Unless otherwise stated, all claims should be proven.

**Question 1.** (20 points)
Let $\Sigma = \{a, b\}$ and $\Sigma' = \Sigma \cup \{\#\}$. Let

$$L_n = \{ u_1 \# u_2 \# u_3 \# \ldots \in (\Sigma')^\omega \mid \exists i \in \mathbb{N} \text{ s.t. } u_i \neq u_{i+1} \}.$$ 

That is, $L_n$ is subset a language of $\omega$-words over $\Sigma'$ which are partitioned to words over $\Sigma$ of size $n$ separated by the $\#$ sign. A word $u_1 \# u_2 \# u_3 \# \cdots$ is in $L_n$ iff there are infinitely many indices $i$ such that the $i$-th sub-word $u_i$’s different than the following sub-word $u_{i+1}$. Describe an NBW $\mathcal{N}$ of size $O(n^2)$ accepting $L_n$, or if you prefer, two NBWs $\mathcal{N}_1$ and $\mathcal{N}_2$ of size $O(n)$ each such that the intersection of their accepted languages is $L_n$.

**Question 2.** (20 points)
Prove or refute (provide a counterexample):

1. Every NBW $\mathcal{A}$ has an equivalent NBW $\mathcal{A}'$ with a single initial state.
2. Every NBW $\mathcal{A}$ has an equivalent NBW $\mathcal{A}'$ with a single accepting state.

**Question 3.** (24 points)
Let $G = (\Sigma, Q, Q_0, \delta, \{F_1, F_2, \ldots, F_k\})$ be a generalized Büchi automaton (NGBW) with $n$ states. Construct an NBW with $O(nk)$ states recognizing the same language.

**Question 4.** (24 points)
Let $\Sigma = \{a, b\}$ and let $L_k = \{w \in \Sigma^\omega \mid \text{both } a \text{ and } b \text{ appear at least } k \text{ times in } w\}$. Let $L_{k,\omega} = L_k \cap \Sigma^* \cap \Sigma^\omega$. An NFW for the language $L_{k,\omega}$ needs at least $k^2$ states.

1. Show that $L_{k,\omega}$ can be recognized by an NBW with at most $2k + 5$ states.
2. Show that $L_{k,\omega}$ can be recognized by an NCW with at most $3k + 5$ states.

**Hint:** Recall that $\Sigma = \{a, b\}$. What does this entail about finitely/infinitely many occurrences of $a$ or $b$ letters in words from $L_{k,\omega}$?

**Question 5.** (24 points)
Let $B$ be an NBW. Let $B'$ be the NFW obtained from $B$ by treating it as an NFW. Prove or give a counterexample:

1. $\llbracket B \rrbracket = R_{\text{ref}}(\llbracket B' \rrbracket)$
2. $\llbracket B \rrbracket = R_{\text{ref}}(\llbracket B' \rrbracket)$ if $\llbracket B \rrbracket \in \mathbb{DBW}$