(a) It suffices to show that for every non-empty open intervals \( I_0 \) and \( J_0 \) of \( A \), \( I_0 \cap J_0 \) be such intervals. For every interval \( J \in J_0 \) let \( \langle I^*_i, g_i^* \mid i \in \omega \rangle \) be such that \( I^*_i \) is an interval of \( A \), \( g_i^* \) is an automorphism of \( A \), \( g_i^*(I^*_i) \subseteq I_i \), and \( \bigcup_{i \in \omega} I^*_i = A \). Let \( \mathcal{J} \) be a countable base for the order topology of \( J \) consisting of open intervals and let \( \mathcal{G}_0 = \{ g_i^* \mid J \in \mathcal{J}, i \in \omega \} \). Then \( I_0 \), \( J_0 \), and \( \mathcal{G}_0 \) satisfy the conditions of part (b). By a symmetric argument there is a family \( \mathcal{H}_1 \) such that \( J_0', I_0 \), and \( \mathcal{G}_1 \) satisfy the conditions of part (b). Hence \( I_0 \), \( J_0 \), and \( \mathcal{G}_0 \cup \{ g^{-1} \mid g \in \mathcal{G}_1 \} \) satisfy the conditions of part (c), hence \( I_0 \approx J_0 \).

(j) For every \( A \in K \) let \( A^* \) be some canonical mixing of \( A \). Let \( A, B \in K \), and we show that \( A \triangleleft B \). Let \( B_1 \) be a nonempty open interval of \( B \), \( A \triangleleft A^* \triangleleft B_1 \), hence let \( g_{x} : A \to B_1 \) be OP. Let \( B_1 = \bigcup_{i \in \omega} B_i \) where \( B_i \approx B_1 \), and let \( g_i : B_1 \to B_i \) be an isomorphism. Let \( h_{B_1, i} = g_i^{-1} \circ g_{x} \), hence \( h_{B_1, i} \subseteq A \times B_1 \). Let \( \{ B_i \mid i \in \omega \} \) be a dense family of non-empty open intervals of \( B \). Clearly \( A, B, \{ h_{B_1, i} \mid i \in \omega \} \) satisfy the conditions of part (b), hence by (b), \( A \triangleleft B \).

It follows from part (i) that \( BA \) holds.

Theorem 6.2. Let \( V \models \text{MA} + \text{OCA} + \text{ISA} \), and let \( A \in K \) be an increasing set dense in \( R \). Then:

(a) \( A, A^* \subseteq A \cup A^* \in K^F \), and every member of \( K^F \) is isomorphic to one of these sets. (Of course \( A \leq A^* \).)

(b) If \( K \models B \leq A \), then \( B \models A \).

(c) If \( A \cup A^* \subseteq B \in K \), then \( B \models A \cup A^* \). (Hence \( A \cup A^* \) is universal in \( K \).

(d) If \( B \in K \) is dense in \( R \), then there is a nond order set \( C \subseteq R \), a countable ordered set \( (L, <) \), and for every \( i \in L \) a member \( A_i \in K^H \) such that \( B \sim C = \sum_{i \in L} A_i \), where \( \sum_{i \in L} \) denotes the ordered sum of linearly ordered sets.

(e) If \( B \in K \), then either \( B \models A + 1 + A^* \) or \( B \models A^* + 1 + A \) or \( B \) can be represented in the form \( B_1 \cup B_2 \) where \( B_1 \cap B_2 = \emptyset \), \( B_1 \models A \) or \( B_1 = \emptyset \), and \( B_2 \models A^* \) or \( B_2 = \emptyset \).

Proof. The proofs of all parts are easy, as an example we prove (b). Let \( K \models B \leq A \). Let \( I, J \) be non-empty open intervals of \( B \) and \( A \) respectively, and let \( f_I^J : I \to J \) be a 1-1 onto function. By OCA, \( f_I^J = \bigcup \{ f_i^J_i \mid i \in \omega \} \), where for every \( i \), \( f_i^J_i \) is monotonic. Since \( B \leq A \) and \( A \) is an increasing set, each \( f_i^J \) is OP. Let \( \mathcal{J} \) be a countable dense set of non-empty open intervals of \( B \), i.e. for every non-empty open interval \( I \) of \( B \) there is \( I \in \mathcal{J} \) such that \( I \leq I \). Similarly let \( \mathcal{G} \) be a countable dense set of non-empty open intervals of \( A \), and let \( \mathcal{G} = \{ f_i^J_i \mid I \in \mathcal{J}, J \in \mathcal{J} \} \) and \( i \in \omega \). Clearly \( B, A \) and \( G \) satisfy the conditions of 6.1(c) and hence \( B \models A \).

Question 6.3. Construct a model of ZFC in which for every \( A, B \in K \), \( A \triangleleft B \), but in which \( BA \) does not hold.

Let \( NA \) denote the following axiom: \( (\forall A, B \in K)(\exists C \in K)(C \triangleleft A \triangleleft C \triangleleft B) \). Clearly \( MA \oplus OCA \Rightarrow (NA \Leftrightarrow \neg ISA) \). In Section 7 we shall see that \( MA \oplus OCA \)}
NA is consistent. We conclude this section by showing that $\text{MA+OCA+NA} \Rightarrow \text{BA}$.

Proposition 6.4. $\text{MA+OCA+NA} \Rightarrow \text{BA}$.

Proof. Let $\lambda, \beta \in K$ and we show that $\lambda \ll \beta$. Let $B_\lambda$ be a non-empty open interval of $\beta$, and by $\text{NA}$ we have that $\lambda \ll \beta$. Let $C_\beta \subseteq B_\beta$ be such that $C_\beta \subseteq K$ and $C_\beta \subseteq C_\beta$. Let $f : \lambda \rightarrow C_\beta$ be a 1-1 function. Then $f$ can be represented as a countable union of monotonic functions $f = \bigcup_{i<\omega} f_i$. Let $g : \lambda \rightarrow C_\beta$ be an OR onto function; for every $i \in \omega$ let $h_{B_\beta,i} = f_i$ if $f_i$ is OP, and $h_{B_\beta,i} = g \circ f_i$ if $f_i$ is OR. Let $\{B_\beta : i \in \omega\}$ be a dense family of intervals of $B$. Then $\lambda \subseteq \beta$. Hence $\lambda \ll \beta$. □

7. Relationship with the weak continuum hypothesis

Our purpose in this section is to present a forcing set which makes two $\aleph_1$-dense sets of real numbers near, that is, given $\lambda, \beta \in K$ we want to add $C \subseteq K$ such that $\lambda \ll \beta$ and $\beta \ll \lambda$. Let $N(A, B) = (\exists C \subseteq K) (\lambda \ll C \ll \beta)$, and let the nearness axiom be as follows.

Axiom NA. $(\forall A, B \in K) N(A, B)$.

Obviously $\text{BA} \Rightarrow \text{NA}$. One can ask whether these two axioms are equivalent. We will show in this section that $\text{BA} \Rightarrow 2^{\aleph_0} = 2^\lambda$, whereas $\text{CON(NA+\lambda < 2^\lambda)}$ holds, so $\text{NA} \Rightarrow \text{BA}$.

Let WCH denote the axiom that $2^{\lambda} < 2^\lambda$. $A \in K$ is prime if for every $\beta \in K : \lambda \ll \beta$; it is universal if for every $\beta \in K : \beta \ll A$. $\{A_i : i \in I\} \subseteq K$ is a prime family, if for every $\beta \in K$ there is $i \in I$ such that $A_i \ll \beta$.

Theorem 7.1 (WCH). (a) $K$ does not contain a prime element. Moreover there is no prime family of power $< 2^{\aleph_0}$.

(b) For every $A \in K$ there is $B \subseteq A$ such that $B \subseteq K$ and $A \not\ll B$.

Proof. Let $T = T^{\aleph_0}$ be the tree of binary sequences of length $< \aleph_1$. Clearly $\|T\| = \aleph_0$. Let $\{a_\alpha : \alpha < \aleph_1\}$ be a 1-1 function from $T$ to $\mathbb{R}$. For every $\eta \in \aleph_0$ let $f_\eta : \alpha < \aleph_1 \rightarrow T$ be a 1-1 function from $\alpha \ll \lambda$ and $\{A_i : i < \lambda\}$ is a prime family. W.l.o.g. for every $i < \lambda$, $A_i$ contains the set of rational numbers. For every $\eta \in \aleph_0$ let $f_\eta : A_\eta \rightarrow A_\eta$ be such that $f_\eta : A_\eta \rightarrow A_\eta$ is an OP function. Clearly for some $\eta \not\ll \nu$, $f_\eta \upharpoonright \aleph_0 \neq f_\nu \upharpoonright \aleph_0$, and $i_\eta = i_\nu$. But then $f_\eta$ differs from $f_\nu$ on at most countably many points. Since $i_\eta = i_\nu$ and $|A_\eta \cap A_\nu| < \aleph_0$, we reach a contradiction.

(b) Suppose by contradiction that $A \in K$, and it is isomorphic to every
element of \( P(A) \cap K \). For every \( B \in P(A) \cap K \) let \( f_B \) be an isomorphism between \( A \) and \( B \), we reach a contradiction in a way similar to what was done in (a).

**Questions 7.2.** (a) Does \( WCH \) imply that for every \( A \in K \) there is \( B \in P(A) \cap K \) such that \( A \nleq B \)?

(b) Does \( WCH \) imply \( K \) does not contain a universal element?

Next we want to show that \( NA \) is consistent with \( WCH \). This brings up a new application of the club method, which we call the 'nearing forcing'. Recall that \( A1 \) is the axiom introduced in Section 6 to replace \( CH \) when we want to apply the club method.

**Lemma 7.3 (A1).** Let \( A, B \in K \). Then there is a c.c.c. forcing set \( P = P_{A,B} \) of power \( \aleph_1 \) such that \( \mathbb{P} \models N(A, B) \).

**Proof.** Let \( M_{A,B} = M \) be a model with universe \( \lambda \), which encodes \( \langle A, \triangleleft \rangle \) and \( \langle B, \triangleleft \rangle \). Let \( N = M^\kappa \) and \( C \) be \( N \)-thin. We identify \( A \cup B \) with \( \lambda \), hence \( A \leq \lambda \) and \( B \leq \lambda \). Since there are two linear orderings on \( \lambda \), we denote \( a \triangleleft b \) to mean that \( a < b \) as reals numbers, and \( a < b \) to mean that \( a < b \) as ordinals. Let \( \{ E_i \mid i < \lambda \} \) be an enumeration of the \( C \)-slices in an increasing order.

