delta > 0$. This constraint is satisfied by all the values of $a$, $b$ and $\Delta$, which satisfy the inequality
$\Delta > \frac{b-a}{2}$, due to the fact that $\Delta > 0$ and $b > a$. Consequently, for graphs with maximal vertex degree of 3, the inequalities $a < b$ and $\frac{b-a}{4} < \Delta < \Delta$ must hold in order to satisfy the requirements of the global potential function.

**Proposition 1.** For any values of $a, b$ and $\Delta$ for which it holds that $a < b$ and $\frac{b-a}{2} < \Delta < \frac{b-a}{2}$, the function $\Phi(v)$ (define by Equation 6) is a global potential function.

**Proof.** First, assume that $u_i(v'_i, v_{-i}) - u_i(v_i, v_{-i}) > 0$ i.e., player $i$ performs an improving step (moves from $v_i$ to $v'_i$). An improving step can occur if either the player’s state before performing the step is $F_l < K$ or that the player’s state is $T_l \geq K$.

If the initial state of the player performing an improving step is $F_l$, it means she has exactly $l$ neighbors performing the action and therefore the change to the player’s share in the global potential is $b - l \cdot \Delta - a$, whereas the change to the global potential resulting from her neighbors’ shares is $-l \cdot \Delta$. Consequently, $b - 2 \cdot l \cdot \Delta - a$ must be greater than 0. Specifically, if $l$ equals 0, then the inequality $b > a$ must hold.

Otherwise, $\Delta$ must be less than $\frac{b-a}{2}$. Since inequality $\Delta < \frac{b-a}{2}$ should hold for every $l < K$, one can conclude that $\Delta < \frac{b-a}{2}$ for $l < K$. Note that in case $K = 1$ the only possible value of $l < K$ is $l = 0$, thus only the inequality $b > a$ is relevant.

The other improving step is from state $T_l$. The transition from state $T_l$ to $F_l$ causes a change of $a - b + l \cdot \Delta$ to the player’s own share in the global potential. The change to the global potential that arises from her neighbors’ shares equals $l \cdot \Delta$, since the player has exactly $l$ neighbors performing the action. To conclude, $\Delta$ must be greater than $\frac{b-a}{2}$. Since this inequality must hold for every $l \geq K$, one can conclude that $\Delta > \frac{b-a}{2} \cdot K$.

For the opposite direction, assume $u_i(v'_i, v_{-i}) - u_i(v_i, v_{-i}) \leq 0$ and show that $\Phi(v'_i, v_{-i}) - \Phi(v_i, v_{-i}) \leq 0$, similarly to the first direction. \(\square\)

The above potential function has an interesting feature: its global maximum is not only a PNE (as every local maximum of the potential function), but it is also the PNE with the lowest social welfare of all PNEs. To see this feature, observe first that in any optimum of the potential function players can only be in state $T_{l<K}$ or in state $F_{l\geq K}$.

\footnote{Note that when $K = 1$ the value of $\Delta$ is only constraint by $\Delta > \frac{b-a}{2}$}
Consider the inequality of Proposition 1

\[
\frac{b - a}{2 \cdot K} < \Delta < \frac{b - a}{2 \cdot (K - 1)}
\] (10)

Combining this inequality with the fact that players in local maxima of the potential function (i.e., PNEs) are in state \( T_l < K \), one can conclude that each player in state \( T_l \) contributes more to the potential than a player in state \( F_l \) (i.e., \( b - l \cdot \Delta > a \)). This in turn implies that the global maximum of the potential function occurs for a state with the maximal number of players performing the action, out of all stable states of the game. However, this is the state with the lowest social welfare out of all PNEs, because in the stable state the utility of a player in state \( T_l \) is lower than the utility of a player in state \( F_l \).

3.2. Complexity of Better Response Dynamics

The fact that the “Best-shot” public goods game is a potential game results in convergence of the better-response dynamics. In the general case the convergence of better-response dynamics is computationally intractable [16], but there are several specific cases for which the convergence to a stable point is polynomial in the number of players [17]. Therefore, it is interesting to bound the run-time of better response in the “Best-shot” public goods games. This can actually be done in a straightforward way because the difference between the maximal and minimal values of the potential function can be easily bounded and so can the size of the change in the potential function value at each better response step.

**Proposition 2.** The difference between the maximal and minimal values of the potential function, which was defined by Equations 6 and 9, is bounded by \( n^2 \cdot (b - a) \).

**Proof.** The share of a player \( i \) in the global potential function depends on the player’s state and equals \( a \) if her state is \( F_l \) or \( b - l \cdot \Delta \) otherwise. Consequently, the maximal value of such a share is \( \max(a, b - l \cdot \Delta) \), whereas the minimal value is \( \min(a, b - l \cdot \Delta) \). One can easily observe that the inequation \( b - l \cdot \Delta \leq b \) holds since \( l \geq 0 \) and \( \Delta > 0 \); such an observation, together with the fact that \( a < b \) (Proposition 1), leads to:

\[
\max(a, b - l \cdot \Delta) \leq \max(a, b) = b
\] (11)
Namely, the upper bound on a share of a single player in the global potential function is $b$.

The lower bound of the player’s share $\min(a, b - l \cdot \Delta)$ in the global potential function can be evaluated by the help of the following inequation:

$$\min(a, b - l \cdot \Delta) \geq \min(a - l \cdot \Delta, b - l \cdot \Delta) = \min(a, b) - l \cdot \Delta > a - n \cdot \Delta \geq a - n \cdot \frac{b - a}{2 \cdot (K - 1)}$$

Given the previous observations, one can evaluate the upper bound of the global potential function by the following equation:

$$\max_{v \in V} \Phi(v) = \sum_{i \in N} \max (a, b - l \cdot \Delta) < \sum_{i \in N} b = n \cdot b$$

whereas the lower bound can be evaluated as follows:

$$\max_{v \in V} \Phi(v) = \sum_{i \in N} \min (a, b - l \cdot \Delta) > \sum_{i \in N} a = n \cdot a - n^2 \cdot \frac{b - a}{2 \cdot (K - 1)}$$

Consequently, the difference between the maximal and minimal values of the global potential function is:

$$\max_{v \in V} \Phi(v) - \min_{v \in V} \Phi(v) < n \cdot b - n \cdot a + n^2 \cdot \frac{b - a}{2 \cdot (K - 1)} < n^2 \cdot (b - a)$$

**Proposition 3.** The minimal improvement in the global potential, resulting from a better response step, is $\frac{b - a}{2 \cdot K}$.

