1 Recap

1. We defined $\omega$-words and $\omega$-languages
2. We defined Büchi automata (NBW)
3. We looked at several examples for $\omega$-languages, their corresponding $\omega$-regular expression and their corr. NBW

We proved that the construction for $L_3 = \{ w \in \{a,b\}^\omega \mid \text{the number of a's in w is either even or infinite} \}$ is correct.

2 Union and Intersection

1. We recalled the construction of union for NFWs and gave a construction for union of NBWs.
   State complexity: $n_1 + n_2$.
   We skipped the proof (as it is identical to the finite word case).
2. We recalled the construction of intersections for NFWs and gave a construction for intersections of NBWs.
   State complexity: $2n_1 n_2$.
   See proof in appendix A.

2.1 A word about determinization

1. Did the constructions preserve determinization?
2. Can we provide a construction for union that preserves determinization? What would be the complexity then?

3 Complementation

1. We recalled the construction of complementation for DFWs, and saw that it does not work for DBWs.
2. Self exercise: Let $A$ be a DBW and $A'$ a DBW obtained by dualizing its accepting states.
   (a) Does $[A'] \subseteq [A]$?
   (b) Does $[A] \subseteq [A']$?
3. We gave a complementation construction for DBWs (inspired by the example of negating the DBW for $\infty a'$s).
   State complexity: $2n$.
4. Does it preserve determinization?

4 Expressiveness

1. Unlike the situation on finite words, in the realm of infinite words non-determinism adds expressive power!
   \textbf{Theorem 4.1} (Landweber 1969).
   \[
   \text{NBW} \supseteq \text{DBW}
   \]
2. Proof by showing that there is no DBW for the language $\neg \infty a$
A Intersection Proof

A.1 Construction

Let $A_i = (\Sigma, Q_i, q_{0i}, \delta_i, F_i)$ and $A_n = (\Sigma, Q_n, q_{0n}, \delta_n, F_n)$ be two DBWs. For $b \in \{1, 11\}$ define $b = 1$ iff $b = 1$. We define $A = (\Sigma, Q_l \times Q_n \times \{1, 11\}, \langle q_{0l}, q_{0n}, 1 \rangle, \delta, F_l \times Q_n \times \{1\})$ where

$$\delta((q_l, q_n, b), \sigma) = \{ \langle q'_l, q'_n, b' \rangle \mid q'_l \in \delta(q_l, \sigma), q'_n \in \delta(q_n, \sigma), b' = \overline{b} \text{ if } q_n \in F_b \text{ and } b' = b \text{ otherwise} \}.$$

A.2 Proof

Proposition A.1. \([A] = [A_i] \cap [A_n]\)

Proof.

⇒ Assume $w \in [A]$. Let $r = \langle q_{0l}, q_{0n}, 1 \rangle \langle q_{1l}, q_{1n}, 1 \rangle \langle q_{2l}, q_{2n}, 1 \rangle \ldots$ be an accepting run of $A$ on $w$. It follows $r_l = q_{0l} q_{1l} q_{2l} \ldots$ and $r_n = q_{0n} q_{1n} q_{2n} \ldots$ are runs of $A_i$ and $A_n$ on $w$, respectively. We claim that these are accepting runs. Since $r$ is accepting, by the definition of the acceptance condition of $A$ we get that there are infinitely many $k$'s such that $\langle q_{ik}, q_{nk}, b_k \rangle$ is in $F_l \times Q_n \times \{1\}$. It follows that there are infinitely many $k$'s such that $q_{ik} \in F_l$. This shows that the run $r_l$ is accepting.

To show that the run $r_n$ is also accepting we use the following lemma.

Lemma A.2. Let $G_l = F_l \times Q_n \times \{1\}$ and $G_n = Q_l \times F_n \times \{a\}$. Let $j, k$ be two indices such that both $\langle q_{ij}, q_{nj}, b_j \rangle$ and $\langle q_{ik}, q_{nk}, b_k \rangle$ are in $G_l$ and $j < k$. Then there exists an index $\ell$ such that $j < \ell < k$ and $\langle q_{\ell j}, q_{\ell n}, b_{\ell} \rangle$ is in $G_n$.

Proof. Let $T_l = Q_l \times Q_n \times \{1\}$ and $T_n = Q_l \times Q_n \times \{a\}$. We call $T_l$ the left copy and $T_n$ the right copy.

By the construction of $A$ whenever we are in the left copy and the first state component is in $F_l$ we move to the right copy. The only way to get back to the left copy from the right copy is that the second state component is in $F_n$. Thus an $\ell$ for which $\langle q_{\ell j}, q_{\ell n}, b_{\ell} \rangle$ is in $G_n$ must exists.

It follows from the lemma, since there are infinitely many $k$'s for which $\langle q_{ik}, q_{nk}, b_k \rangle \in G_n$, that there are also infinitely many $\ell$'s for which $\langle q_{i\ell}, q_{n\ell}, b_{\ell} \rangle \in G_n$. It follows that there are infinitely many $\ell$'s such that $q_{n\ell} \in F_n$. Thus the run $r_n$ is accepting as well.

⇐ Assume $w \in [A_i]$ and $w \in [A_n]$. Then there exists accepting runs $r_l$ and $r_n$ of $A_i$ and $A_n$ on $w$, respectively. Their extension to a run $r$ of $A$ on $w$ following the definition of the transition relation $\delta$ induces an accepting run. \(\square\)