Let \( P = \{ f \in P_{\lambda}(A \times B) \mid f \) is an OP function, \( \text{Dom}(f) \cup \text{Rng}(f) \) is \( C \)-separated, for every \( a \in \text{Dom}(f) \) there is \( i < \lambda \) and \( 0 < N \in \alpha \) such that \( a \in E_i \) and \( f(a) \in E_{i+n} \), and for every distinct \( a, b \in \text{Dom}(f) \) if \( a < b \) then \( f(a) < b \). \( f \leq g \) if \( f \models g \). For later use we denote \( P = P_{A,B,C} \).

We prove that \( P \) is c.c.c. Let \( \{ f_\alpha \mid \alpha < \lambda \} \subseteq P \). W.l.o.g. \( f_\alpha = \langle \langle a_{\alpha,0}, a_{\alpha,1} \rangle, \ldots, \langle a_{\alpha,2n-2}, a_{\alpha,2n-1} \rangle \rangle \) where \( i < j \) implies \( a_{\alpha,i} < a_{\alpha,j} \), there is \( n \leq \lambda \) such that for every \( \alpha < \beta < \lambda \), and for every \( i < 2n \): if \( i < 2m \), then \( a_{\alpha,i} = a_{\alpha,0} \) and if \( 2m < i \) then \( a_{\alpha,i} < a_{\alpha,m} \). In addition we can assume that there are pairwise disjoint closed intervals \( V_0, \ldots, V_{2n-1} \) such that for every \( \alpha < \lambda \) and \( i < 2n \), \( a_{\alpha,i} \in V_i \).

We regard each \( f_\alpha \) as an element in \( (A \times B)^\kappa \). Let \( F \) be the topological closure of \( \{ f_\alpha \mid \alpha < \lambda \} \) in \( (A \times B)^\kappa \). Let \( p \in |N| \) be such that \( F \) is definable from \( p \), and every rational interval is definable from \( p \). Let \( \gamma_0 \in C \) be such that \( C \cap [\gamma_0, \lambda] \subseteq N \). Let \( f_\alpha \) be such that \( \gamma_0 \leq a_{\alpha,2n} \). We denote \( a_{\alpha,1} \) by \( a_{\alpha} \).

We will next duplicate \( f_\alpha \). The new element in the duplication argument is the use of the following fact: If \( a_1 \neq a_2, b_1 \neq b_2 \) are real numbers, then either \( \langle (a_1, b_1), (a_2, b_2) \rangle \) or \( \langle (a_1, b_1), (a_2, b_2) \rangle \) is an OP function. This fact replaces the assumption that \( X \) does not contain uncountable 0-colored sets in the proof of \( SOCA \), and the need to preassign colors in the proof of \( OCA \).

We define by a downward induction formulas \( \varphi_{2n}, \varphi_{2n-1}, \ldots, \varphi_m \), and we assume by induction that for every \( i : M^\kappa \models \varphi_i[a_0, \ldots, a_{i-1}] \), and that the only parameters of \( \varphi_i \) is \( p \). Let \( \varphi_0 = \langle x_0, \ldots, x_{2n-1} \rangle \in F \). Suppose \( \varphi_{i+1} \) has been defined.
Since $M \models \varphi_{i+1}(a_0, \ldots, a_i)$,

$$M \models \left( \exists x_{i+1} \left( x_{i+1} \neq x_i \land \bigwedge_{i=0}^{i} \varphi_{i+1}(a_0, \ldots, a_{i-1}, x_{i+1}) \right) \right).$$

Hence there are disjoint rational intervals $U_{i,0}, U_{i,1}$ such that

$$M \models \bigwedge_{i=0}^{i} \left( \exists x_{i+1} \in U_{i,0} \right) \varphi_{i+1}(a_0, \ldots, a_{i-1}, x_{i+1}).$$

Let

$$\varphi_i \equiv \bigwedge_{i=0}^{i} \left( \exists x_{i+1} \in U_{i,0} \right) \varphi_{i+1}(x_0, \ldots, x_{i-1}, x_{i+1}).$$

Let $m \leq i < n$, and let us consider the intervals $U_{2i,0}, U_{2i,1}, U_{2i+1,0}, U_{2i+1,1}$. Let $\varepsilon_i \in \{0, 1\}$ be such that whenever $(a_1, b_1) \in U_{2i,0} \times U_{2i+1,0}$ and $(a_2, b_2) \in U_{2i+1,1} \times U_{2i+1,1}$, then $(a_1, b_1), (a_2, b_2)$ is an OP function. Using $\varphi_{2m}, \ldots, \varphi_{2n}$ we can inductively choose sequences $(a_0, \ldots, a_{2m-1}), (a_1, \ldots, a_{2n-1}) \in F$ and for every $m \leq i < n$,

$$a_2i \in U_{2i,0}, \quad a_{2i+1} \in U_{2i+1,1}, \quad a_{2i+1} \in U_{2i+1,1} \quad \text{and} \quad a_{2n+1} \in U_{2i+1,1}.$$

Since $F$ is the closure of \{f_\alpha \mid \alpha < \kappa_3\}, there are $\beta$ and $\gamma$ such that for every $m \leq i < n$,

$$a_{n,2i} \in U_{2i,0}, \quad a_{n,2i+1} \in U_{2i+1,1}, \quad a_{n,2i} \in U_{2i+1,1} \quad \text{and} \quad a_{n,2i+1} \in U_{2i+1,1}.$$

It follows that $f_\beta \cup f_\gamma \in P$. We have thus proved that $P$ is c.c.c. \(\square\)

**Corollary 7.4.** NA + OCA is consistent.

**Proof.** Combine the methods of 3.2 and 7.3. \(\square\)

Let $cov(\lambda, \kappa)$ mean that $\kappa < \lambda$, and there is a family $D \subseteq P_\kappa(\lambda)$ such that $|D| = \lambda$ and for every $c \in P_\kappa(\lambda)$ there is $d \in D$ such that $c \subseteq d$.

**Theorem 7.4.** Let $V \models CH$ and let $\lambda$ be a regular cardinal in $V$ such that $\lambda < 2^\lambda = \mu$. Then $\kappa_2 < \lambda$ and $\varphi(\kappa_2, \kappa_2)$ holds. Then there is a forcing set $\mathcal{P}$ such that $\mathbb{P}$-NA $+$ $(2^\kappa = \lambda) \land (2^{\kappa_2} = \mu)$.

**Remark.** Clearly $cov(\kappa_2, \kappa_1)$ holds, hence one can start with the universes $L_\alpha$ and construct $V$ in which $2^{\kappa_2} = \kappa_1$ and $2^{\kappa_1} = \kappa_3$, and then by 7.3 get a universe satisfying $\kappa_2 < \lambda$ and $\varphi(\kappa_2, \kappa_2) = (2^{\kappa_1} = \lambda) \land (2^{\kappa_2} = \mu)$.

**Proof.** Let $V, \lambda, \mu$ be as in the theorem. Let $P = P_{\kappa_2}(\lambda)$, $G \subseteq P$ be a generic filter and $W = V[G]$. Using the fact that $P$ is $\kappa_2$-c.c. and does not collapse $\kappa_1$, it is easy to see that $W \models cov(\lambda, \kappa_2)$. Let $W_\alpha = V[G \cap P_{\kappa_2}(\alpha)]$ and $C_\alpha$ be the $(\alpha + 1)$st club which is added to $V$ be
$P_{\alpha}(\lambda)$. Let $\{\tau_\alpha \mid \alpha < \lambda\} \subseteq P_{\alpha}(\lambda \times \lambda)$ be such that for every $\sigma \in P_{\beta}(\lambda \times \lambda)$, $\{ \alpha \mid \sigma \subseteq \tau_\alpha \}$ is unbounded in $\lambda$. If $R$ is a forcing set and $R \subseteq \lambda$, then each $\tau_\alpha$ can be regarded as an $\alpha$-name of subset of $\lambda$ of power $\aleph_1$.

We define by induction on $\alpha < \lambda$ a finite support iteration $\langle (R_\alpha \mid \alpha < \lambda), \{\rho_\alpha \mid \alpha < \lambda\}, \{\eta_\alpha \mid \alpha < \lambda\} \rangle$ and $\{\xi_\alpha \mid \alpha < \lambda\}$ such that: (a) for every $\alpha$, $|R_\alpha| < \lambda$ and $\rho_\alpha$ is c.c.c., (b) $\rho_\alpha$ is a 1-1 function from $R_\alpha$ into $\lambda$, and if $\alpha < \beta$, then $\rho_\beta \subseteq \rho_\alpha$; we denote by $\Omega_\alpha$ the forcing set which is the image of $R_\alpha$ under $\rho_\alpha$; (c) for some $\gamma < \lambda$, $\xi_\alpha \in W_\gamma$ and $\forall \alpha (\xi_\alpha)$ "$\xi_\alpha$ is a 1-1 function from a subset of $\gamma$ onto $\lambda$, and if $\alpha < \beta$, then $\xi_\beta \subseteq \xi_\alpha$".

If $h$ is an isomorphism between forcing sets $S_1$ and $S_2$, and $\tau$ is an $S_1$-name, let $h(\tau)$ denote the image of $\tau$ under $h$, hence $h(\tau)$ is an $S_2$-name.

Let $\Omega_\alpha$ be a trivial forcing set. For a limit ordinal $\delta$ let $R_\delta = \bigcup_{\alpha < \delta} R_\alpha$ and $h_\delta = \bigcup_{\alpha < \delta} h_\alpha$. There is some $\gamma < \lambda$ such that $Q_\delta \subseteq W_\gamma$ hence if $5.3$, if $H$ is a $\Omega_{\alpha}$-generic filter, then $\mathcal{R}_{\gamma}^{W_\delta}[\mathcal{R}] = \mathcal{R}_{\gamma}^{W_\delta}[\mathcal{R}]$. It thus follows that there is $\xi_\delta \in W_\gamma$ such that $\forall \Omega (\xi_\delta)$ is a 1-1 function from a subset of $\lambda$ onto $R$, and for every $\alpha < \delta$, $\xi_\delta \subseteq \xi_\alpha$.

Suppose $\alpha$, $h_\alpha$, $\xi_\alpha$ have been defined, and we define $\rho_\alpha$. Let $\xi_\alpha \in W_\gamma$. Since $|\Omega_\alpha|, |\tau_\alpha| < \lambda$ and $\rho_\alpha$ is $\mathcal{R}_{\gamma}$-c.c.c., there is $\gamma_1 < \lambda$ such that $Q_\alpha \tau_\alpha \in W_\gamma$. Let $\gamma = \gamma_0 \cup \ldots \cup \gamma_\alpha \cup \gamma_\alpha$.