**Proof.** There are two types of improving steps when applying better-response dynamics to a PGG – from $F$ to $T$ if player $i$’s state is $F_{i < K}$ and from $T$ to $F$ if $\text{state}_i(v) = T_{i \geq K}$. For the sake of clarity we will first consider the case $K \geq 2$ and refer to the adjustments for the $K = 1$ case subsequently. In case $K \geq 2$, Proposition 1 restricts the value of $\Delta$ to be in the range $\frac{b - a}{2 \cdot K} < \Delta < \frac{b - a}{2 \cdot (K - 1)}$. Let us fix the value of $\Delta$ to be $(b - a) \cdot \frac{2 \cdot K - 1}{2 \cdot (K - 1) \cdot K}$, which is the center of the restricted range.
Consider the case that the player’s state, before changing her strategy, is $F_{l<K}$. In such a case, the change in the global potential, is $b - a - 2 \cdot l \cdot \Delta$. Since the upper bound of $l$ is $K - 1$ and $\Delta$ is positive, the following inequality holds:

$$b - a - 2 \cdot l \cdot \Delta \geq b - a - 2 \cdot (K - 1) \cdot (b - a) \cdot \frac{2 \cdot K - 1}{4 \cdot (K - 1) \cdot K}$$

$$= b - a - (b - a) \cdot \frac{2 \cdot K - 1}{2 \cdot K}$$

$$= (b - a) \cdot \frac{2 \cdot K - 2 \cdot K + 1}{2 \cdot K}$$

$$= \frac{b - a}{2 \cdot K}$$

(14)

A change to the global potential of $a - b + 2 \cdot l \cdot \Delta$ will occur if the player’s state before the strategy update is $T_{l \geq K}$. Such a change results in the following inequality:

$$a - b + 2 \cdot l \cdot \Delta \geq a - b + 2 \cdot K \cdot (b - a) \cdot \frac{2 \cdot K - 1}{4 \cdot (K - 1) \cdot K}$$

$$= a - b + (b - a) \cdot \frac{2 \cdot K - 1}{2 \cdot (K - 1)}$$

$$= (b - a) \cdot \frac{2 \cdot K - 2 \cdot K + 2}{2 \cdot (K - 1)}$$

$$= \frac{b - a}{2 \cdot (K - 1)}$$

$$> \frac{b - a}{2 \cdot K}$$

Consequently, the minimal improvement to the global potential, as a result of a better-response step, is $\frac{b - a}{2 \cdot K}$ when $K \geq 2$.

In case $K = 1$ the only possible value of $l$ is zero when the player’s state is $F_{l<K}$, which makes Equation 14 trivial. If the player’s state before the strategy update is $T_{l \geq K}$, the value $\Delta = b - a$ can be used to obtain the lower bound of $\frac{b - a}{2 \cdot K}$.

\[ \square \]

**Corollary 4.** Better-response dynamics for the “best-shot” public goods game converges in at most $2 \cdot K \cdot n^2$ improvement steps.

The maximal number of improvement steps can be computed by dividing the maximal difference of the values of the potential function (i.e., $n^2 \cdot (b - a)$) by the minimal improvement in the potential from any improvement step (i.e., $\frac{b - a}{2 \cdot K}$). Therefore, the correctness of Corollary 4 is a direct result of Proposition 2 and 3.
4. Equilibrium Efficiency vs. Stability

The fact that the general version of the “best-shot” public goods game is a potential game guarantees the existence of at least one PNE for each game. However, it is possible that a game will have multiple equilibria. Consider the example of a PGG that is based on the underlying graph depicted in Figure 2, where $K = 3$ and the costs of performing the action are equal for all players (i.e., $\forall i \in N, C_i = C$). The underlying graph consists of a clique, involving exactly $K = 3$ central players (represented by solid nodes), and a set of eight peripheral players (represented by dashed nodes), each of which is connected to all the nodes of the clique.

It is easy to see that the above PGG has exactly two equilibria states. In one stable state all central players choose $T$ as their strategy, while all peripheral players choose $F$. In the other stable state all central players choose $F$, while all others choose $T$. The social welfare of the first PNE is $3 \cdot (3 - C) + 8 \cdot 3 = 33 - 3 \cdot C$, whereas the second PNE has a social welfare of $3 \cdot 3 + 8 \cdot (3 - C) = 33 - 8 \cdot C$. From a global perspective, the network as a whole is better off in the first outcome, which maximizes social welfare (SW). We relate to such an outcome as an efficient one.

Naturally, one would like to address the question of the efficiency of the outcomes that are produced by better-response dynamics. In particular, it is interesting to compare their efficiency to that of the stable outcome that maximizes social welfare. This relates closely to the concept of Price of Anarchy (PoA) \[18\]. The price of anarchy
measures the bound on the degradation of the efficiency of a system due to the selfish
behavior of its agents. It is defined to be the ratio between the optimal solution (i.e., of
maximal SW) and the worst equilibrium:

\[ \text{PoA} = \frac{\max_{v \in V_1 \times \ldots \times V_n} SW(v)}{\min_{v \in V_1 \times \ldots \times V_n | v \text{ is a PNE}} SW(v)} \] (16)

The PoA of the above example of a PGG is

\[ 1 < \frac{33 - 3 \cdot C}{33 - 8 \cdot C} < \frac{6}{5} \] (17)

following directly from the fact that \( 0 < C < 1 \). It is easy to see that the first equi-
librium above is the most efficient outcome of the game. Given the configuration of
Figure 2 with an unboundedly large number of peripheral players, the limit of the PoA
is \( \frac{K}{(K-1)} \), and can be unboundedly large for \( K = 1 \).

Another interesting concept regarding the relationship between efficiency and sta-
bility is that of the Price of Stability (PoS) \(^{[18]}\). The price of stability measures the
ratio between the optimal solution and the best equilibrium:

\[ \text{PoS} = \frac{\max_{v \in V_1 \times \ldots \times V_n} SW(v)}{\max_{v \in V_1 \times \ldots \times V_n | v \text{ is a PNE}} SW(v)} \] (18)

PoS conveys the efficiency gap between the globally most efficient outcome and the
most efficient stable outcome.

Figure 3: Underlying graph of a PGG with high PoS.

To get some intuition about the price of stability of a PGG one can use the graph
depicted in Figure 3 and assume that the costs of all actions are equal \( \forall i \in N, C_i = C \).
The players participating in the game can be split into two distinct groups \( N_1 \) and \( N_2 \) of equal size so that \( N_1 \cup N_2 = N \), \( N_1 \cap N_2 = \emptyset \) and \( |N_1| - |N_2| \leq 1 \). For each group one can create a graph like Figure 2 where an edge exists between every two central players of both groups. The overall graph in Figure 3 depicts the case with \( |N| = 22 \) and \( K = 3 \). For this case the optimal outcome has a social welfare of \( 2 \cdot K \cdot (K - C) + (|N| - 2 \cdot K) \cdot K \). Clearly, this outcome is not stable, as any of the central players could improve her utility by not performing the action. When considering any stable outcomes, the social welfare of the most efficient PNE is \( K \cdot (K - C) + \left( \frac{|N|}{2} - K \right) \cdot K + \left( \frac{|N|}{2} - K \right) \cdot (K - C) + K \cdot K \). The most efficient PNE has one group in which the central players take the action and let all their peripheral players gain the most, and another group in which only the peripheral players take the action and let the central players gain the most. The price of stability in this case is therefore:

\[
PoS = \frac{|N| \cdot K - 2 \cdot K \cdot C}{|N| \cdot K - \frac{|N|}{2} \cdot C}
= 1 + \left( \frac{|N| - 4 \cdot K}{|N|} \right) \cdot C
\]

Equation 19

It is easy to see from Equation 19 that if the number of the participating agents \( |N| \) is unboundedly large, then the the limit of the PoS is \( 1 + \frac{C}{2K - C} < \frac{3}{2} \). Both of the above examples serve to demonstrate that there is still a wide efficiency gap (up to 50% in the last example) between the globally most efficient outcome and the most efficient stable outcome. This gap can be thought of as the tradeoff between efficiency and stability. Nevertheless, this gap can be bridged by the algorithm proposed in the next sections.

