Improved Bounds on the Average Distance to the Fermat-Weber Center of a Convex Object

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Abstract

We show that for any convex object \( Q \) in the plane, the average distance between the Fermat-Weber center of \( Q \) and the points in \( Q \) is at least \( 4\Delta(Q)/25 \), and at most \( 2\Delta(Q)/(3\sqrt{3}) \), where \( \Delta(Q) \) is the diameter of \( Q \). We use the former bound to improve the approximation ratio of a load-balancing algorithm of Aronov et al. [1].

1 Introduction

The Fermat-Weber center of an object \( Q \) in the plane is a point in the plane, such that the average distance from it to the points in \( Q \) is minimal. For an object \( Q \) and a point \( y \), let \( \mu_Q(y) \) be the average distance between \( y \) and the points in \( Q \), that is, \( \mu_Q(y) = \int_{x\in Q} \|xy\| \, dx/\text{area}(Q) \), where \( \|xy\| \) is the Euclidean distance between \( x \) and \( y \). Let \( \mathcal{FW}_Q \) be a point for which this average distance is minimal, that is, \( \mu_Q(\mathcal{FW}_Q) = \min_y \mu_Q(y) \), and put \( \mu_Q^* = \mu_Q(\mathcal{FW}_Q) \). The point \( \mathcal{FW}_Q \) is a Fermat-Weber center of \( Q \).

It is easy to verify, for example, that the Fermat-Weber center of a disk \( D \) coincides with the center \( o \) of \( D \), and that the average distance between \( o \) and the points in \( D \) is \( \Delta(D)/3 \), where \( \Delta(D) \) is the diameter of \( D \). Carmi, Har-Peled, and Katz [3] studied the relation between \( \mu_Q^* \) and the diameter of \( Q \), denoted \( \Delta(Q) \). They proved that there exists a constant \( c_1 \), such that, for any convex object \( Q \), the average distance between a Fermat-Weber center of \( Q \) and the points in \( Q \) is at least \( c_1 \Delta(Q) \), and that the largest such constant \( c_1^* \) lies in the range \([1/7, 1/6]\).

In this paper, we both improve the above bound on \( c_1^* \), and tightly bound a new constant \( c_2^* \); see below. More precisely, we first significantly narrow the range in which \( c_1^* \) must lie, by proving (in Section 2) that \( 4/25 \leq c_1^* \leq 1/6 \). Next, we consider the question what is the smallest constant \( c_2^* \), such that, for any convex object \( Q \), \( \mu_Q \leq c_2^* \Delta(Q) \). We prove (in Section 3) that \( 1/3 \leq c_2^* \leq 2/(3\sqrt{3}) \). A useful corollary obtained from these results is that the average distance to the center of the smallest enclosing circle of a convex \( n \)-gon \( P \) is less than \( 2.41 \times \mu_P^* \).

The Fermat-Weber center of an object \( Q \) is a very significant point. The classical Fermat-Weber problem is: Find a point in a set \( F \) of feasible facility locations, that minimizes the average distance to the points in a set \( D \) of (possibly weighted) demand locations. If \( D \) is a finite set of points, \( F \) is the entire plane, and distances are measured using the \( L_2 \) metric, then it is known that the solution is algebraic [2]. See Wesolowsky [8] for a survey of the Fermat-Weber problem.

Only a few papers deal with the continuous version of the Fermat-Weber problem, where the set of demand locations is continuous. Fekete, Mitchell and Weinbrecht [4] presented algorithms for computing an optimal solution for \( D = F = P \) where \( P \) is a simple polygon or a polygon with holes, and the distance between two points in \( P \) is the \( L_1 \) geodesic distance between them. Carmi, Har-Peled and Katz [3] presented a linear-time approximation scheme for the case where \( P \) is a convex polygon.