In $W_\gamma$, $\forall \Omega (\xi_\alpha(\tau_\alpha) \in P_{\lambda}(\mathcal{R}))$, and hence there is a $Q_\alpha$-name $\tau'_\alpha \in W_\gamma$ such that $\tau_\alpha \subseteq \tau'_\alpha$ and in $W_\gamma$, $\forall \Omega \xi_\alpha(\tau'_\alpha) \in K$. By the same argument as in 5.4, if $H$ is $Q_\alpha$-generic over $W$ and $A = v_{\alpha}(\xi_\alpha(\tau'_\alpha))$, then $C_\alpha$ is $M_{\alpha, \lambda}$-thin. (Recall that $M_{\alpha, \lambda}$ is a model with universe $\mathcal{R}_\alpha$ which encodes $(\alpha, \prec)$.) Hence there is a $Q_\alpha$-name $\rho'_\alpha$ such that $|\rho'_\alpha| = \mathcal{R}_\alpha$ and $\forall \Omega \rho'_\alpha = P_{\mathcal{R}_{\alpha, \lambda}}$. (The notation $P_{\mathcal{R}_{\alpha, \lambda}}$ was introduced in the beginning of the proof of 7.3.) Let $\rho_\alpha = h_\alpha(\rho'_\alpha)$, hence $\rho_\alpha$ is an $\alpha$-name of a forcing set. Let $R_{\alpha+1} = R_\alpha + \rho_\alpha$ and $h_{\alpha+1}$ be a 1-1 function from a subset of $\lambda$ onto $R_{\alpha+1}$, and $h_{\alpha+1} \supseteq h_\alpha$. $\xi_{\alpha+1}$ is defined as in limit case.

Let $H$ be an $\alpha$-generic filter, and $H' = h_\alpha(H)$. Let $U = W[H]$ and for every $\alpha < \lambda$, $\xi_\alpha = v_{\alpha}(\xi_\alpha)$. Clearly $\xi_\alpha$ is a 1-1 function from a subset of $\lambda$ onto $\mathcal{R}_{\alpha}$ and $\xi_\alpha = \bigcup_{\alpha < \delta} \xi_\alpha$. Let $A_1, A_2 \in K[\mathcal{R}_{\alpha}]$, $\sigma_1 = \mathcal{R}_{\alpha}(A_1)$, $\sigma_2 = \mathcal{R}_{\alpha}(A_2)$, and $\tau'$ be $Q_\alpha$-names such that $v_{\alpha}(\tau') = \sigma_1$, $\tau'$ can be regarded as an element of $\mathcal{R}_{\alpha}(\lambda \times \lambda)$ by identifying $\tau'$ with the set $(\alpha, \beta) \mid \alpha \cap \beta \in \tau' \cap \mathcal{R}_{\alpha}$ where $\mathcal{R}_{\alpha}$ is the canonical name for $\beta$. Let $\alpha < \lambda$ be such that: $\tau \subseteq \tau' \subseteq \tau_\alpha$, $\tau \subseteq \tau' \subseteq \tau_\alpha$, and $\sigma_1 \cup \sigma_2 \subseteq \text{Dom}(\xi_\alpha)$. Let $A = v_{\alpha}(\xi_\alpha(\tau'))$. Hence

$$A \supseteq v_{\alpha}(\xi_\alpha(\tau')) = \alpha \cup (v_{\alpha}(\xi_\alpha(\tau')) \supseteq \alpha \cup (v_{\alpha}(\xi_\alpha(\tau')) = A_1 \cup A_2.$$
We have seen that \( U \models (\forall A_1, A_2 \in K) \neg \exists \alpha (A_1, A_2) \). It remains to show \( U \models 2^{\aleph_0} = \lambda \land 2^{\aleph_0} = \mu \). Since \( V \models 2^{\aleph_0} = \mu \), \( |P_{\aleph_0}(\lambda)^+ R_\lambda| = \mu \), it does not collapse \( \aleph_1 \) and is \( \kappa_\varepsilon \) c.c.c. \( 2^\kappa = \mu \) holds in \( U \) too. Since \( \text{cov}(\kappa, \kappa_1) \) holds in \( W \), then \( W \models \lambda \kappa_\varepsilon = \lambda \cdot \kappa_\varepsilon = \lambda \); \( |R_\lambda| = \lambda \), hence \( U \models 2^{\aleph_0} = \lambda \). It is easy to see that for every \( \alpha < \beta \), \( W[H \cap R_\beta] - W[H \cap R_\alpha] \) contains a real, hence \( U \models 2^{\aleph_0} = \lambda \). □

Let \( A, B \in K \). We say that \( A \) and \( B \) are densely near to one another (DN\((A, B)\)), if there is an OP function \( f \subseteq A \times B \) such that \( \text{Dom}(f), \text{Rng}(f) \in K \), and \( \text{Dom}(f), \text{Rng}(f) \) are dense in \( A \) and \( B \) respectively. Let DN\((A, B)\) = \((\forall A, B \in K)\) DN\((A, B)\).

**Theorem 7.5.** NA \( \Rightarrow \) DNA.

We need some lemmas and terminology. Let \( a, b \in \mathbb{R}^+, a < b \) if there is \( n_0 \in \omega \) such that for every \( n \geq n_0 \), \( a(n) < b(n) \), and \( \{ n \mid a(n) < b(n) \} \) is infinite. An element of \( \mathbb{R}^+ \) which is not eventually zero is indistinguishable from the interval \((0, 1)\) which it represents. \( \aleph_1 + \aleph_1 \) denotes the linear ordering which is the sum of \( \aleph_1 \), \( \langle \rangle \) and \( \langle \aleph_1 \rangle \).

Hausdorff using ZFC only constructed a sequence \( \{ a_i \mid i \in \aleph_1 \} \subseteq \mathbb{R} \) such that: (a) if \( i < j \in \aleph_1 \), then \( a_i < a_j \); and (b) there is no \( a \in \mathbb{R}^+ \) such that for every \( i \in \aleph_1 \), and \( j \in \aleph_1 \), \( a < a_i \). We call such a set a Hausdorff set.

**Proposition 7.6.** (a) Let \( A \subseteq \mathbb{R} \) be a Hausdorff set, and let \( B \subseteq \mathbb{R} \) be countable. Then there is a \( G_\delta \)-set \( G \) such that \( G \supseteq B \) and \( G \cap (A - B) = \emptyset \).

**Proof.** Easy and well known. □

**Proposition 7.7 (NA).** Let \( A \subseteq \mathbb{R} \) be uncountable. Then for every countable \( B \subseteq \mathbb{R} \) there is an open set \( U \) such that \( U \supseteq B \) and \( |A - U| \gg \aleph_1 \).

**Proof.** Let \( A, B \) be as above, and let \( C \) be a Hausdorff set. Let \( f \subseteq C \times A \) be an uncountable OP function, let \( F \) be the closure of \( f \) in \( \mathbb{R} \times \mathbb{R} \), and let \( D = F^{-1}(B) \). Clearly for every \( b \in B \), \( |F^{-1}(b)| \leq 2 \), and hence \( D \) is countable. Let \( V \) be an open set in \( \mathbb{R} \) such that \( V \supseteq D \) and \( |\text{Dom}(f) - V| \gg \aleph_1 \). W.l.o.g. if \( b \in B \) and \( F^{-1}(b) = \{ d_1, d_2 \} \), then \( V \) contains the closed interval determined by \( d_1 \) and \( d_2 \). (This can be assumed since the above interval does not intersect \( \text{Dom}(f) - D \). Let \( \{ V_i \mid i \in \omega \} \) be the partition of \( V \) into pairwise disjoint open intervals. We can further assume that the endpoints of each \( V_i \) belong to \( \text{cl}(\text{Dom}(f)) \). (This is to say that every open interval \( I \) is contained in an open interval \( J \) with endpoints in \( \text{cl}(\text{Dom}(f)) \) such that \( I \cap \text{cl}(\text{Dom}(f)) = J \cap \text{cl}(\text{Dom}(f)) \).) Let \( V_i = (c_i, d_i) \), \( a_i = \min(F(c_i)) \), \( b_i = \max(F(d_i)) \) and \( U_i = (a_i, b_i) \). It is easy to check that \( U = \bigcup_{i \in \omega} U_i \supseteq B \) and \( |\text{Rng}(f) - U| \gg \aleph_1 \). □

**Proof of Theorem 7.5.** Assume NA, and let \( A, B \in K \). W.l.o.g. \( A, B \) are dense in \( \mathbb{R} \). Hence we have to construct an OP function \( f \subseteq A \times B \) such that \( \text{Dom}(f), \text{Rng}(f) \) are dense in \( \mathbb{R} \) and belong to \( K \).
We call an OP function \( f \) extendible, if \( \text{Dom}(f) \) belongs to \( K \) and, the closure of \( f \) in \( \mathbb{R} \times \mathbb{R} \) is an OP function.

Let us first see that if \( A, B \in K \), then there is an extendible function \( f \in A \times B \). Let \( g \subseteq A \times B \) be an uncountable OP function. Let \( G \) be the closure of \( g \) in \( \mathbb{R} \times \mathbb{R} \). Let \( C = \{ a \in \mathbb{R} | \text{there is } a' \in \mathbb{R} \text{ such that } a' \neq a \text{ and } G(a) \cap G(a') \neq \emptyset \} \), and \( D = \{ a \in \mathbb{R} | \text{there is } a' \in \mathbb{R} \text{ such that } a' \neq a \text{ and } G^{-1}(a) \cap G^{-1}(a') \neq \emptyset \} \). Obviously \( C \) and \( D \) are countable. Let \( V \) be an open set containing \( C \) such that \( \text{Dom}(g) \cap V \) is uncountable, and let \( g_1 = g \upharpoonright (\text{Dom}(g) \cap V) \). Let \( U \) be an open set containing \( D \) such that \( \text{Rng}(g_1) \cap U \) is uncountable, and let \( f_1 = g_1 \upharpoonright (\text{Dom}(g_1) \cap g_1^{-1}(U)) \). It is easy to check that \( f_1 \) is uncountable and its closure is an OP function. Let \( f \) be a restriction of \( f_1 \) to an element of \( K \), then \( f \) is as desired. This concludes the proof of the above claim.