5. Finding Efficient Equilibria – Preliminaries

Despite the efficiency gap that was described in the previous section one may be able to find efficient equilibria in the “best-shot” public goods game by using an extended version of better-response dynamics. In the extended scenario each player is endowed with the ability to propose a payoff transfer at each step, in response to a neighbor’s proposal of her better response. For the simple version of the PGG where \( K = 1 \), it is shown next that the players’ ability to propose payoff transfers enables the extended version of better-response dynamics to converge to a stable state of greater
(or equal) efficiency than that of the initial, not necessarily stable, outcome. When the extended better-response dynamics is applied to the general version of PGG ($K \geq 2$), there is no such theoretical guarantee for the efficiency. Nevertheless, our experimental evaluation in Section 7 demonstrates that in practice the efficiency is improved for all the evaluated game instances.

The following subsection describes the mechanism of transfers of payoffs among players, which is a key element in the stability enforcing algorithm that is presented in Section 6. Next, a novel stability enforcing procedure that relies on transfers of payoffs is proposed. Finally, it is shown how this procedure may be complemented by existing heuristics that find initial outcomes of high efficiency. These components form the complete procedure for finding efficient PNEs in the “best-shot” public goods game.

5.1. Transfers of payoffs

Transfers of payoffs enable “best-shot” public goods games to be transformed from the inside, by endowing players with the possibility of sacrificing part of their payoff in order to convince other players to play a certain strategy. The transfer function between players enables each player to pay (or receive payment from) each one of its neighbors.

Given a “best-shot” public goods game with an underlying interactions network $G = \{N, E\}$, let us define the transfer function $\tau : N \times N \times V_1 \times \ldots \times V_n \rightarrow \mathbb{R}^+$ so that $\tau_{i,j}(v)$ denotes the payment being transferred from player $i$ to player $j$ if outcome $v$ is played. In order to take the transfer of payoffs into consideration while deciding on the action to take, a player’s utility must also reflect the change in her payoff caused by the transfers. The net loss that is incurred on player $i$ in outcome $v$ when using the transfer function $\tau$ is defined as follows:

$$\tau_i(v) := \sum_{j \in N} (\tau_{i,j}(v) - \tau_{j,i}(v))$$

We restrict our attention to transfer functions from neighbors, that is, if $j \notin N_i$ then $\forall v \in V_1, \ldots, V_n : \tau_{i,j}(v) = 0$. 

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The introduction of transfers of payoffs leads to an updated definition of the utility that player $i$ obtains from outcome $v$ (given a transfer function $\tau$):

$$u^\tau_i(v) := u_i(v) - \tau_i(v)$$  \hspace{1cm} (21)

Since the use of transfers of payoffs is motivated by the attempt to secure a stable state (PNE) with certain properties, it is required to differentiate between outcomes that can be transformed into a PNE by the use of transfers of payoffs from those that cannot.

**Definition 1.** An outcome $v$ is side-payments enforceable (SPE) if there exists a transfer function $\tau$, such that:

$$\forall i \in N, \exists v'_i \in V_i : u_i(v'_i, v_{-i}) > u^\tau_i(v_i, v_{-i})$$

Let us explain the intuition behind the ability of transfers of payoffs to obtain efficient outcomes: A player can offer a neighboring player compensation, which relies on the second player’s action, such that the compensation effectively reflects any impact that the second player’s action has on the first (paying) player’s utility. Take for example the following case, in which the benefit to player $i$ is $x$ if player $j$ takes action $v'_j$ rather than action $v_j$ (i.e., $u_i(v'_j, v_{-j}) - u_i(v_j, v_{-j}) = x$). Assume also that the cost to player $j$ for taking action $v'_j$ rather than $v_j$ is only $y$ (i.e., $u_j(v_j, v_{-j}) - u_j(v'_j, v_{-j}) = y$) where $x > y$. Consider a transfer of payoff $z$, where $x \geq z \geq y$, by player $i$ to player $j$ in case player $j$ plays $v'_j$ instead of $v_j$ (i.e., $\tau_{i,j}(v'_j, v_{-j}) = z$). Any $z$ such that $x \geq z \geq y$ will provide a sufficient incentive for player $j$ to take the action $v'_j$, because

$$u^\tau_j(v'_j, v_{-j}) = u_j(v'_j, v_{-j}) + z \geq u_i(v'_j, v_{-j}) + u_j(v_j, v_{-j}) - u_j(v'_j, v_{-j}) = u_j(v_j, v_{-j})$$

Additionally, player $i$ has a sufficient incentive to secure the transfer of payoff $z$ to player $j$ due to the fact that

$$u^\tau_i(v'_j, v_{-j}) = u_i(v'_j, v_{-j}) - z \geq u_i(v'_j, v_{-j}) - u_i(v'_j, v_{-j}) + u_i(v_j, v_{-j}) = u_i(v_j, v_{-j})$$

In a “best-shot” public goods game a player $i$ has an incentive to compensate its neighbor $j$ only in case player $j$ benefits from deviation from its strategy $T$. In this case
j’s deviation (from strategy $T$ to $F$) affects $i$’s utility, and the payoff loss incurred to player $i$ is 1 (according to Equation 2). Therefore, the utility loss of player $i$, termed $\ell_i$, which can be thought of as the maximal payment (i.e., transfer of payoff) that player $i$ may be willing to sacrifice in order to convince player $j$ to stay with strategy $T$ is defined as follows:

$$
\ell_i(v) := \begin{cases} 
1, & \text{if } state_i(v) = T_{0<l<K} \\
1, & \text{if } state_i(v) = F_{0<l<K} \\
C_i, & \text{if } state_i(v) = F_K \\
0, & \text{otherwise}
\end{cases}
$$

(22)

In the case of $state_i(v) = T_{l<K}$ the change of $j$’s strategy from $T$ to $F$ will decrease $i$’s payoff by 1. This arises from the fact that before player $j$ changes its strategy the utility of player $i$ is $u_i(v) = l + 1 - C_i$, while after $j$’s change of strategy it will be decreased to $u_i(v_j = F, v_{-j}) = l - C_i$. Therefore, player $i$ is willing to secure a payoff of at most 1 to player $j$. Similarly to the previous case, if $state_i(v) = F_{l<K}$, player $i$’s utility will be also decreased by 1, which results in $i$’s willingness to propose a payoff of at most 1 to player $j$ in order to convince player $j$ to stay with its choice $T$. When $state_i(v) = F_K$, $j$’s change of strategy from $T$ to $F$ will decrease $i$’s payoff to $K - 1$. But $i$’s utility may be increased to $K - C_i$ by its own unilateral deviation to $T$, in a following step. Therefore, the maximal payoff that player $i$ will be willing to secure for player $j$ is $C_i$.