Aronov et al. [1] considered the following load-balancing problem. Let \( D \) be a convex region and let \( \mathcal{P} = \{p_1, \ldots, p_m\} \) be a set of \( m \) points representing \( m \) facilities. One would like to divide \( D \) into \( m \) equal-area subregions \( R_1, \ldots, R_m \), so that region \( R_i \) is associated with point \( p_i \), and the total cost of the subdivision is minimized. Given a subdivision, the cost \( \kappa(p_i) \) associated with facility \( p_i \) is the average distance between \( p_i \) and the points in \( R_i \), and the total cost of the subdivision is \( \sum_i \kappa(p_i) \).

Aronov et al. discussed the structure of an optimal subdivision, and also presented an \((8 + \sqrt{2\pi})\)-approximation algorithm, under the assumption that the regions \( R_1, \ldots, R_m \) must be convex and that \( D \) is a rectangle. Our improved bound on the constant \( c_1^* \), allows us (in Section 4) to improve the above approximation ratio.

2 \[ 4/25 \leq c_1^* \leq 1/6 \]

Carmi, Har-Peled and Katz [3] showed that there exists a convex polygon \( P \) such that \( \mu_P^* \leq \Delta(P)/6 \). This immediately implies that \( c_1^* \leq 1/6 \). We prove below that \( c_1^* \geq 4/25 \). Our proof is similar in its structure to the proof of [3].

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Theorem 2.1. Let $P$ be a convex object. Then $\mu_P \geq 4\Delta(P)/25$.

Proof: Let $FW_P$ be a Fermat-Weber center of $P$. We need to show that $\int_{x \in P} \|xFW_P\|\, dx \geq \frac{4\Delta(P)}{27} \text{area}(P)$. We do this in two stages. In the first stage we show that for a certain subset $P'$ of $P$, $\int_{x \in P'} \|xFW_P\|\, dx \geq \frac{4\Delta(P)}{27} \text{area}(P)$. This implies that for any convex object $Q$, $\mu_Q \geq 4\Delta(Q)/27$. In the second stage we apply this intermediate result to a collection of convex subsets of $P - P'$ that are pairwise disjoint to obtain the claimed result. This latter stage is essentially identical to the second stage in the proof of [3]; it is included here for the reader's convenience.

We now describe the first stage. Let $s$ be a line segment of length $\Delta(P)$ connecting two points $p$ and $q$ on the boundary of $P$. We may assume that $s$ is horizontal and that $p$ is its left endpoint, since one can always rotate $P$ around, say, $p$ until this is the case.

Figure 1: Proof of intermediate result.

Let $P^\alpha$ be the polygon obtained from $P$ by shrinking it by a factor of $\alpha$, that is, by applying the transformation $f(a,b) = (a/\alpha, b/\alpha)$ to the points $(a,b)$ in $P$. We place a copy $R_1$ of $P^{3/2}$, such that $R_1$ is contained in $P$ and has a common tangent with $P$ at $q$. Similarly, we place a copy $R'_1$ of $P^{3/2}$, such that $R'_1$ is contained in $\partial P$ and has a common tangent with $P$ at $p$; see Figure 1(a). Clearly, $\text{area}(R_1) = \text{area}(R'_1) = \frac{1}{3} \text{area}(P)$.

Let $R_2 = R_1 \cap R'_1$. We place a copy $R_3$ of $R_2$, such that $R_3$ is contained in $R_1$ and has a common tangent with $R_1$ at $q$. Similarly, we place a copy $R'_3$ of $R_2$, such that $R'_3$ is contained in $R'_1$ and has a common tangent with $R'_1$ at $p$. Let $R_4 = R_1 - (R_2 \cup R_3)$ and $R'_4 = R'_1 - (R_2 \cup R'_3)$; see Figure 1(b).