Let \( \{ I_i | i \in \omega \} \) be a list of all rational intervals. We define by induction a sequence of extendible functions \( \{ f_i | i \in \omega \} \) where \( f_0 \subseteq A \times B \). Let \( f_0 \subseteq A \times B \) be any extendible function. Suppose \( f_i \) has been defined. If \( |\text{Dom}(f_i) \cap I_i| \geq 2 \), let \( f_{i+1} = f_i \). Suppose otherwise, then using the fact that \( f_i \) is extendible it is easy to see that there are non-empty open intervals \( J_1, J_2 \) such that \( J_1 \subseteq I_i, J_1 \cap \text{Dom}(f_i) = \emptyset \), and for every \( a \in \text{Dom}(f_i) \): if \( a < J_1 \), then \( f_i(a) < J_2 \), and if \( a < J_1 \), then \( J_1 < f_i(a) \). Let \( g = (A \cap J_1) \times (B \cap J_2) \) be an extendible function such that the endpoints of \( J_1 \) and \( J_2 \) do not belong to \( \text{Dom}(g) \) and \( \text{Rng}(g) \) respectively. Let \( f_{i+1} = f_i \cup g \). It is easy to see that \( f_{i+1} \) is extendible.

We define \( f_{i+1} \) from \( f_i \) analogously in order to assure that \( |\text{Rng}(f_{i+1}) \cap I_i| \geq \aleph_1 \).

It is easy to see that \( \bigcup_{i \in \omega} f_i \) is an OP function such that \( f \subseteq A \times B \) and \( \text{Dom}(f), \text{Rng}(f) \) belong to \( K \) and are dense in \( \mathbb{R} \). \( \Box \)

**Question 7.8.** Let \( \text{NA}^- = (\forall A, B \in K) (N(A, B) \cup N(A, B^\complement)) \). Does \( \text{NA}^- \) imply that \( (\forall A, B \in K) (\exists I, J) (I \text{ and } J \text{ are intervals } \wedge (\text{DN}(A \cap I, B \cap J) \vee \text{DN}(A \cap I, (B \cap J)^\complement))) \)?

8. A weak form of Martin's axiom, the consistency of the incompactness of the Magidor-Malitz quantifiers

In this section we deal with two separate problems. The first is to construct a model of set theory in which the Magidor–Malitz quantifier is countably incompact. This question was raised by Malitz. It was first solved by Shelah (unpublished) using methods of Avraham. The first solution involved properties of Suslin trees which are expressible by sentences in the Magidor–Malitz language (MML). So it was possible to show that CH did not imply the countable compactness of MML. On the other hand, the solution that we present here shows that MA + \((\aleph_1 + 2^{\aleph_0})\) does not imply the countable compactness of MML.

The second question we are concerned with is whether the axioms like BA, NA, OCA etc. imply MA\(_\aleph_0\). The answer to this question is negative; in fact, forcing sets constructed with the aid of the club method do not destroy Suslin
trees. We will formulate some strong form of the chain condition which preserves Susslin trees but can still change the satisfaction of sentences in MML.

We start with the incompactness of MML.

In [1] the notion of a $k$-entangled set of real numbers was introduced, and it was proved (Theorem 6) that for every $k > 0$ $\text{MA}_{\omega_1}$ + "There exists a $k$-entangled set" is consistent.

We will use a similar construction in order to get a model of ZFC in which MML is not countably compact. (Using c.c.c.-indestructible S-spaces, K. Kunen showed that the incompactness of MML holds in every model of MA obtained from a ground model satisfying CH by a c.c.c. forcing.)

Let us recall some definitions. A $k$-place configuration is a sequence $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_{k-1})$ of zeroes and ones. Let $a, b$ be sequences of real numbers of length $k$. $(a, b)$ has configuration $\varepsilon$ ($\varepsilon[a, b]$) if for every $i < k$: if $\varepsilon_i = 0$, then $a_i < b_i$, and if $\varepsilon_i = 1$, then $b_i < a_i$.

**Definition 8.1.** (Shelah [1]). Let $A \subseteq \mathbb{R}$ and $|A| = \aleph_1$. $A$ is $k$-entangled, if for every sequence $(a_i : i < \aleph_1) \subseteq A^k$ of 1-1 pairwise disjoint sequences, and for every $k$-place configuration $\varepsilon$ there are $i, j < \aleph_1$ such that $\varepsilon[a_i, a_j]$.

Note that if $A$ is $k$-entangled, then it is $l$-entangled for every $l < k$.

The following easy claim appears in [1].

**Proposition 8.2.** $\text{MA}_{\omega_1}$ implies that there is no $A$ such that $A$ is $k$-entangled for every $k > 0$.

The incompactness of MML follows from 8.2 and the following main theorem.

**Theorem 8.3.** $\text{MA}_{\omega_1} + (\forall k > 0) (\exists A) (A$ is $k$-entangled) is consistent.

Let $V$ be a model of ZFC which satisfies the axiom mentioned in 8.3. We show that MML is countably incompact in $V$.

Let $L$ be a language containing unary predicates $(P_i : i \in \omega)$, for every $n > 0$ on $(n + 1)$-place predicate $R_n$, a unary predicate $Q$ and a binary predicate $<$. Let $T$ be the following theory in MML.

(1) $P_0$ is uncountable; $<$ is a linear ordering of $P_0$.

(2) $P_0 \subseteq P_0$ and is a countable dense subset relative to $<$.

(3) For $n > 0$, $R_n$ is a 1-1 function from $P_0$ onto $P_n$ (i.e. $P_n$ represents the set of $n$-tuples from $P_0$).

(4) $(P_0, <)$ is $k$-entangled for every $k > 0$. (The reader should check that (4) can be expressed by an MML-theory.)

By 8.2, $\text{MA}_{\omega_1}$ implies that $T$ is not consistent but since for every $k \in \omega$ there is a $k$-entangled set, every finite subset of $T$ is consistent.

We now proceed to the proof of 8.3.
Definition 8.4. (a) Let \( k = (k_0, \ldots, k_{n-1}) \) be a sequence of positive natural numbers, \( k = \sum_{i \leq n} k_i \) and \( A = (A_{i_0}, \ldots, A_{i_{n-1}}) \) be a sequence of uncountable subsets of \( \mathbb{R} \). We say that \( A \) is \( k \)-entangled if for every sequence \( \{a_i \in I \leq \aleph_1 \}_I \subseteq A_{i_0} \times \cdots \times A_{i_{n-1}} \) of pairwise disjoint 1-1 sequences, and every \( (n_1) \)-place configuration \( e_1 \), there are \( i, j < \aleph_1 \) such that \( \vec{e}(a_i, a_j) \).

(b) Let \( k = \{k_i \mid i \in \omega \} \) be a sequence of positive natural numbers and \( A = \{A_i \mid i \in \omega \} \) be a sequence of uncountable subsets of \( \mathbb{R} \). \( A \) is \( k \)-entangled if for every \( n \in \omega \), \( A \uparrow n \) is \( k \uparrow n \)-entangled.

The exact form of 8.3 that we prove is the following.

Lemma 8.5. Suppose \( V \uparrow \mathop{CH} + (2^\aleph_0 = \aleph_2) \), and suppose \( A \) is an \( \omega \)-sequence such that \( A \) is \( k \)-entangled. Then there is a c.c.c. forcing set \( P \) of power \( \aleph_2 \) such that \( \Vdash_p " A \) is \( k \)-entangled, \( 2^\aleph_0 = \aleph_2 \) and \( MA \) holds".

Proof. The general framework is the usual one, and we will not repeat it. In each atomic step of the iteration we will apply the explicit contradiction method introduced in Section 2, that is, we are given a c.c.c. forcing set \( Q \) of power \( \aleph_1 \), if \( \Vdash_Q " A \) is \( k \)-entangled", then \( Q \) is the next stage in the iteration. If however there is some \( q \in Q \) which forces that \( A \) is not \( k \)-entangled, then we devise a c.c.c. forcing set \( R \) of power \( \aleph_1 \) such that \( \Vdash_R " Q \) is not c.c.c. and \( A \) is \( k \)-entangled" and add \( R \) as the next stage in the iteration. Hence the central claim in the proof is the following.

Claim 1 (CH). Let \( A \) be \( k \)-entangled, let \( Q \) be a c.c.c. forcing set, \( q \in Q \) and \( q \Vdash_Q " A \) is not \( k \)-entangled". Then there is a c.c.c. forcing set \( R \) of power \( \aleph_1 \) such that \( \Vdash_R " Q \) is not c.c.c. and \( A \) is \( k \)-entangled".

Proof. The proof is very similar to the proof of Theorem 6 in [1]. However technically the present proof is somewhat more complicated.

Let \( A, k, Q \) and \( q \) be as in the lemma. Let \( M \) be a model with universe \( \aleph_1 \), which encodes \( A \) and enough set theory. Let \( < \) denote the linear ordering of ordinals in \( M \), and \( < \) denote the linear ordering of real numbers in \( M \). By replacing \( q \) by a more informative condition we can w.l.o.g. assume that there is an \( n \in \omega \), and \( n \)-place configuration \( \rho = (\rho_0, \ldots, \rho_{n-1}) \) and a \( Q \)-name \( \tau \) such that \( q \Vdash_Q " \tau \) is an uncountable set of pairwise disjoint 1-1 sequences from \( \prod_{i \leq n} A_i \), and for every \( a, b \in \tau \), \( \neg \rho[a, b] \)"

Let \( k = \sum_{i \leq n} k_i \). It is easy to construct a sequence \( \langle q_{\alpha} a(\alpha, 0), \ldots, a(\alpha, k) \rangle \ | \ \alpha < \aleph_1 \rangle \) such that: (1) for every \( \alpha, q_{\alpha} \geq q \); (2) for every \( \alpha \) and \( j, q_{\alpha} \Vdash_Q a(\alpha, j) \in \tau \); (3) \( \langle a(\alpha, j) \ | \ \alpha < \aleph_1, j \leq k \rangle \) is a family of pairwise disjoint 1-1 sequences; and (4) for every \( \alpha < \aleph_1 \), there are ordinals \( \beta(\alpha, 0) < \cdots < \beta(\alpha, k) \) in \( C_M \) such that: for every \( j \leq k \), \( \beta(\alpha, j) = a(\alpha, j) \), for every \( j < k \), \( a(\alpha, j) < \beta(\alpha, j + 1) \), and for every \( \alpha' < \alpha \), \( a(\alpha', k) < \beta(\alpha, 0) \). (\( a < \beta \) means that all the elements of \( a \) are less than \( \beta \) etc.)
Let \( a(\alpha) = a(\alpha, 0) \cdots a(\alpha, k) \), and let \( a(\alpha, j) = \langle a(\alpha, j, 0), \ldots, a(\alpha, j, k-1) \rangle \). By further uniformization we can assume that there is a family of pairwise disjoint open rational intervals \( \{U(j, i) \mid j \leq k, i < k \} \) such that for every \( \alpha, j, i, a(\alpha, j, i) \in U(j, i) \).