6. Stability Enforcing Algorithm

The stability enforcing algorithm is composed of two consecutive stages. Starting from an initial outcome $v$, the first stage finds an improved outcome $v^*$ that is guaranteed to be SPE. In other words, $v^*$ is an outcome for which there exists a transfer function $\tau$ that transforms it into a stable state. Such a transfer function can be efficiently found by the algorithm. In the second stage the computed transfers to $v^*$ are applied and the improved outcome is transformed into a stable state (i.e., a PNE).
6.1. The simple version of PGG

Algorithm 1 conducts the first stage for the simple version of PGG, where \( K = 1 \). Similarly to standard better response, it is assumed that the players of the game play in some fixed and predefined order. Each player in its turn executes the `onStrategySelect()` procedure that deals with unstable states \( (F_0, T_{l \geq 1}) \). The procedure ensures that each player \( i \) in state \( T_{l \geq 1} \) will change its strategy to \( F \), except in cases where other players have incentive and sufficient funds to convince player \( i \) to remain with its current strategy \( T \). This is done by player \( i \) sending a message `strategy-change` to all of its neighbors (line 4), which in turn may propose a compensation (i.e., payoff transfer \( \tau_{j,i}(v) \)) so as to convince player \( i \) to stay with choice \( T \) (lines 8-9). This method enables to compensate player \( i \) by transfer of payoffs and in this way to enforce stability.

In addition, the function `onStrategySelect()` ensures that no player will stay in state \( F_0 \), by changing the choice of each player in this state from \( F \) to \( T \) (line 2). Algorithm 1 terminates when a pass through all the players of the game will not yield any change in the players’ strategies.

**Algorithm 1** Finding an improved outcome.

```plaintext
onStrategySelect()
1: if \( \text{state}_i(v) = F_0 \) do
2: update: \( v_i \leftarrow T \)
3: if \( \text{state}_i(v) = T_{l \geq 1} \) do
4: send(`strategy-change`) to each \( j \in N_i \), and await for the replies
5: let \( \tau_{j,i}(v) \) be the reply from neighbor \( j \in N_i \)
6: if \( C_i > -\tau_i(v) \) do
7: update: \( v_i \leftarrow F \)

when received(`strategy-change`)
8: update: \( \tau_{i,j}(v) \leftarrow \ell_i(v) \)
9: reply \( \tau_{i,j}(v) \)
```

According to Equation 22, only players in state \( F_1 \) can propose payoff transfers to players in state \( T_{l \geq 1} \). If the payoffs from such players (\( \tau_{j,i}(v) \)) sums up to a value
equal to or greater than $C_i$, then the overall compensation to player $i$ is sufficient and she will remain with strategy $T$. Otherwise (lines 6-7) she will update her strategy to $F$.

Note that the transfers of payoff do not affect the social welfare of outcomes. Therefore, one can omit the payoff transfers (i.e., take into consideration only the “original” utilities of the players) when reasoning about the social welfare. However, it is essential to take the payoff transfers into consideration when reasoning about the stability of an outcome.

**Proposition 5.** For every initial outcome $v \in V$, the social welfare of $v^*$, which is the end result of running Algorithm[1] on $v$, is greater than or equal to the social welfare of $v$.

**Proof.** In order to prove that the social welfare of $v^*$ is not lower than that of $v$ one can show that each change of strategy, resulting from the execution of function $onStrategySelect()$, has two possible consequences. Either it immediately improves the social welfare, or it will eventually improve the social welfare before Algorithm[1] terminates. According to Algorithm[1] a player $i$ will change its strategy if either $state_i(v) = F_0$ or $state_i(v) = T_l \geq 1 \land C_i > -\tau_i(v)$ (lines 2 and 7 respectively). Therefore, only these cases will be considered.

Suppose that player $i$’s state is $F_0$. This means that $i$’s choice is to avoid action ($F$) and the number of its neighbors performing the action is zero. In such a case, $i$’s utility is 0, and can be improved to $1 - C_i$ if player $i$ changes its strategy to $T$. Such a change cannot decrease the utility of any other player, and therefore, the change in social welfare will be at least $1 - C_i - 0$, which is positive since $0 < C_i < 1$.

In the case that player $i$’s state is $T_l \geq 1$, it would benefit from changing its strategy to $F$. Player $i$ will deviate from its strategy $T$ to $F$ only in case $C_i > -\tau_i(v)$, which means that $C_i > \sum_{j \in N_i|state_j(v) = F_1} C_j$. Such a change in $i$’s strategy will result in an increment of $C_i$ in its utility, but may decrease the utility of $i$’s neighbors. The utility of each player $j \in N_i$, which stands to lose from the change of $i$’s strategy, will be decreased by 1 (according to Equation[2]). Therefore, the change in social welfare
resulting from \(i\)’s deviation, will be \(C_i - \sum_{j \in N_i | \text{state}_j(v) = F_1} 1\). Since the updated state of \(i\)’s neighbors affected by this deviation is \(F_0\), Algorithm 1 will not terminate until their utility will be improved (either by self deviation or by a neighbor’s deviation to \(T\)). In the case where \(j\)’s utility is increased by its own deviation (from \(F\) to \(T\)) the increment to the social welfare will be at least \(1 - C_j\) (as in the previous case \(F_0\)). Otherwise, one of \(j\)’s neighbors will change its strategy from \(F\) to \(T\), which will in turn increase the social welfare by at least 1. Combining both cases, the social welfare will increase by at least \(1 - C_j\). Consequently, Algorithm 1 will not terminate until the social welfare will be improved by at least \(C_i - \sum_{j \in N_i | \text{state}_j(v) = F_1} 1 + \sum_{j \in N_i | \text{state}_j(v) = F_1} (1 - C_j) = C_i - \sum_{j \in N_i | \text{state}_j(v) = F_1} C_j > 0\). Therefore, if player \(i\) that is currently in state \(T_1\), deviates from \(F\) to \(T\), then eventually the social welfare will increase. 

**Corollary 6.** Algorithm 1 converges in at most \(2 \cdot K \cdot n^2\) improvement steps.

Algorithm 1 allows only a subset of the strategy changes that can be done by best-response dynamics. Therefore, each strategy change that is the result of running Algorithm 1 increases the value of the potential function defined by Equation 6 and 9. Consequently, the correctness of Corollary 6 follows directly from Corollary 4.

**Proposition 7.** An outcome \(v^*\), yielded by Algorithm 1 is side-payments enforceable.

**Proof.** Algorithm 1 will terminate when a pass through all the players of the game will not yield changes in the players’ strategies. Therefore, the state of the players at \(v^*\) will be either \(F_{\geq 1}\) or \(T_{\geq 0}\). Consequently, in order to show that \(v^*\) is side-payments enforceable, one needs to prove that no player will benefit from a unilateral deviation, given the payoff transfer function \(\tau\). The payoff transfer function, computed during the run of Algorithm 1, will be used for this purpose.