We know that, regardless of the exact location of $FW_P$, the distance between $FW_P$ and the points in $R_3$ plus the distance between $FW_P$ and the points in $R'_3$ is greater than $\frac{2\Delta(P)}{3} \text{area}(R_3)$, and the distance between $FW_P$ and the points in $R_4$ plus the distance between $FW_P$ and the points in $R'_4$ is greater than $\frac{2\Delta(P)}{3} \text{area}(R_4)$. More precisely,

$$\int_{x \in R_3} \|xFW_P\|\, dx + \int_{x \in R'_3} \|xFW_P\|\, dx \geq \frac{2\Delta(P)}{3} \text{area}(R_3)$$

and

$$\int_{x \in R_4} \|xFW_P\|\, dx + \int_{x \in R'_4} \|xFW_P\|\, dx \geq \frac{2\Delta(P)}{3} \text{area}(R_4).$$

Since $\text{area}(R_1) = \text{area}(R_3) - (\text{area}(R_2) \cup \text{area}(R_3)) = \frac{1}{3} \text{area}(P)/2 \text{area}(R_2) - \frac{2}{3} \text{area}(R_2)$, we obtain our intermediate result

$$\int_{x \in P} \|xFW_P\|\, dx \geq \int_{x \in R_3} \|xFW_P\|\, dx + \int_{x \in R_4} \|xFW_P\|\, dx + \int_{x \in R_5} \|xFW_P\|\, dx + \int_{x \in R'_3} \|xFW_P\|\, dx + \int_{x \in R'_4} \|xFW_P\|\, dx + \int_{x \in R'_5} \|xFW_P\|\, dx \geq \frac{2\Delta(P)}{3} \text{area}(R_3) + \frac{2\Delta(P)}{3} \text{area}(R'_3) + \frac{2\Delta(P)}{3} \text{area}(R_4) + \frac{2\Delta(P)}{3} \text{area}(R'_4).$$

This intermediate result immediately implies that for any convex object $Q$, $\mu_Q \geq 4\Delta(Q)/27$. In the second stage we show that the 27 in the denominator can be replaced by 25.

Figure 2: Proof of improved result.

Consider Figure 2. We draw the axis-aligned bounding box of $P$. The line segment $s$ (whose length is $\Delta(P)$) divides the bounding box of $P$ into two rectangles, $abqp$ above $s$ and $pqdc$ below $s$. We divide each of these rectangles into two parts (a lower part and an upper part), by drawing the two horizontal lines $l$ and $l'$. Let $R_5$ denote the intersection of $P$ with the upper part of the upper rectangle, and let $R'_5$ denote the intersection of $P$ with the lower part of the lower rectangle.

Let $e$ be any point on the segment $ab$ that also lies on the boundary of $R_5$. We mention several facts concerning $R_5$ and $R'_5$. $R_5 \cap R'_5 = \phi$, $R_5 \cap R_3 = \phi$, $R_5 \cap R'_3 = \phi$, $R'_5 \cap R_1 = \phi$, and $R'_5 \cap R'_3 = \phi$. Notice also that $\Delta(R_5)$, $\Delta(R'_5) \geq \Delta(P)/3$, since, e.g., the line segment $l \cap R_5$ contains the base of the triangle that is obtained by intersecting the triangle $pqeq$ with $R_5$, and the length of this base is $\Delta(P)/3$.

We observe that $\text{area}(R_5) + \text{area}(R'_5) \geq \text{area}(P)/3$ by showing that $\text{area}(R_5) \geq \text{area}(P \cap abqp)/9$ and that $\text{area}(R'_5) \geq \text{area}(P \cap pqdc)/9$. Let $g, h$ be the two points on the line $l$ that also lie on the boundary of $R_5$. Let $l(s)$ be the line containing $s$, and let $T$ be the triangle defined by $l(s)$ and the two line segments connecting $e$ to $l(s)$ and passing through $g$ and through $h$, respectively. Let $T_2$ denote the triangle $gch$. 