We say that \( q_\alpha \) and \( q_\beta \) are explicitly contradictory if for some \( j \leq k \), \( \models \rho[a(\alpha, j), a(\beta, j)] \) or \( \models [a(\alpha, j), a(\beta, j)] \). Clearly if \( q_\alpha \) and \( q_\beta \) are explicitly contradictory, then they are incompatible in \( Q \). Let \( R = \{ \sigma \in P_{\aleph_0}(K) \mid \text{for every } \alpha \neq \beta \in \sigma, q_\alpha \text{ and } q_\beta \text{ are explicitly contradictory} \} \). Clearly \( \sigma \preceq \eta \) if \( \sigma \subseteq \eta \).

We also show that \( R \) is c.c.c., there is a standard argument which shows that there is \( r_0 \in R \) such that \( r_0 \models \text{"} A \text{ is uncountable"} \).\( ^* \) where \( A \) is the standard name for \( \{ q_\alpha \mid \alpha \in \mathbb{G} \} \). Hence by replacing \( R \) by \( R' = \{ r \in R \mid r \models r_0 \} \) we can conclude that \( \models R \models \text{"} Q \text{ is not c.c.c."} \).

The proof that \( \models R \models \text{"} A \text{ is } k \text{-entangled"} \) already includes the arguments appearing in the proof that \( R \) is c.c.c., so we omit the latter.

Suppose by contradiction there is \( r \in R \) such that \( r \models \text{"} A \text{ is not } k \text{-entangled"} \). W.l.o.g. \( r = 0 \), and \( m \geq n \) is such that \( \models R \models \text{"} A \upharpoonright m \text{ is not } (k \upharpoonright m) \text{-entangled"} \). Let \( e \) be a configuration and \( \eta \) be an \( R \)-name such that \( \models r \models \text{"} \eta \text{ is an uncountable set of } 1 \text{-1 pairwise disjoint sequences from } \prod_{i < n} A_i \upharpoonright \), and there are no \( a, b \in \eta \) such that \( \models e[a, b] \).

Let \( \langle r_\alpha, b_\alpha \rangle \mid \alpha \in \mathbb{K} \rangle \) be such that \( r_\alpha \models \text{"} b_\alpha \in \eta \text{ and } \{ b_\alpha \mid \alpha \in \mathbb{K} \} \text{ is a family of pairwise disjoint sequences"} \).

We now have to uniformize the sequence \( \{ r_\alpha, b_\alpha \mid \alpha \in \mathbb{K} \} \) as much as possible. We do not repeat the details of this process which have already appeared so many times.

Suppose \( r_\alpha = \{ \gamma(\alpha, 0), \ldots, \gamma(\alpha, l-1) \} \) where the \( \gamma \)'s appear in an increasing order. Let us assume for simplicity that the \( r_\alpha \)'s are pairwise disjoint. For every \( \alpha \) we form the following sequence \( c_\alpha \) which belongs to a finite product of \( A_i \)'s. Let \( a_\alpha = a(\gamma(\alpha, 0)) \cdots a(\gamma(\alpha, l-1)) \) and \( c_\alpha = a_\alpha \upharpoonright b_\alpha \). Let \( F = \text{\textit{cl}}(\{ c_\alpha \mid \alpha \in \mathbb{K} \}) \), and \( \delta_0 \in C_\alpha \) be such that \( F \) is definable in \( M \) by some parameter \( \leq \delta_0 \). Let \( \alpha \) be such that \( \delta_0 \leq c_\alpha \).

We want to duplicate \( \langle r_\alpha, b_\alpha \rangle \). Before stating the exact claim we need additional notations. Let \( d(i, j) = a(\gamma(\alpha, i), j), \quad d(i) = d(i, 0) \cdots d(i, k), \) and \( a, b, c \) denote respectively \( a_\alpha, b_\alpha \) and \( c_\alpha \).

**Duplication claim.** There are \( c^0, c^1 \in F \) such that

\[
c = d^0(0, 0) \cdots d^0(0, k) \cdots d^0(l-1, 0) \cdots d^0(l-1, k) \upharpoonright b,
\]

\[i = 1, 2\]

and (1) \( \models e[b^0, b^1] \), (2) for every \( i < l \), there is \( j \leq k \) such that \( \models \rho[d^0(i, j), d^1(i, j)] \).

The contradiction follows easily from the duplication claim by choosing a \( c_\alpha \).
near enough to \( c^0 \) and a \( c^n \) near enough to \( c^1 \), because for such \( \beta \) and 
\( \gamma: r_0 \cup r_n \in R \) but \( \epsilon[b_{\beta_n}, b_{\beta_{n-1}}] \).

**Proof of the duplication claim.** Let \( \leq_1 \) denote the lexicographic order on \( l \times (k+1) \)
let \( t = l(k+1) \) and let \( s_0 <_1 s_1 <_1 \cdots <_1 s_{n-1} \) be an enumeration of \( l \times (k+1) \). By
the construction, there are \( \beta(s) = \beta_j \) in \( C_M \) such that \( \beta_0 < d(s_0) < \beta_1 < \cdots < \beta_{n-1} < d(s_{n-1}) \). Let \( E(s) = E_j = \{ \beta_0, \beta_{n+1} \} \). We next divide \( b \) and \( \epsilon \) into parts in the
following way. \( b = (b_0, \ldots, b_{l-1}, b_0, \ldots, b_n) \) where \( b_0, \ldots, b_{n-1} \) belong to some \( A_i \) where \( i < n \) and \( b_n, \ldots, b_0 \) belong to some \( A_i \) where \( n < i < m \). Let \( e_i(s) = e_{\epsilon, j} \) be
the restriction of \( b \) to those coordinates \( w \) for which \( b_w \in E_j \cap \bigcup_{n=1}^{m} A_i \) and let \( f_j \)
be the restriction of \( b \) to those coordinates \( w \) such that \( b_w \in E_j \cap \bigcup_{n=1}^{m} A_i \). Let \( e_{\epsilon, j} \) be the restriction of \( \epsilon \) such that \( \text{Dom}(e_{\epsilon, j}) = \text{Dom}(e_{\epsilon, j}) \), and similarly \( \rho \) is a
restriction of \( \epsilon \) such that \( \text{Dom}(\rho_{\epsilon, j}) = \text{Dom}(f_{\epsilon, j}) \). Let \( \epsilon = \bigcup_{i \leq n} e_{\epsilon, j} \).

By the entanglement of \( A \) we can construct \( c^0, c^1 \in F \) having the form
\[
c^i = d'(s_0) \cdots d'(s_{n-1}) \bigcup_{i \leq n} e_{\epsilon, j} \bigcup_{i \leq n} f_{\epsilon, j} \bigcup_{i \leq n} f_{\epsilon, j} \]
such that for every \( j < t \); if \( e_{\epsilon} = A \), then \( \epsilon[\rho[d'(s_0), d'(s_1)] \) and \( \rho[f_{\epsilon, j}^0, f_{\epsilon, j}^1] \); and if \( e_{\epsilon} = A \), then \( \epsilon[e_{\epsilon, j}, e_{\epsilon, j}^1] \) and \( \rho[f_{\epsilon, j}^0, f_{\epsilon, j}^1] \).

This is proved by the usual duplication argument, that is, we first define by a
downward induction some formulas \( \psi_{n-1}, \ldots, \psi_0 \), and then starting with \( \psi_0 \) we
construct by induction on \( j, d'(s_0), e_{\epsilon, j}, f_{\epsilon, j} \).

Since \( \epsilon \) contains only \( k \) elements whereas for every \( i \) we have \( k+1 \) \( d(i, j) \)'s it
follows that for every \( i < l \), there is \( j \) such that \( e(i, j) = A \). Hence for every \( i \), there
is \( j \) such that \( \epsilon[\rho[d'(i, j), d'(i, j)] \). Hence the duplication claim is proved. This
concludes the proof of claim 1, and hence the proof of 8.5.

There is still a gap between 8.3 and 8.5. In order that 8.3 will follow we still
need the following easy claim.

**Claim.** There is a universe \( V \) satisfying CH and \( 2^\omega = \aleph_2 \) such that there is a
sequence \( A \) in \( V \) which is \( k \)-entangled, where \( k = \langle 1, 2, 3, \ldots \rangle \).

In fact (CH) implies that there is \( A \) in \( K \) such that \( A \) is \( k \)-entangled for every \( k \),
and hence if \( \{ A_i \mid i \in \omega \} \) is a partition of \( A \) into uncountable sets, then \( \{ A_i \mid i \in \omega \} \)
is \( k \)-entangled for every \( k \).

However, instead, we can start with a universe \( V \) satisfying CH and \( 2^\omega = \aleph_2 \),
and then add to \( V \) a set of \( \aleph_1 \) Cohen reals. It is shown in [1] that such an \( A \) is
\( k \)-entangled for every \( k \).

We turn now to the proof that Suslin trees are preserved under forcing sets
constructed by the club method.

**Definition 8.6.** Let \( P \) be a forcing set. \( P \) has the strong countable chain condition
(P is s.c.c.) if for every uncountable \( A \subseteq P \), there is \( B \subseteq A \) and \( \{ B_{ij} \mid i \in \omega, j = 0, 1 \} \)
such that $B$ is uncountable, for every $i$ and $j$, $B_{ij} \subseteq B$; and (1) for every $i \in \omega$, $b_0 \in B_{i0}$ and $b_1 \in B_{i1}$, $b_0$ and $b_1$ are compatible, (2) for every uncountable $C \subseteq B$, there is $i$ such that $B_{i0} \cap C = \emptyset$ and $B_{i1} \cap C \neq \emptyset$.

Remarks. Clearly every s.c.c. forcing set is c.c.c. Kunen and Tall [10] defined the following property of forcing sets. $P$ has property $S$ if every uncountable subset of $P$ contains an uncountable subset of pairwisecompatible conditions. Clearly $S \Rightarrow$ s.c.c. Let $\text{MMML}_{\omega}$ be the portion of $\text{MML}$ in which only the quantifier $Q^2$ which bounds two variables is used. If $\varphi \in \text{MMML}_{\omega}$, $M \models \varphi$ and $P$ has property $S$, then $\forces_P (M \models \varphi)$. If $P$ is s.c.c. this is not longer true, since e.g. $N(A, B)$ can be expressed by an $\text{MMML}_{\omega}$ sentence about the model $M = (A \cup B, \langle, A, B \rangle)$. Now suppose $\neg N(A, B)$ holds in $V$, and let $P$ the nearing forcing of $A$ and $B$. Then $P$ is s.c.c., and $\forces_P \neg N(A, B)$.