Consider the case \(\text{state}_i(v^*) = F_1\). This means that \(u_i(v^*) = 1\) and consequently \(u^*_i(v^*) = 1 - \tau_i(v^*)\). In this case player \(i\) has exactly one neighbor \(j\) s.t. \(\text{state}_j(v^*) = T_{\geq 0}\) and the maximal payoff transfer, secured by player \(i\) for player \(j\), is \(\tau_{i,j}(v^*) = C_i\). This results in \(i\)’s net loss of \(\tau_i(v^*) = C_i\). Therefore, \(u^*_i(v^*) \geq 1 - C_i = u_i(v_i = T, v^*_v - i),\) which means that player \(i\) cannot benefit from unilateral deviation.
Now consider the case \( \text{state}_i(v^*) = \mathbb{F}_{l>1} \). In this case player \( i \) does not secure a payoff transfer for any of its neighbors (according to Equation 22). Therefore, \( u_i^* (v^*) = u_i (v^*) = 1 \), which means that \( u_i^* (v^*) > 1 - C_i = u_i (v_i = T, v^*_{-i}) \). Similarly, when \( \text{state}_i(v^*) = T_0 \), player \( i \)'s net loss is zero (\( \tau_i(v^*) = 0 \)) and it will not benefit from unilateral deviation.

Finally, if \( \text{state}_i(v^*) = T_{l \geq 1} \) then after the termination of Algorithm 1 it must hold that \( -\tau_i(v) \geq C_i \). Therefore, \( u_i^* (v^*) = 1 - C_i - \tau_i(v^*) \geq 1 = u_i (v_i = F, v^*_{-i}) \) which means that player \( i \) cannot benefit from unilateral deviation also in this case. Having considered all possible states of player \( i \) in outcome \( v^* \), it holds that \( v^* \) is side-payments enforceable.

**Observation 8.** An outcome \( v \) that maximizes social welfare is side payments enforceable.

Following Proposition 8, it is clear that applying Algorithm 1 to an outcome \( v \) that maximizes social welfare returns the exact same outcome \( v^* = v \). Therefore, this outcome is SPE as shown by Proposition 7.

### 6.2. The general version of PGG

Algorithm 2 below, is an extended version of Algorithm 1 that handles the general version of the “best-shot” public goods game. The main difference between the algorithms is that for the general version of PGG there is no guarantee regarding the improvement of the social welfare. Nevertheless, the experimental evaluation of Section 7 demonstrates that in practice Algorithm 2 yields stable outcomes of greater efficiency than that of the initial outcome. The lack of theoretical guarantees regarding the efficiency of the outcomes stems from the fact that according to Algorithm 1 each player may secure a transfer of payoffs to at most one of its neighbors. This observation does not hold in the general case when \( K > 1 \). Similarly to the simple version, Algorithm 2 terminates when a pass through all the players of the game does not yield any changes to the players’ strategies.

The function onStrategySelect() of Algorithm 2 is executed by each player according to some fixed predefined order. After the execution of the function by player
Algorithm 2 Finding an improved outcome (extended).

\begin{algorithm}
\textbf{onStrategySelect}()
1: \textbf{if} \((\text{state}_i(v) = F_{l < K})\) \textbf{do}
2: \quad \text{update: } v_i \leftarrow T
3: \textbf{if} \((\text{state}_i(v) = T_{l \geq K})\) \textbf{do}
4: \quad \text{send(strategy-change) to each } j \in N_i
5: \quad \text{let } \tau_{j,i}(v) \text{ be the reply from neighbor } j \in N_i
6: \textbf{if} \(C_i > -\tau_i(v)\) \textbf{do}
7: \quad \text{update: } v_i \leftarrow F
\end{algorithm}

\textbf{when received(strategy-change)}
8: \textbf{choose } \tau_{i,j}(v)
9: \textbf{reply } \tau_{i,j}(v)

\(i\), the player will not benefit from unilateral deviation. This property is achieved either by the change of \(i\)'s strategy or by promised compensation to player \(i\) by its neighbors. In order to ensure that the outcome, yielded by Algorithm 2 is indeed SPE we restrict our attention only to \textit{admissible} payoff transfer functions according to the following definition.

\textbf{Definition 2.} \textit{Given an outcome }\(v\) \textit{of a “best-shot” public goods game, the payoff transfer function }\(\tau\) \textit{is admissible if for every player }\(i\) \textit{the following conditions hold:}

1. \(\forall i \in N, j \in N_i, \tau_{i,j}(v) \leq \ell_i(v)\)
2. \(\text{if } \text{state}_i(v) = F_K \text{ then } \tau_i(v) \leq C_i\)

\textbf{Corollary 9.} Algorithm 2\textit{ converges in at most }\(2 \cdot K \cdot n^2\) \textit{improvement steps.}

Algorithm 2 allows only a subset of strategy changes out of those that can occur during the run of better-response dynamics (similarly to Algorithm 1). Consequently, the correctness of Corollary 9 follows directly from Corollary 4 that relates to the simple PGG case.

In order to get some intuition about the expected improvement in social welfare of the resulting outcome, consider the following example:
Example 1. Consider an example PGG with 8 agents connected by the network depicted in Figure 4a, where $k = 2$, and equal costs for all players (i.e., $\forall i \in N, C_i = C$). Figure 4a also presents a specific initial outcome of the game (i.e., $v_1 = T, v_2 = T, v_3 = T, v_4 = F, v_5 = T, v_6 = T, v_7 = T$ and $v_8 = F$). In this example, Algorithm 2 traverses the players in the order of their indexes. Additionally, the use of an admissible payoff transfer function is assumed.

![Figure 4a](image)

(a) Initial state of the game.  
(b) End result of Algorithm 2.

The first player that invokes function onStrategySelect() is player 1. The state of the player is $T_2$, because exactly two of its neighbors chose $T$ as their action. Therefore, player 1 sends a strategy-change message to all of its neighbors (i.e., players 2 and 3). Since state2(v) = state3(v) = T3, $\ell_2(v) = 0$ and $\ell_3(v) = 0$, both neighbors reply with zero. Thus, player 1 changes her strategy from $T$ to $F$.

The next player in order is 2; the state of player 2 at this stage is $T_2$ and therefore she sends a strategy-change message to players 1, 3, 4 and 7. The replies of both players 3 and 7 are zero (state3(v) = $T_2$ and state7(v) = $T_3$). Consider now the replies of players 1 and 4, which are both at state $F_2$ at this stage. This means that
\( \ell_1 = \ell_4 = C \), and therefore the sum of the replies of players 1 and 4 should be at least
\( C \) (say, \( \tau_{1,2}(v) = \tau_{4,2}(v) = \frac{C}{2} \)). According to these replies, the net gain of player 2 is
greater/equal to \( C \) (i.e., \(-\tau_2(v) \geq C\)). Therefore, player 2 remains with her choice \( T \).

The result of the execution of function \( \text{onStrategySelect}() \) by player 3 is similar
to that of player 2. The state of player 3 is \( T_2 \) which leads to four messages \( \text{strategy-change} \)
to be sent to players 1, 2, 4 and 6. The sum of non-zero messages (from players
1 and 4) can be larger than \( C \) (e.g., rationally preferable by 1 and 4), which will prevent
player 3 from changing her strategy.

Player 4 does not change its strategy since \( \text{state}_4(v) = F_2 \) and the net loss of
player 4 at this stage is \( -\tau_4(v) \leq C \). The results of the subsequent invocations of function
\( \text{onStrategySelect}() \) are as follows: player 5 changes her strategy to \( F \) for the same
reasons of player 1, whereas players 6, 7 and 8 will remain with their choices for the
same reasons of players 2, 3 and 4, respectively. An additional run over all players,
traversed again according their indexes, does not yield any changes in their strategies,
which leads to the termination of Algorithm 2 (see Figure 4).