Clearly \( T_2 \subseteq R_5 \). Put \( Q = R_5 - T_2 \). Then, \( \text{area}(R_5) = \text{area}(T_2) + \text{area}(Q) = \text{area}(T)/9 + \text{area}(Q) \). Therefore, \( \text{area}(R_5) \geq (\text{area}(T) + \text{area}(Q))/9 \geq \text{area}(P \cap abq)/9 \). We show that \( \text{area}(R_5) \geq \text{area}(P \cap pqdc)/9 \) using the “symmetric” construction. Since \( (P \cap abq) \cup (P \cap pqdc) = P \) we obtain that \( \text{area}(R_5) + \text{area}(R_5') \geq \text{area}(P)/9 \).

It is also easy to see that \( \Delta(R_2) = \Delta(P)/3 \) and \( \text{area}(R_2) \geq \text{area}(P)/9 \). This is because \( P^3 \subseteq R_2 \) and \( \text{area}(P^3) = \text{area}(P)/9 \), where \( P^3 \) is the polygon obtained from \( P \) by shrinking it by a factor of 3. Now using the implication of our intermediate result we have

\[
\int_{x \in R_5} \|xFW_p\| \, dx + \int_{x \in R_5} \|xFW_p\| \, dx + \\
+ \int_{x \in R_5} \|xFW_p\| \, dx + \int_{x \in R_5} \|xFW_p\| \, dx + \\
+ \int_{x \in R_5} \|xFW_p\| \, dx + \int_{x \in R_5} \|xFW_p\| \, dx + \\
+ \int_{x \in R_5} \|xFW_p\| \, dx + \int_{x \in R_5} \|xFW_p\| \, dx + \\
\geq \frac{4\Delta(P)}{27} \text{area}(P) + \frac{8\Delta(P)}{729} \text{area}(P) = \\
\frac{116\Delta(P)}{729} \text{area}(P).
\]

Therefore

\[
\int_{x \in P} \|xFW_p\| \, dx \geq \int_{x \in R_5} \|xFW_p\| \, dx + \\
+ \int_{x \in R_5} \|xFW_p\| \, dx + \int_{x \in R_5} \|xFW_p\| \, dx + \\
+ \int_{x \in R_5} \|xFW_p\| \, dx + \int_{x \in R_5} \|xFW_p\| \, dx + \\
+ \int_{x \in R_5} \|xFW_p\| \, dx + \int_{x \in R_5} \|xFW_p\| \, dx + \\
\geq \frac{4\Delta(P)}{27} \text{area}(P) + \frac{8\Delta(P)}{729} \text{area}(P) = \\
\frac{116\Delta(P)}{729} \text{area}(P).
\]

At this point we may conclude that for any convex object \( Q \), \( \mu_Q^* \geq 116\Delta(Q)/729 \). So we repeat the calculation above using this result for the regions \( R_5, R_5' \) and \( R_5 \) (instead of using the slightly weaker result, i.e., \( \mu_Q^* \geq 4\Delta(Q)/27 \)). This calculation will yield a slightly stronger result, etc. In general, the result after the \( k \)-th iteration is \( \mu_Q^* \geq c_k \Delta(Q) \), where \( c_k = 4/27 + 2c_{k-1}/27 \) and \( c_0 = 4/27 \). It is easy to verify that this sequence of results converges to \( \mu_Q^* \geq 4\Delta(Q)/25 \).

**Corollary 2.2.** Let \( P \) be a non-convex \( \beta \)-fat polygon, i.e., the ratio between the area of a minimum-area enclosing ellipse and the area of a maximum-area enclosed ellipse is at most \( \beta \), for some constant \( \beta \). Then \( \mu_P^* \geq 4\Delta(P)/(25\beta^2) \).