It seems that "$P$ is s.c.c. and $\forces_P \varphi$ is s.c.c." does not imply that $P \ast \varphi$ is s.c.c. However, since we are dealing with the preservation of $\text{MML}$ sentences we can avoid this problem by using the following lemma.

Lemma 8.7. Let $M \in V$ be a model and $\varphi \in \text{MML}$ be a sentence of the form $\neg Q^2 x_1 \cdot \cdot \cdot x_n R(x_1, \ldots, x_n)$ where $R$ is a relation symbol. Let $\langle \langle P_i | i \in \omega \rangle, \{ \tau_i | i \in \omega \rangle \rangle$ be a finite support iteration of c.c.c. forcing sets such that $\alpha$ is a limit ordinal, and for every $\beta < \alpha$, $\forces_{P_\beta} (M \models \varphi)$. Then $\forces_{P_\alpha} (M \models \varphi)$.

Proof. Suppose by contradiction that $p \forces_{P_\alpha} (M \models \neg \varphi)$. Then there is a $P_\beta$-name $\tau$ such that $p \forces_{P_\beta} \langle \tau \rangle$ is uncountable and every $n$-tuple from $\tau$ satisfies $R$". Let $\langle \langle p_\eta, a_i | i \in \eta \rangle \rangle \subseteq P_\eta \times |M|$ be such that for every i, $p_i \not\forces_{P_\eta} \varphi$, if $i \neq j$ then $a_i \not\forces_{P_\eta} \varphi$, and $p_i \forces_{P_\beta} A_i \in \tau$, where $A_i$ is the canonical name of $a_i$. For $q \in P_\eta$, let $\sup(q)$ be the support of $q$, hence $\sup(q) \subseteq P_\eta \alpha$. If there is $\beta$ such that $\sup(p) \subseteq \beta$ and for all $P_\eta$'s, $\sup(q) \subseteq \beta$, then $\forces_{P_\alpha} (M \models \neg \varphi)$, which is a contradiction. Hence the cofinality of $\alpha$ must be $\omega_1$, and we can w.l.o.g. assume that $\sup(p_i) \supseteq \Delta$ -system of the following form: $\sup(p_i) = \sigma \cup \sigma_i$ where for every $i \neq j$, $\sigma_i \cap \sigma_j = \emptyset$, and for every $\beta \in \sigma$ and $\gamma \in \sigma_\beta \gamma < \beta$. Let $\beta$ be such that for $\sigma \subseteq \beta$, $\sup(p) \subseteq \beta$ and for every $i$ and $\gamma \in \sigma_\beta \gamma < \gamma$. Let $\tau' = \langle \langle p_i \mid \beta, A_i \rangle \rangle | i \in \eta \rangle$. $\tau'$ is a $P_\eta$-name. Clearly for every $i_1, \ldots, i_n < \eta$, there is $q \in P_\eta$ such that $q \not\forces_{P_\eta} p_i$ if there is $q \in P_\eta$ such that $q \not\forces_{P_\eta} p_i \mid \beta$, $\tau'$ is uncountable and every $n$-tuple from $\tau'$ satisfies $R$". Hence $\forces_{P_\eta} (M \models \neg \varphi)$, a contradiction. □

Recall that our aim is to show that Suslin trees are preserved under forcing sets constructed by the club method. Let $(T, \langle \rangle)$ be a Suslin tree in $V$. In order that it will remain a Suslin tree after forcing with some forcing set $P$ it has to satisfy in $V^P$ the following sentence

$\varphi = \neg Q^2 x y \langle \langle \neg x < y \wedge \neg y < x \rangle \rightarrow x = y \rangle$. 


By the last lemma if \( P = P_{\omega} \) where \( \{ P_i | i < \alpha \} \) is a finite support iteration of c.c.c forcing sets, then it suffices to show that for every \( i < \alpha \), \( P_i \) "If \( \langle T, < \rangle \) is Suslin tree, then \( \Vdash P_i \langle T, < \rangle \) is a Suslin tree." So we prove two claims.

**Lemma 8.8.** If \( P \) is s.c.c. and \( T \in V \) is a Suslin tree, then \( \Vdash P \" T is a Suslin tree " \).

**Observation 8.9.** If \( P \) is constructed by the club method, then it is s.c.c.

Observation 8.9 is not an exact mathematical statement since we have never defined what it means for a forcing set \( P \) to be constructed with the aid of the club method. However, the reader can easily check that in every case in which we applied the club method the resulting forcing set was indeed s.c.c.

**Proof of 8.8.** Let \( P \) and \( T \) be as above and suppose by contradiction that there is \( p^0 \in P \) such that \( p^0 \Vdash \" T is not a Suslin tree " \). Then there is a \( P \)-name \( \tau \) such that \( p^0 \Vdash \tau \) is an uncountable antichain in \( T \). Let \( \langle p_i, q_i | i < \kappa \rangle \subseteq P \times T \) be such that for every \( i < \kappa \), \( p_i \Vdash \tau \), and if \( j \neq i \), then \( q_i \neq q_j \). W.l.o.g. there are \( \{ B_i | i \in \omega, j = 0, 1 \} \) such that \( B_i \subseteq \{ p_i | i < \kappa \} \), for every \( q_i \in B_{\omega}, j = 0, 1 \), and \( q_0 \) and \( q_1 \) are compatible, and for every uncountable \( B \subseteq \{ p_i | i < \kappa \} \) there is \( i \in \omega \) such that \( B \cap B_i \neq \emptyset \) for \( j = 0, 1 \). Let \( A = \{ a_i | i < \kappa \} \) and \( A_j = \{ q_{a_i} | p_a \in B_i \} \). Hence we have the following situation: (1) \( A \) is an uncountable subset of \( T \); (2) if \( i \in \omega \), \( a^0 i \in A_{i0} \) and \( a^1 i \in A_{i1} \), then \( a^0 \) and \( a^1 \) are incompatible in \( T \); and (3) if \( A \) is an uncountable subset of \( A \), then for some \( i \in \omega \), \( A^0 \cap A_{i0}, A^1 \cap A_{i1} \neq \emptyset \).

We will now show that if \( T \) is a Suslin tree, then there are no \( A \) and \( \{ A_j | i \in \omega, j = 0, 1 \} \) as above.

Let \( T \) be a Suslin tree and \( h : T \rightarrow P(\omega) \) such that if \( a < b \), then \( h(a) \subseteq h(b) \). We show that there is \( a \in T \) such that for every \( b > a \), \( h(b) = h(a) \). Suppose not. Let \( \langle a_i, b_i | i < \kappa \rangle \subseteq T \times T \) be such that for every \( i < \kappa \), \( a_i < b_i \) and \( h(a_i) \subseteq h(b_i) \), and if \( i \neq j \), then \( a_i \neq a_j \). W.l.o.g. there is \( n \in \omega \) such that for every \( i < \kappa \), \( n \in h(b_i) \) but \( n \notin h(a_i) \). Let \( i \) and \( j \) be such that \( b_i < b_j \). Hence \( h_i < h_j \) however \( n \in h(b_i) \) and \( n \notin h(A_j) \), contradicting the monotonicity of \( h \).

Returning to our original claim, suppose by contradiction \( T \) is a Suslin tree and \( A_i = \{ A_j | i \in \omega, j = 0, 1 \} \) satisfy (1), (2) and (3) above. Since \( A \) is a Suslin tree, we can assume that \( T = A \). Let \( h : T \rightarrow P(\omega \times \omega) \) be defined as follows: \( h(a) = \{ (b, j) | b \leq a \} \) if there is \( b \leq a \) such that \( b \in A_{j} \). Hence, if \( b = a \), then \( h(b) = h(a) \). Let \( a \) be such that for every \( b > a \), \( h(b) = h(a) \), and \( A^1 = \{ b | b > a \} \). Since \( A^1 \) is uncountable there are \( i \in \omega \) and \( b_i \in A^1 \) such that \( b_i \in A_{i0} \) and \( b_i \in A_{i1} \). Hence \( (i, 0) \in h(b_i) \) and \( (i, 1) \in h(b_i) \). Since \( a < b_i \), there follows \( (i, 0), (i, 1) \in h(a) \), but this means that there are \( a_0, a_1 \leq a \) such that \( a_j \in A_{i0} \), \( j = 1, 2 \), and \( a_0, a_1 \) are comparable, and this contradicts (2). Hence the lemma is proved.

Let \( M \) be the axiom saying that for every s.c.c forcing set \( P \) and every family \( \{ D_i | i < \kappa \} \) of dense subsets of \( P \), there is a filter of \( P \) intersecting every \( D_i \).
Corollary 8.10. (a) \(-\text{SH} + \text{MSA}_\kappa\) is consistent.
(b) \(-\text{SH} + \text{TCA}\) is consistent.

\textbf{Proof.} (a) follows from 8.7 and 8.9, and (b) follows from 8.7, 8.8, 8.9. In fact in
(b) one can replace TCA by any consistent conjunction of axioms whose consistency was proved by the club method. Also in (b) one can add \(2^\kappa \models \mathbb{N}_\omega\). \(\square\)

9. The isomorphizing forcing, and more on the possible structure of \(K\)

The new tool to be presented in this section is the isomorphizing forcing. Given \(A, B \in K\) we construct a forcing set \(P_{A,B}\) which makes \(A\) and \(B\) isomorphic.

Baumgartner [2] constructed a \(P_{A,B}\) as above in order to prove the consistency of BA. However, since our construction is more canonical, it is easier to combine it with the other methods we have presented.

In this section we use \(P_{A,B}\) in order to get (seemingly) a strengthening of BA.

by combining the new technique with other methods we obtain a variety of consistency results on the structure of \(K\).

Let \(\mathcal{G}\) be a partition of \(N_\omega \times N_\omega\). We define the graph \(G_{\mathcal{G}}\). The set of vertices of the graph \(V_{\mathcal{G}}\) is \(\{E \in \mathcal{G} \mid (\exists a, b)(a, b) \in \sigma \text{ and } (a \in E \text{ or } b \in E)\}\). The set of edges is \(\sigma\), and \(E_1, E_2\) which belongs to \(V_{\mathcal{G}}\) are connected by \((a, b) \in \sigma\), if \(a \in E_1\) and \(b \in E_2\) or \(b \in E_1\) and \(a \in E_2\). When \(\mathcal{G}\) is fixed and \(\sigma\) varies, we denote \(G_{\mathcal{G}}\) and \(V_{\mathcal{G}}\) by \(G_\sigma\) and \(V_\sigma\) respectively.