Note that the social welfare of the initial outcome is \( 8 \cdot 2 - 6 \cdot C \); this is due to
the fact that each player has at least two neighbors taking the action and there are six
players with strategy \( v_i = T \). The social welfare of the outcome yielded by Algorithm 2
is \( 8 \cdot 2 - 4 \cdot C \), which is higher. Additionally, it is easy to see that the resulting outcome
is side payments enforceable (SPE). Namely, the outcome \( v = (F, T, T, F, F, T, T, F) \)
can be stabilized by using the payoff transfers that were contracted (i.e., offered and
accepted) during the run of Algorithm 2.

Next, the stability of the outcome yielded by Algorithm 2 is investigated, followed
by an example of an admissible payoff transfer function.

**Proposition 10.** The use of an admissible payoff transfer function during the run of
Algorithm 2 ensures that the resulting outcome \( v^* \), is SPE.

**Proof.** The correctness of Proposition 10 follows directly from the termination condi-
tion of Algorithm 2. First, one can observe that after the algorithm terminates there
are no players in states \( F_{<K} \). Consequently, the only players that may want to change
their strategy are players that are in states \( F_K, F_{>K}, T_{<K} \) or \( T_{>K} \).
Consider the case \( state_i(v^*) = F_K \), which means that \( u_i(v^*) = K \). Therefore, \( u_i^T(v^*) = K - \tau_i(v^*) \). A unilateral deviation of player \( i \) will result in outcome \( v' = (v_i = T, v^*_{-i}) \), in which \( i \)'s utility will be \( u_i(v') = K - C_i \). According to the definition of an admissible payoff transfer function, if \( state_i(v) = F_K \) then \( i \)'s net loss \( (\tau_i(v)) \) must be at most \( C_i \). Therefore, \( u_i^T(v^*) = K - \tau_i(v^*) \geq K - C_i = u_i(v') \), which means that player \( i \) will not benefit from unilateral deviation (from \( F \) to \( T \)).

Now consider the case \( state_i(v^*) = F_{l > K} \). According to the definition of admissible payoff transfer functions, \( i \)'s net loss is zero \( (\tau_i(v^*) = 0) \), since \( \ell_i(v) = 0 \). Therefore, \( u_i^T(v^*) = u_i(v^*) = K > K - C_i = u_i(v_i = T, v^*_{-i}) \). Clearly, player \( i \) has no incentive to change her strategy. Similar reasoning settles the case of \( state_i(v) = T_{l < K} \).

Finally, \( state_i(v^*) = T_{l \geq K} \) can exist for some player \( i \) after the termination of Algorithm 2 only if \( C_i \leq -\tau_i(v^*) \). Consequently, \( u_i^T(v^*) = K - C_i - \tau_i(v^*) \geq K = u_i(v_i = T, v^*_{-i}) \), which essentially means that player \( i \) can be convinced to remain with its choice \( T \).

Next, an example of an admissible payoff transfer function is presented. Note that according to Algorithm 2, transfers of payoffs should only be proposed to players with \( state_i(v) = T_{l \geq K} \). Additionally, note that only the player’s state and its cost for taking the action limit the value of the payoff. As a result, a player \( j \) may decide on the following payoff transfer without consulting other players:

\[
\tau_{j,i}(v) := \begin{cases} 
\ell_j(v), & \text{if } state_j(v) \neq F_K \\
\max(\ell_j(v) - \tau_j(v), 0), & \text{if } state_j(v) = F_K 
\end{cases}
\]

(23)

In other words, each player will secure the maximal payoff allowed according to the admissibility restriction (Definition 2). Let us proceed to see how an SPE outcome can be stabilized using the payoff transfers of Equation 23.

**Example 2.** Reconsider the outcome \( v^* \) of Example 1 as yielded by Algorithm 2 (Figure 4b). One can easily see that only players 2, 3, 6 and 7 should be incentivized in order to stabilize outcome \( v^* \). According to Equation 23, player 1 can secure a payoff of \( C \) to player 2 \( (\tau_{1,2} = C) \) in order to convince her to remain with her choice \( T \). At
this stage \( \tau_1(v^*) = C \). Subsequently the maximal payoff transfer that player 1 may secure to player 3 is zero (since \( \text{state}_1(v^*) = F_2 \) and \( \max (\ell_1(v^*) - \tau_1(v^*), 0) = \max (C - C, 0) = 0 \)). Nevertheless, player 4 may secure a transfer of \( C \) to player 3 (\( \tau_{4,3} = C \)) and as a result player 3 will not benefit from unilateral deviation to strategy \( F \). Similar payoff transfers are proposed by the respective players on the right-hand side of the graph (players 5-8). One can observe that the payoff transfers \( \tau_{1,2} = C, \tau_{4,3} = C, \tau_{5,6} = C \) and \( \tau_{8,7} = C \) will indeed stabilize outcome \( v^* \).

7. Experimental Evaluation

The proposed stability enforcing algorithm of Section 6 was proven to ensure stability (Propositions 7 and 10). The objective of this experimental evaluation is to empirically demonstrate the efficiency of the resulting stable states. A theoretical guarantee regarding the efficiency of Algorithm 1 (i.e., \( K = 1 \)) was given in Proposition 5. Nevertheless, in all the conducted experiments, the efficiency of the outcomes \( (v^*) \) yielded by Algorithm 2 (with various values of \( K \)) was always greater than the efficiency of the original outcomes \( (v) \). One way to demonstrate the efficiency of the method is to compare it to that of best-response dynamics. Despite the conceptual differences between the two methods, both of them ultimately converge to a stable point – best-response dynamics to a PNE and the proposed procedure to an SPE.

The first set of experiments considers the efficiency of the resulting stable outcomes of both methods keeping in mind the PoA and PoS of the games (see Section 4). For this purpose the PNEs that maximize and minimize the social welfare are considered, these PNEs were found by exhaustive search. The second set of experiments studies how do the parameters of the problem (i.e., the number of players \( N \), the value of \( K \) and the average number of neighbors \( m \)) influence the performance of the two methods. The third set of experiments studies the impact of the order of the players on the resulting outcome. The use of fixed orders of players is reminiscent of “serial dictatorship” procedures, which are notorious for their possibly “unfair” results. Therefore, studying the fairness of both methods is important.
7.1. Problem Generation

In order to generate various PGG instances for the experimental evaluation, we use two types of underlying social networks:

1. **Random networks** – a random graph is obtained by starting with a set of $n$ isolated nodes and adding successive edges between them at random. In our experiments the model of Erdős and Rényi \cite{19} was used to generate random networks. Instances of these networks were constructed by generating $n$ vertices, and for each pair of vertices $i, j$ the edge $\{i, j\}$ was added with a probability $p^3$.

2. **Scale-free networks** – in a scale-free graph, the distribution of node degrees follows a power law (at least asymptotically), $n_d \propto d^{-\gamma}$, where $N_d$ is the number of nodes of degree $d$ and $\gamma > 0$ is a constant (typically $2 < \gamma < 3$). The model of Barabási and Albert \cite{20} was used to generate random scale-free networks. The network begins with an initial connected network of $m_0 = 3$ nodes. New nodes are added to the network one at a time. Each new node is connected to $m$ existing nodes with a probability that is proportional to the number of links that the existing nodes already have.

