1 Recap

1. We defined $\omega$-words and $\omega$-languages
2. We defined Büchi automata (NBW)
3. We looked at several examples for $\omega$-languages, their corresponding $\omega$-regular expression and their corr. NBW

2 Union and Intersection

1. We recalled the construction of union for NFWs and gave a construction for union of NBWs.
   State complexity: $n_1 + n_2$.
   We skipped the proof (as it is identical to the finite word case).
2. We recalled the construction of intersections for NFWs and gave a construction for intersections of NBWs.
   State complexity: $2n_1n_2$.
   See proof in appendix A.

2.1 A word about determinization

1. Did the constructions preserve determinization?
2. Can we provide a construction for union that preserves determinization? What would be the complexity then?

3 Complementation

1. We recalled the construction of complementation for DFWs, and saw that it does not work for DBWs.
2. Self exercise: Let $A$ be a DBW and $A'$ a DBW obtained by dualizing its accepting states.
   (a) Does $[A'] \subseteq [A]$?
   (b) Does $[A] \subseteq [A']$?
3. We gave a complementation construction for DBWs (inspired by the example of negating the DBW for $\infty a'$s).
   State complexity: $2n$.
4. Does it preserve determinization?

4 Expressiveness

1. Unlike the situation on finite words, in the realm of infinite words non-determinism adds expressive power!
   Theorem 4.1 (Landweber 1969).
   \[
   \text{NBW} \supseteq \text{DBW}
   \]
2. Proof by showing that there is no DBW for the language $\text{\neg}\infty a$ (i.e. the language of all $\omega$-words over \{a, b\} with a finite number of a's).
\section*{A Intersection Proof}

\subsection*{A.1 Construction}

Let $A_i = (\Sigma, Q_i, q_{0i}, \delta_i, F_i)$ and $A_u = (\Sigma, Q_u, q_{0u}, \delta_u, F_u)$ be two DBWs.

We define $A = (\Sigma, Q_l \times Q_u \times \{1, 11\}, \langle q_{00}, q_{01}, 1 \rangle, \delta, F_l \times Q_u \times \{1\})$ where

$$\delta((q_{0}, q_{0}, b), \sigma) = \{ (q'_{0}, q'_{1}, b') \mid q'_{0} \in \delta(q_{0}, \sigma), \: q'_{1} \in \delta(q_{0}, \sigma), \: b' = \bar{b} \text{ if } q_{0} \in F_{b} \text{ and } b' = b \text{ otherwise } \}.$$

\subsection*{A.2 Proof}

\begin{proposition}
\[ [A] = [A_i] \cap [A_u] \]
\end{proposition}

\begin{proof}

Assume $w \in [A]$. Let $r = \langle q_{00}, q_{01}, b_{0} \rangle \langle q_{11}, q_{12}, b_{1} \rangle \langle q_{21}, q_{22}, b_{2} \rangle \ldots$ be an accepting run of $A$ on $w$.

It follows $r_i = q_{00} q_{11} q_{22} \ldots$ and $r_u = q_{00} q_{11} q_{22} \ldots$ are runs of $A_i$ and $A_u$ on $w$, respectively.

We claim that these are accepting runs.

Since $r$ is accepting, by the definition of the acceptance condition of $A$ we get that there are infinitely many $k$'s such that $\langle q_{0k}, q_{0k}, b_{k} \rangle$ is in $F_l \times Q_u \times \{1\}$.

It follows that there are infinitely many $k$'s such that $q_{0k} \in F_i$.

This shows that the run $r_i$ is accepting.

To show that the run $r_u$ is also accepting we use the following lemma.

\begin{lemma}
Let $G_i = F_i \times Q_u \times \{1\}$ and $G_u = F_i \times Q_u \times \{u\}$.

Let $j, k$ be two indices such that both $\langle q_{ij}, q_{ij}, b_{j} \rangle$ and $\langle q_{ik}, q_{ik}, b_{k} \rangle$ are in $G_i$ and $j < k$.

Then there exists an index $\ell$ such that $j < \ell < k$ and $\langle q_{i\ell}, q_{i\ell}, b_{\ell} \rangle$ is in $G_u$.
\end{lemma}

\begin{proof}

Let $T_i = Q_l \times Q_u \times \{1\}$ and $T_u = Q_l \times Q_u \times \{u\}$.

We call $T_i$ the left copy and $T_u$ the right copy.

By the construction of $A$ whenever we are in the left copy and the first state component is in $F_l$ we move to the right copy. The only way to get back to the left copy from the right copy is that the second state component is in $F_u$. Thus an $\ell$ for which $\langle q_{i\ell}, q_{i\ell}, b_{\ell} \rangle$ is in $G_u$ must exists.

It follows from the lemma, since there are infinitely many $k$'s for which $\langle q_{0k}, q_{0k}, b_{k} \rangle \in G_i$, that there are also infinitely many $\ell$'s for which $\langle q_{i\ell}, q_{i\ell}, b_{\ell} \rangle \in G_u$.

It follows that there are infinitely many $\ell$'s such that $q_{i\ell} \in F_u$.

Thus the run $r_u$ is accepting as well.

Assume $w \in [A_i]$ and $w \in [A_u]$.

Then there exists accepting runs $r_i$ and $r_u$ of $A_i$ and $A_u$ on $w$, respectively.

Their extension to a run $r$ of $A$ on $w$ following the definition of the transition relation $\delta$ induces an accepting run.
\end{proof}

\end{proof}