We say that a graph \(G\) is cycle free if it does not contain cycles, i.e. it does not contain a sequence of vertices \(a_1, \ldots, a_n\) and a set of distinct edges \(e_1, \ldots, e_n\) such that \(e_i\) connects \(a_i\) and \(a_{i+1}\) and \(e_n\) connects \(a_n\) and \(a_1\). Let \(C \subseteq N_\omega\) be a club, let \(\mathcal{G}^C\) denote the set of \(C\)-slices and let \(\{E^C_\sigma | \sigma \in \mathcal{G}\}\) be an enumeration of \(\mathcal{G}\) in an increasing order. We regard the set \(E = \{a \mid a < \min(C)\}\) as a \(C\)-slice, hence \(E = E^C_\emptyset\). \(E^C_\sigma\) and \(E^C_\tau\) are near, if for some \(n \in \omega\), \(i + n = j\) or \(j + n = i\). Let \(a \in N_\omega\). \(E^C(a)\) denotes the member of \(\mathcal{G}^C\) to which \(a\) belongs. \(E^C(a)\) is abbreviated by \(E(a)\) when \(C\) is fixed.

Let \(<\) be a linear ordering of a subset of \(N_\omega\), \(C \subseteq N_\omega\) be a club and \(A, B \in N_\omega\). We define \(P = P(C, <, A, B)\):

\[P = \{f \in P_{\mathcal{G}}(A \times B) | f \text{ is an OP function with respect to } <, G^C_\sigma \text{ is cycle free, and if } f(a) = b \text{ then } E^C(a) \text{ and } E^C(b) \text{ are near} \} \]

\(f < g \text{ if } f \subseteq g.\)

\textbf{Theorem 9.1.} Let \(A, B \in K\), and \(M\) be a model such that \(|M| = N_\omega\). There is a linear ordering \(<\) on \(N_\omega\) definable in \(M\) such that \((N_\omega, <)\) is embeddable in \((\mathbb{R}, <)\), and \((A \cup B, <)\) is embeddable in \((N_\omega, <)\). w.l.o.g \(A, B \subseteq N_\omega\), and \(N_\omega \text{ and the usual linear ordering } < \text{ of } N_\omega \text{ are definable in } M\). Let \(C\) be \(M^\omega\)-thin, and suppose further that for every \(C\)-slice, \(E\), \((A \cap E, <)\) and \((B \cap E, <)\) are dense in \((A, <)\) and \((B, <)\) respectively. Then (a) \(P = P(C, <, A, B)\) is c.c.c.; and (b) \(\models A \equiv B\).
Remarks. (a) Baumgartner [2] proved a similar theorem but he used (CH).

(b) Note that we did not have to assume that A and B are definable in M.

Proof. (b) Let $f \in P$ and $a \in A - \text{Dom}(f)$. We show that there is $g \equiv f$ such that $a \in \text{Dom}(g)$. $V_f$ is finite, hence there is a C-slice $E$ such that $E \notin V_f$, $E \neq E(a)$, and $E$ and $E(a)$ are near. Since $B$ is dense in itself and $B \cap E$ is dense in $B$, there is $b \in B \cap E$ such that $g \equiv f \cup \{(a, b)\}$ is OP. By the choice of $E$, $g \in P$, hence $g$ is as required. Similarly if $b \in B - \text{Rng}(f)$, then there is $g \in P$ such that $b \in \text{Rng}(g)$. This proves (b).

(a) If $V$ and $W$ are sets of pairs of real numbers we say that $(V, W)$ is OP if for every $(\langle a, 0 \rangle, \langle w, b \rangle) \in V$ and $\langle \langle v, 0 \rangle, \langle w, 0 \rangle \rangle \in W$, $\langle \langle b, v \rangle, \langle w, 0 \rangle \rangle$ is also in $V$. Analogously we define the notion $(V, W)$ is order reversing (OR). Note that if $U_i, i = 0, \ldots, 3$, are pairwise disjoint intervals, then $\langle U_0 \times U_1, U_2 \times U_3 \rangle$ is either OP or OR.

Let $\{f_\alpha \mid \alpha < \kappa_i\} = F_\alpha \subseteq P$. As in the previous cases we uniformize $F_\alpha$ as much as possible. We thus assume that $F_\alpha$ is a $\Delta$-system, and it will suffice to deal with the case when the kernel of $F_\alpha$ is empty. Hence let us assume that $f_\alpha = \{(\langle a(\alpha, 0), a(\alpha, 1) \rangle, \ldots, \langle a(\alpha, 2n - 1), a(\alpha, 2n - 1) \rangle)\}$ where the $a(\alpha, 2i)$'s are distinct, and if $\alpha, \beta, \gamma \in \text{Dom}(f_\alpha) \cup \text{Rng}(f_\alpha)$ and $d \in \text{Dom}(f_\beta) \cup \text{Rng}(f_\beta)$, then $E(c) \neq E(d)$. This last condition assures that if $f_\alpha \cup f_\beta$ is OP, then $f_\beta \cup f_\beta \in P$.

Let $a(\alpha) = (a(\alpha, 0), \ldots, a(\alpha, 2n - 1))$, $F = \{a(\alpha) \mid a < \kappa_i\}$ and $F$ be the topological closure of $F_i$ in $(\langle \kappa_i^+, \kappa_i \rangle)^{2^{n}}$. It will be convenient (however not necessary) to assume that all the $a(\alpha, i)$'s are distinct, hence w.l.o.g. we assume that $A \cap B = \emptyset$. Let $D \in \mathcal{M}$ be such that $F$ is definable from $D$ in $M^*$, and there is some countable open base of $(\kappa_i^+, \kappa_i)$ consisting of intervals whose elements are definable from $D$ in $M^*$. Let $\gamma_0 \in \mathcal{C}$ be such that $\mathcal{C} \cap [\gamma_0, \kappa_i) \subseteq \{\alpha \mid (\exists \mathcal{N} \prec M) (D \in \mathcal{N})$ and $\mathcal{N} \cap \kappa_i = \alpha\}$. Let $f_\alpha = f$ be such that for every $i < 2n, \gamma_i \equiv a(\alpha, i)$. We denote $a(\alpha, i)$ by $a(i), a(\alpha) = a$ and $W = \text{Dom}(f_\alpha) \cup \text{Rng}(f_\alpha)$. Let $E, \ldots, E^{k-1}$ be the set of C-slices which intersect $W$, arranged in an increasing order. Let $x = (x(0), \ldots, x(2n - 1))$ be a sequence of variables. For every $s < k$, let $R_s = \{i \mid a(i) \in E^s\}$, $r_s = a \upharpoonright R_s$ and $x_s = x \upharpoonright R_s$. Hence $\bigcup_{s < k} a_s = a$ and $\bigcup_{s < k} x_s = x$.

We are now ready for the duplication argument. We define by a downward induction on $s = k, \ldots, 0$ formulas $\rho_{s}(x_0, \ldots, x_{s-1})$ such that the only parameter in $\rho_{s}$ is $D$, and $M^* \models \rho_{s}[a_0, \ldots, a_{s-1}]$. Let $\phi_k = \bigcup_{s < k} \rho_{s} \subseteq F$. Suppose $\phi_{s+1}$ has been defined and we define $\phi_s$. Clearly by our assumptions

$$M^* \models (\exists x_0, x_1) \left( \text{Rng}(x_0) \cap \text{Rng}(x_1) = \emptyset \land \bigwedge_{i=0}^{s} \varphi_{s+1}(a_0, \ldots, a_{s-1}, x_i) \right).$$

For every $i \in R_s$ and $i = 0, 1$, let $U^i_i$ be an interval definable from $D$ such that: (1) the $U^i_i$'s are pairwise disjoint; (2) if $i, i' \in R_s$ and $i' \not\in \{0, 1\}$, then $U^i_i \cap U^i_{i'} = \emptyset$; and

$$(3) \quad M^* \models (\exists x_0, x_1) \left( \bigwedge_{i=0}^{s} \left( x_i \in \bigcap_{i \in R_s} U^i \right) \land \bigwedge_{i=0}^{s} \varphi_{s+1}(a_0, \ldots, a_{s-1}, x_i) \right).$$
Let
\[ \varphi_2 = (\exists x_2^2, x_3^2) \left( \bigwedge_{i=0}^{1} \left( x_i^r \in \bigcup_{i=1}^1 U_i^r \right) \land \bigwedge_{i=0}^{1} \varphi_{i+1}(x_0, \ldots, x_{i-1}, x_i) \right). \]

We prove that we can find for every \( s < k \), \( l(s) \in \{0, 1\} \) such that for every \( i < n \): if \( 2i \in R_i \) and \( 2i + 1 \in R_i \), then \( \langle U_2^{(0)} \times U_1^{(0)}, U_2^{(1)} \times U_1^{(1)} \rangle \) is OP. Recall that the graph \( G_i \) has vertices \( E^{(0)}, \ldots, E^{(k)} \); the edges of \( G_i \) are \( (a(0), a(1)), \ldots, (a(2n-2), a(2n-1)) \); \( E^{(0)} \) is connected to \( E^{(i)} \) by \( (a(2i), a(2i + 1)) \) if \( a(2i) \in E^{(i)} \) and \( a(2i + 1) \in E^{(i+1)} \) or \( a(2i) \in E^{(i+1)} \) and \( a(2i + 1) \in E^{(i)} \); and \( G_i \) is cycle free. Let \( S \subseteq k \) be such that for every component \( T \) of \( G_i \) there is a unique \( s \in S \) such that \( E^{(s)} \in T \). Let \( S_i = \{ s \in k \mid \text{there is } t \in S \text{ and a 1-2 path in } G_i \text{ of length } j \text{ connecting } E^{(s)} \text{ with } E^{(t)} \} \).

Since \( G_i \) is cycle free the \( S_i \)s are pairwise disjoint and moreover for every \( i \in S_{i+1} \) there is a unique edge in \( G_i \) which connects \( E^{(i)} \) with some element of \( \{ E^{(s)} \mid s \in S_i \} \).

For every \( s \in S_0 \) define \( l(s) = 0 \). Suppose \( l(s) \) has been defined for every \( s \in \bigcup_{s \leq n} S_m \). Let \( t \in S_{i+1} \); let \( (a(2i), a(2i + 1)) \) be the unique edge connecting \( E^{(i)} \) to an element of \( \{ E^{(s)} \mid s \in S_i \} \), and w.l.o.g. suppose that \( s \in S_0 \) and \( a(2i) \in E^{(i)} \) and \( a(2i + 1) \in E^{(i+1)} \). Define \( l(t) \) in such a way that \( \langle U_2^{(0)} \times U_1^{(0)}, U_2^{(1)} \times U_1^{(1)} \rangle \) will be OP. We have defined \( l(s) \) for every \( s \in k \), and it is easy to check that \( \{ l(s) \mid s \in k \} \) is as required.