While the construction processes of these networks are quite different from each other, they both share two important parameters – the number of nodes in the network ($n$) and the average number of neighbors (edges) for each node ($m$). For each instance a random problem was generated. First, a random social network was generated using each of the above models. Next, a “best-shot” public goods game was constructed according to the rules described in Section 2, where the cost of assigning $T$ was drawn uniformly from the range $[0, 1)$. Finally, an initial outcome and the order over the players were randomly chosen from a uniform distribution. Both of the evaluated methods – the stability enforcing algorithm (Section 6) and better-response dynamics (Section 3.1) – were applied to this outcome in order to achieve a stable solution.

\footnote{Note that in such a case the average number of edges equals $\binom{n}{2} \cdot p$.}
7.2. Metrics for evaluation

The performance of the two methods was evaluated by three metrics:

1. **Social welfare** – the utilitarian sum (the efficiency of the outcome)
   
   \[ SW(v) = \sum_{i \in N} u_i(v) \]

2. **Gini coefficient** – a measure of inequality among players’ utilities (the fairness of the solution)
   
   \[ GI(v) = \frac{\sum_{i \in N} \sum_{j \in N} |u_i(v) - u_j(v)|}{2 \sum_{i \in N} \sum_{j \in N} u_j(v)} \]

3. **Number of improvement steps** – the number of times a strategy of one of the players was changed. This measures the speed of convergence and serves to also assess the scalability of the method.

7.3. Experimental results

The first set of experiments evaluates the difference between the social welfare yielded by the two methods – the stability enforcing algorithm and better-response dynamics – in comparison to the PNE that maximizes social welfare and the PNE that minimizes social welfare. For these experiments the average number of neighbors is fixed to \( m = 4 \), for \( K = 1 \), and \( m = 5 \) when \( K \) is 3. Due to the small size of these networks we were able to compute the best and worst PNEs by exhaustive search, i.e., by enumerating all possible outcomes. Figures 5 and 6 depict the improvement in social welfare and in the Gini index respectively. The baseline for improvement in social welfare is the social welfare of the initial (random) outcome.

It is easy to see that the average efficiency of the outcomes obtained by the proposed method is higher than those obtained by better-response dynamics (Figure 5) for both cases (\( K = 1 \) and \( K = 3 \)). For the simple version, where \( K = 1 \), the improvement in SW of Algorithm 1 is relatively small (Figure 5a). However, for general PGGs with \( K = 3 \) the improvement in social welfare obtained by Algorithm 2 is quite close to that of the best PNEs (Figure 5b). In contrast, the efficiency of the outcomes obtained by better-response dynamics is only slightly higher than that of the worst PNEs. This
Improvement in social welfare
Players
Better-response
Improvement procedure
Best PNE
Worst PNE

(a) $K = 1$ and $m = 4$.

(b) $K = 3$ and $m = 5$.

Figure 5: Social welfare of stable outcomes on random networks.

observation emphasizes the motivation for the use of payoff transfers when searching for efficient stable solutions of “best-shot” public goods games.

(a) $K = 1$ and $m = 4$.

(b) $K = 3$ and $m = 5$.

Figure 6: Fairness of stable outcomes on random networks.

The Gini index of the best PNEs is better (lower) than that of the worst PNEs (Figures 6a and 6b). This relates to the fact that in order to maximize social welfare one needs to minimize the number of players that choose to take the action. In such a case the number of players that suffer a cost of $C_i$ is minimal, which in turn maximizes the fairness of the solution.

The picture changes when one examines the Gini coefficient of the outcomes yielded by the proposed method; while in the case of $K = 1$ the fairness of outcomes is similar to those of better-response dynamics, the results for $K = 3$ are of higher Gini coefficient, even higher than that of the worst PNEs. This phenomenon may stem from the fact that players do not cooperate while reasoning about the payoff transfers. Namely, multiple players may secure a payoff to the same player $i$ which results in the net gain
of player $i$ ($\tau_i(v^*)$) being much higher than its cost for taking the action ($C_i$) leading to a deterioration in the fairness of the solution.

The next set of experiments studies the influence of the parameters of the “best-shot” public goods games on the performance of the methods. First, the effect of the size of the underlying network on the efficiency of the resulting outcomes. For these experiments the number of players is varied in the range \{100, \ldots, 1000\}. Next, the value of $m$ is varied in the range \{3, \ldots, 10\}, for a fixed number of players ($n = 1000$). Studied finally is the impact of the value of $K$ on the results of the methods. This set of experiments empirically evaluates the efficiency and the fairness of stable solutions of PGGs on large random and scale-free networks. These networks are much larger than those used in the first set of experiments, and therefore results of the best and worst PNEs are not provided here, since these are intractable.

Figure 7 depicts the improvement in social welfare of the outcomes yielded by both methods on large random networks. When $K = 1$, the proposed method produces outcomes which about 5% higher social welfare improvement than those provided by better-response dynamics (Figure 7a), which is consistent with the results of the first set of experiments (Figure 5a). When $K = 3$, the results of the proposed method are approximately 16% more efficient than those of better-response dynamics (Figure 5b). Note that in this case better-response dynamics yields outcomes that are $\approx 1\% - 2\%$ less efficient than the initial outcome. This deterioration in the social welfare can be
explained by the properties of the potential function described in Section 3.1 (i.e., the maximal value of the potential function represents a PNE of minimal social welfare).

Figure 8 presents the improvement in social welfare on large scale-free networks. It is easy to see that the type of underlying network has almost no impact on the ability of the proposed method to improve the social welfare in both cases ($K = 1$ and $K = 3$). Moreover, it provides an additional incentive to use payoff transfers when $K = 3$ since the decrease in social welfare of better-response dynamics is over 15%.

Figure 9 presents a measure of the fairness of the resulting outcomes of the two methods on both types of underlying networks with $K = 3$. The results for $K = 1$ are very similar to the respective results of the first set of the experiments (Figure 6a).
and are thus omitted. Figure 9 clearly shows that the ability of the proposed method to provide outcomes of higher efficiency than those of better-response dynamics comes at the expense of fairness.

![Graphs showing number of improvement steps vs. players for random and scale-free networks.](a) Random networks. (b) Scale-free networks.)

Figure 10: Number of improvement steps to convergence, $K = 3$ and $m = 5$.

A major factor regarding the applicability of each of the evaluated methods is their run-time performance. Figure 10 shows that the average number of improvement steps is linear in the number of players for both methods ($K = 3$). These empirical results are in sync with Corollaries 4 and 9 which provide polynomial upper-bounds on the number of improvement steps until convergence of both methods. Additionally, note that the proposed method performs approximately half the number of the improvement steps of better-response dynamics until convergence. This comes as no surprise since the proposed method performs only a subset of the strategy changes that are allowed by better-response dynamics.