**Proof:** As in [3], except that we apply the improved bound of Theorem 2.1.

**3** \( 1/3 \leq \mu^* \leq 2/(3\sqrt{3}) \)

As mentioned in the introduction, the average distance between the Fermat-Weber center of a disk \( D \) (i.e., \( D \)'s center) and the points in \( D \) is \( \Delta(D)/3 \), where \( \Delta(D) \) is the diameter of \( D \). This immediately implies that \( \mu^* \geq 1/3 \).

We first state a simple lemma and a theorem of Jung that are needed for our proof.

**Lemma 3.1.** Let \( R, Q \) be two (not-necessarily convex) disjoint objects, and let \( p \) be a point in the plane. Then, \( \mu_{(R \cup Q)}(p) \leq \max \{ \mu_R(p), \mu_Q(p) \} \).

**Proof:**

\[
\mu_{(R \cup Q)}(p) = \frac{\int_{x \in R \cup Q} \|px\| \, dx}{\text{area}(R \cup Q)} = \\
= \frac{\int_{x \in R} \|px\| \, dx + \int_{x \in Q} \|px\| \, dx}{\text{area}(R) + \text{area}(Q)} = \\
\leq \frac{\text{area}(R) \cdot \mu_R(p) + \text{area}(Q) \cdot \mu_Q(p)}{\text{area}(R) + \text{area}(Q)} \\
\leq \frac{\text{area}(R) + \text{area}(Q)) \cdot \text{max} \{ \mu_R(p), \mu_Q(p) \}}{\text{area}(R) + \text{area}(Q)} \leq \text{max} \{ \mu_R(p), \mu_Q(p) \}.
\]

**Theorem 3.2 (Jung’s Theorem [5, 6]).** Every set of diameter \( d \) in \( \mathbb{R}^n \) is contained in a closed ball of radius \( r \leq d \sqrt{\frac{n}{2n(n+1)}} \). In particular, if \( R \) is a convex object in the plane, then the radius of the smallest enclosing circle \( C \) of \( R \) is at most \( \Delta(R)/\sqrt{3} \), where \( \Delta(R) \) is the diameter of \( R \).

**Theorem 3.3.** For any convex object \( R \), \( \mu_R^* \leq 2\Delta(R)/(3\sqrt{3}) \).

**Proof:** Let \( R \) be a convex polygon. Let \( C \) be the smallest enclosing circle of \( R \), and let \( \partial R \) be its center and let \( r \) be its radius. Notice that \( o \in R \), since \( R \) is convex. We divide \( R \) into 8 regions \( R_1, \ldots, R_8 \) by drawing four line segments through \( o \), such that each of the 8 angles formed around \( o \) is of 45°; see Figure 3(a). Clearly, for each \( R_i, o \in R_i \) and \( \Delta(R_i) \leq r \).

We first prove that for each region \( R_i \), \( \mu_{R_i}(o) \leq 2\Delta(R_i)/3 \). (This is done by adapting the proof of Lemma 3.1 of Aronov et al. [1].) Consider Figure 3(b). Let \( p \) be the point on the arc \( cd \), such that the regions \( Q_1 \) and \( Q_2 \) obtained by drawing the segment \( opf \) are of equal area. (\( Q_1 \) is the region \( opd \) and \( Q_2 \) is the difference between the segment \( opf \) and the
region opx, where x is the intersection point between o and the boundary piece pb.) Similarly, let e be the point on the arc cd, such that the regions Q₃ and Q₄ obtained by drawing the segment oe are of equal area. (Q₃ is the region oay and Q₄ is the difference between the sector oep and the region oyp, where y is the intersection point between oe and the boundary piece ap.)

Now, on the one hand, since opb is convex, x is the farthest point from o in Q₁, and, on the other hand, x is the closest point to o in Q₂. Hence, any point in Q₂ is farther from o than any point in Q₁. Thus we get that \( \mu_{opb}(o) = \mu_{Q'(Q₂)}(o) \leq \mu_{Q'(Q₁)}(o) = \mu_{op}(o) = 2\|op\|/3 = 2\Delta R_i/3 \). We show that \( \mu_{op}(o) \leq 2\Delta R_i/3 \) using the "symmetric" analysis. Since \( opb \) and oap are disjoint convex objects, then, by Lemma 3.1, \( \mu_{R}(o) = \mu_{opb, oap}(o) \leq 2\Delta R_i/3 \).

We now show that \( \mu_{R}(o) \leq 2\Delta(R)/(3\sqrt{3}) \), immediately implying that \( \mu_{R} \leq 2\Delta(R)/(3\sqrt{3}) \). By Theorem 3.2, we know that \( r \leq \Delta(R)/\sqrt{3} \). We also know that for each \( R_i \), \( \Delta(R_i) \leq r \). Thus, \( \mu_{R_i}(o) \leq 2\Delta(R_i)/3 \leq 2r/3 \leq 2\Delta(R)/(3\sqrt{3}) \).

We now apply Lemma 3.1 to obtain that

\[
\mu_{R}(o) \leq \max \left\{ \mu_{(R₁∪R₂∪R₃∪R₄)}(o), \mu_{(R₁∪R₂∪R₃∪R₄)}(o) \right\} \leq \max \left\{ \mu_{(R₁∪R₂∪R₃∪R₄)}(o), \mu_{(R₁∪R₂)}(o) \right\} \leq \max \left\{ \mu_{(R₁∪R₂)}(o), \mu_{(R₁∪R₂)}(o) \right\} \leq \mu_{(R₁∪R₂)}(o), \mu_{(R₁∪R₂)}(o) \leq \mu_{(R₁∪R₂)}(o), \mu_{(R₁∪R₂)}(o) \leq \mu_{(R₁∪R₂)}(o), \mu_{(R₁∪R₂)}(o) \leq \mu_{(R₁∪R₂)}(o), \mu_{(R₁∪R₂)}(o) \leq 2\Delta(R)/(3\sqrt{3}).
\]

**Corollary 3.4.** Let \( P \) be a convex \( n \)-gon. Then one can compute in linear time a point \( p \), such that \( \mu_{P}(p) \leq \frac{25}{6\sqrt{3}}\mu_{R} \).

**Proof:** We apply Megiddo’s linear-time algorithm for computing the smallest enclosing circle \( C \) of \( P \) [7]. Let \( p \) denote the center of \( C \), then, by Theorem 2.1

\[
\mu_{P}(p) \leq \frac{2\Delta(P)/(3\sqrt{3})}{4\Delta(P)/25} = \frac{25}{6\sqrt{3}} \mu_{R}.
\]

Corollary 3.4 gives us a very simple linear-time constant-factor approximation algorithm for finding an approximate Fermat-Weber center in a convex polygon. A less practical linear approximation scheme for finding such a point was presented by Carmi et al. [3].

4 Application

We consider the load balancing problem studied by Aronov et al. [1]. Let \( D \) be a convex region and let \( \mathcal{P} = \{p₁, \ldots, pₘ\} \) be a set of \( m \) points representing \( m \) facilities. The goal is to divide \( D \) into \( m \) equal-area convex regions \( R₁, \ldots, Rₘ \), so that region \( R_i \) is associated with point \( p_i \), and the total cost of the subdivision is minimized. The cost \( \kappa(p_i) \) associated with facility \( p_i \) is the average distance between \( p_i \) and the points in \( R_i \), and the total cost of the subdivision is \( \sum \kappa(p_i) \).

Assuming \( D \) is a rectangle that can be divided into \( m \) squares of equal size, Aronov et al. present an \( O(m^3) \)-time algorithm for computing a subdivision of cost at most \((8 + \sqrt{2π})\) times the cost of an optimal subdivision. By applying Theorem 2.1 in the analysis of their algorithm, we obtain a better approximation ratio, namely, \( (\frac{25}{6\sqrt{3}} + \sqrt{2π}) \). For further details, see the full version of this paper.

**References**