The average number of neighbors seems to play a critical role in the ability to improve social welfare. Figure 11 shows that the increase in social welfare reduces as the average number of neighbors grows in both types of underlying networks. Nevertheless, while better-response dynamics yields outcomes of lower efficiency than the initial (random) outcomes, the proposed method manages to improve the efficiency of the outcomes even for larger numbers of neighbors.

Interestingly, the fairness of the outcomes increases as the average number of neighbors grows (Figure 12). This may be because larger neighborhoods require less players.
to acquire the goods in a stable solution. Still, the solutions yielded by the proposed method are of higher Gini coefficient than those obtained by better-response dynamics, similarly to the previous cases.

One can clearly observe that the average number of improvement steps increases as the average number of neighbors grows (Figure 13). This observation correlates with Equation 13 which plays a crucial role in the theoretical bound on the number of improvement steps to convergence. As expected, the proposed method performs less improvement steps than better-response dynamics for all the evaluated values of $m$.

The ability to improve the efficiency of the initial outcome depends also on the value of $K$ (Figure 14). The proposed method outperforms better-response dynamics, in terms of efficiency, for all the evaluated values of $K$ in both types of the underlying networks.
networks. Additionally, the application of Algorithm 2 always produces a positive improvement in the social welfare. The proposed improvement procedure can provide outcomes with up to 20\% higher efficiency than those of better-response.

The fairness of the solutions provided by better-response dynamics is almost not affected by the value of K (Figure 15). In contrast, the Gini coefficient of the outcomes provided by the proposed method increases with the value of K. A possible reason for this can be the fact that large values of K increase the probability of a player i to receive a payoff compensation \((state_i(v^*) = T_{I > K})\) from its neighbors. Since players do not cooperate during the decision process regarding the payoff transfer, the fairness may deteriorate with each additional player that receives a compensation.
Corollaries 4, 6 and 9 state that the number of improvement steps until convergence is linear in the value of $K$. However, Figure 16 shows that the average number of improvement steps can be almost unaffected by the value of $K$ (better-response dynamics on random networks) or even drop down with increase of $K$ (proposed method for $K \leq 5$). This behavior may stem from the fact that only a small fraction of the players in the initial (random) outcomes can benefit from unilateral deviation.

The final set of experiments studies the impact of the traversal order over the players for both methods. In these experiments three different orders are used – a random order, an ascending order of costs and a descending order of costs. The influence of
the different orders on the performance of the methods for $K > 1$ was insignificant, therefore the reported results are only for $K = 1$.

![Figure 17](image1.png)

**Figure 17:** Social welfare of stable outcomes on large networks, $K = 1$ and $m = 4$.

Figure 17a presents the change in social welfare for random underlying networks. It is easy to see that the descending order of costs provides the most significant improvement in social welfare for both proposed method and better-response dynamics, while ascending order of costs results in the minimal change of social welfare. This results from the fact that the majority of improvement steps performed by players, were from strategy $T$ to strategy $F$. Figure 17b provides similar results for better-response dynamics. The changes in the improvement in social welfare, made by the proposed method ($K = 1$), are less significant and almost negligible.

![Figure 18](image2.png)

**Figure 18:** Fairness of stable outcomes, $K = 1$ and $m = 4$.

The fairness of the solutions, yielded by both methods, is maximal when descending order of costs is used. This result resembles the previous observations – outcomes that maximize social welfare are of higher fairness than those of lower efficiency. Addi-
tionally, outcomes of better-response dynamics are of lower Gini index than the stable solutions yielded by the proposed method.

8. Conclusions

The “best-shot” public goods game (PGG) is a graphical game, where agents wish to use some good and can either purchase it (for a finite cost) or use it for free if some agent in their neighborhood obtains the good (cf. [3, 4]). The utility for agents who use the good without purchasing it is maximal, is lower to agents who use the good and purchase it, and is the lowest for agents who do not purchase it and have no neighbor who does. For the general version of the PGG the utility of agents increases if they have several neighbors who purchase the good.

The general version of PGG is shown to be an ordinal potential game. This implies the convergence of better-response dynamics to a stable state (a pure Nash equilibrium) [10]. The better-response dynamics is shown to converge in a number of steps that is bounded by $2 \cdot K \cdot n^2$, where $n$ is the number of agents and $K$ is the maximal number of neighbors that can increase the utility of a neighboring agent by purchasing the good. Despite its ability to converge to a stable state, it is shown that better-response tends to converge to outcomes of low efficiency. Therefore, an algorithm that searches for stable outcomes of high efficiency is desired.

An iterative distributed search algorithm for finding a stable outcome of improved efficiency in “best-shot” public goods games has been proposed. The algorithm iterates among all players in the game in a fixed order and each player in its turn exchanges messages with the players that interact with it (e.g., its neighbors in the game). Messages propose the selected strategy of the player and in response other players, that are affected by it, may offer transfer of funds in order to secure a desired outcome. One can think of the proposed improvement procedure as an extension to the well known better-response mechanism (cf. [21, 22]) that uses transfers of funds.

The proposed algorithm is guaranteed to converge to a state that is of higher efficiency than its initial state, for the simple version of PGG. This is in contrast to standard best response that does not address efficiency at all. The final state to which the algo-
Algorithm converges can be stabilized by the use of transfers of funds among the players who participate in the game. A transfer function that can achieve stability is computed by the players running the algorithm during its run. Since the transfers that secure stability form a binding contract among all players, that is computed during the iterative search, it is natural to think of them as side-payments [13].

An extensive experimental evaluation shows an improvement in social welfare, and one that is significantly higher than those achieved by the better-response dynamics, for general version of the PGG. These results emphasize the importance of the proposed improvement procedure despite the fact that there is no theoretical guarantee regarding the efficiency of the resulting outcomes. Additionally, both better-response and the improvement procedure have a polynomial bound on the number of the improvement steps, that enables to apply them to large scale social networks.

It is interesting to consider the results on PGGs from the point of view of potential games. It was shown in Section [1] that one can guarantee efficient stable solutions to the general version of PGGs by using the mechanism of side-payments within the framework of best response. This raises the immediate question whether similar algorithms that use side-payments can converge to stable solutions of improved efficiencies also in games that are not potential games. It turns out that this question can be answered in the affirmative, though with a weaker guarantee on efficiency, also in non-potential games. This is the focus of a further study that is now being completed.

The algorithms proposed by the present study fall into the class of distributed local search (cf. [23, 24, 25, 26]) and in particular for asymmetric distributed constraints problems (ADCOPs) which are equivalent to multi-agents games (cf. [27, 28]). It is important to note the major difference between existing distributed local search algorithms and the algorithms proposed by the present paper. First and foremost, local search algorithms on either DCOPs or ADCOPs are not guaranteed to converge [23, 26]. Second, standard local search algorithms may converge to a stable state (e.g., a PNE) on asymmetric DCOPs but they have no guarantee on the efficiency of the resulting solution. One can describe the proposed algorithms as enlarging the search space by adding transfers of funds, thereby enabling the guarantee of convergence to stable solutions of higher efficiency. A last important difference between all distributed local
search algorithms and the INEA algorithm is that standard local search requires co-
operation among all agents in order to find a solution, while the INEA algorithm uses
steps of best-response which require no cooperation among the participating agents and
thereby fits well into a framework of selfish (or competitive) agents that participate in
a typical PGG game.
References