Using the \( \varphi_i \)s we will now construct two members of \( F \). Since \( M^{(0)} \models \varphi_0 \), there is \( b_0 \) such that \( M^{(0)} \models b_0 \in \prod_{i=0}^{n} U_{i+1}^{(0)} \land \varphi_1(b_0) \). Suppose \( b_0, \ldots, b_{i-1} \) have been defined in such a way that \( M^{(0)} \models \varphi_i(b_0, \ldots, b_{i-1}) \); hence by the definition of \( \varphi_i \) there is \( b_i \) such that \( M^{(0)} \models b_i \in \prod_{i=0}^{n} U_{i+1}^{(0)} \land \varphi_{i+1}(b_0, \ldots, b_i) \). According to this definition we obtain \( b_0, \ldots, b_{i-1} \) such that \( \bigcup_{s \leq k} b_s \in F \) (this is assured by \( \varphi_0 \)) and for every \( s < k \), \( b_s \in \prod_{i=0}^{n} U_{i+1}^{(0)} \).

Similarly we can define \( b_1, s < k \), such that \( \bigcup_{s \leq k} b_s \in F \) and \( b_1 \in \prod_{i=0}^{n} U_{i+1}^{(0)} \).

For \( i = 0, 1 \), let \( b^i = \bigcup_{s \leq k} b_s \) and \( b^i = (b^i(0), \ldots, b^i(2n - 1)) \). By the construction, for every \( i < n \), \( \{ (b^i(2i), b^i(2i + 1)), (b^i(2i + 1), b^i(2i)) \} \) is OP. Since \( F = \cl(\langle a(\beta) \mid \beta \in \gamma \rangle) \), there are \( \beta, \gamma \) such that for every \( s < k \),
\[ a(\beta) \upharpoonright R_s \in \prod_{i=0}^{n} U_{i+1}^{(0)} \quad \text{and} \quad a(\gamma) \upharpoonright R_s \in \prod_{i=0}^{n} U_{i+1}^{(0)}. \]

Thus for every \( i < n \), \( \{ (a(\beta, 2i), a(\beta, 2i + 1)), (a(\gamma, 2i), a(\gamma, 2i + 1)) \} \) is OP. By the method \( F_0 \) was uniformized. If \( i \neq 0 \), then \( \{ (a(\beta, 2i), a(\beta, 2i + 1)), (a(\gamma, 2i), a(\gamma, 2i + 1)) \} \) is OP. Hence \( f_s \cup f_r \) is OP, and by the uniformization its graph is cycle free, so \( f_s \in f_r \in P \). This concludes the proof of 9.1. \( \square \)

Let \( A = \{ A_i \mid i < \aleph_0 \} \subseteq K \) be a family of pairwise disjoint sets such that for every \( i < \aleph_0 \), \( A_i \) is dense in \( \bigcup_{j < \aleph_0} A_j \). Let \( M(A) \) be a model whose universe is \( A \upharpoonright \bigcup_{j < \aleph_0} A_j \). \( M(A) \) has a binary relation which denotes the linear ordering which \( A \) inherits from \( R_i \), and it has unary predicates which denote \( A_i \). A model of the above form is called a \( K \)-shuffle.

**Axiom BAL.** Every two \( K \)-shuffles are isomorphic.

**Theorem 9.2.** \( MA + BAL \uplus 2^{\aleph_0} \models \) \( \aleph_0 \) is consistent.
Proof. The proof follows from the methods developed so far. We start with a universe $V_0 \models \text{GCH}$ and construct first a universe $V \supseteq V_0$ in which $A_1$ holds and $2^{\aleph_0} = \aleph_2$. Then step by step we isomorphize all pairs of $K$-shuffles. We thus have to show the following claim.

Claim (A1). Let $M(A)$ and $M(B)$ be $K$-shuffles, then there is a c.c.c. forcing set $P_{A,B}$ of power $\aleph_1$ such that $\Vdash_{P_{A,B}} M(A) \equiv M(B)$.

Proof. Let $M$ be a model whose universe is $\aleph_1$ and which encodes $A$ and $B$. Let $C$ be $M^* \text{-thin}$. Let $\Lambda = \bigcup_{i < \omega} A_i$ and $B = \bigcup_{i < \omega} B_i$. Let $P = \{ f \in P_{\aleph_0}(A \times B) \mid f \text{ is an OP function, for every } i < \omega, f(A_i) \subseteq B_i, \text{ and } f^{-1}(B_i) \subseteq A_i \}, G^P$ is cycle free, and if $f(a) = b$, then $E^C(a)$ and $E^C(b)$ are near. $f \leq g$ if $f \subseteq g$. It follows from the proof of 9.1 that $P_{A,B}$ is as required. □

Question 9.3. Prove that $\text{BA} \Rightarrow \text{BA1}$.

One can ask whether $\text{BA}$ can be strengthened to say that every two members of $K$ are isomorphic by a differentiable OP function. This strengthening is inconsistent with ZFC.

Proposition 9.4. There are $A, B \in K$ dense in $\mathbb{R}$, such that for every uncountable 1-1 function $f \subseteq A \times B$ there is $a \in \text{Dom}(f)$ such that every neighborhood of $a$ contains uncountably many elements of $\text{Dom}(f)$ and

$$\lim_{\text{Dom}(f) \ni a \to a} \frac{f(b) - f(a)}{b - a} \text{ does not exist.}$$

Proof. For $r \in \mathbb{Q}$ we construct $A_r, B_r \subseteq \mathbb{R}$ such that: (1) $A_r, B_r$ are countable and contain $r$; (2) $A_r, B_r$ have the order type of the rationals; and (3) for every $M$ there is $\delta > 0$ such that for every $r, s \in \mathbb{Q}$ and for every $(a_1, b_1), (a_2, b_2) \in A_r \times B_r$ if $a_1 \neq a_2$ and $b_1 \neq b_2$ and $|a_1 - a_2|, |b_1 - b_2| < \delta$, then

$$\frac{|b_2 - b_1|}{|a_2 - a_1|} > M \text{ or } \frac{|a_2 - a_1|}{|b_2 - b_1|} > M.$$

The construction of such a system of sets is done inductively, and if we define $A = \bigcup_{r \in \mathbb{Q}} \text{cl}(A_r)$ and $B = \bigcup_{r \in \mathbb{Q}} \text{cl}(B_r)$, then it is easy to see that $A$ and $B$ are as required. □

Question 9.5. Is it consistent that there is $A \in K$ such that every two dense subsets of $A$ which belong to $K$ are isomorphic by a differentiable function?

So far we have presented several techniques for constructing forcing sets. It seems that there is a large group of consistency results concerning the structure of
K that can be proved using the methods presented. However, we did not try to
find an exact formulation to the scope of consistency results that can be proved
using these methods. Instead we bring several examples how the various methods
can be combined to yield universes in which K has quite a diverse range of
properties.

It seems to us that the techniques that have been presented so far, suffice in
order to prove any consistency result about the structure of K which is consistent
with MA and $2^\kappa=\aleph_2$. However, we did not make an attempt to formulate what
are exactly those consistency results about K which can be proved by our models.
In the sequel we prove some consistency results in which we apply the previous
methods, and which hopefully exemplify the power of the above methods.

Let $A \perp B$ mean that there is no $C \in K$ such that $C \preceq A$ and $C \preceq B$, in this case
we say that $A$ and $B$ are far; $A \parallel B$ denotes that $A \perp B$ and $A \perp B^*$, and this case
we say that $A$ and $B$ are monotonically far (M-far).

Suppose that $G \parallel H$, and that we want to isomorphize $A$ and $B$ without
destroying the M-farness of $G$ and $H$. In the following lemma we show how this
can be done.

**Lemma 9.6** (A1). Let $\lambda < 2^{\kappa_1}$; for every $i < \lambda$ let $G_i, H_i \in K$ be such that $G_i \parallel H_i$.
Let $A, B \in K$ be such that for every $i < \lambda$, $A \parallel G_i$ and $B \parallel G_i$. Then there is a c.c.c.
forcing set $P$ of power $\aleph_1$ such that $\Vdash P A \equiv B$, and for every $i < \lambda$, $\Vdash_P G_i \parallel H_i$ if $G_i$ is
increasing, then $\Vdash_P "G_i is increasing",$ and if $G_i$ is 2-entangled then $\Vdash_P "G_i is
2-entangled"$.

**Proof.** We first construct a model which encodes all the information we need. Let
$h : A \cup B \rightarrow K$ be a 1-1 function, and for every $i < \lambda$, let $h_i : A \cup B \cup G_i \cup H_i \rightarrow K_i$
be a 1-1 onto function containing $h$. Let $M$ be the following model: $|M| = K \cup \lambda$;
$M$ has a three-place relation $R = \{(i, \alpha, \beta) \mid h_i^{-1}(\alpha) < h_i^{-1}(\beta)\}$, we denote $\alpha < \beta$ to
mean that $(i, \alpha, \beta) \in R$; $M$ has unary predicates which represent $h(A)$ and $h(B)$; and
finally $M$ has the binary relations $S_G = \{(i, \alpha) \mid \alpha \in h_i(G_i)\}$ and $S_H = \{(i, \alpha) \mid \alpha \in h_i(H_i)\}$.

Let $\alpha <^0 \beta$ denote that $h_i^{-1}(\alpha) < h_i^{-1}(\beta)$, hence $<^0$ is definable in $M$. Let $C$ be
$M^\sigma$-thin, and let $P = P(C, <^0, A, B)$ be as defined in 2.1. By 9.1, $P$ is c.c.c., and it
isomorphizes $A$ and $B$. We next show that for every $i$, $\Vdash_P G_i \parallel H_i$. Suppose by
contradiction $p \Vdash_P \neg(G_i \perp H_i)$. We denote $G = G_0, H = H_0$ and $<^0 < i$. By abuse
of notation we assume $A \cup B \cup G \cup H = K_i$. Let $\tau$ be a $P$-name such that $p \Vdash_P "\tau is an
uncountable OP function and $\tau \in G \times H$. W.l.o.g. $p = 0$. Let
$(\langle f_\alpha, \langle a_\alpha, b_\alpha \rangle \rangle)_{\alpha < \kappa_1}$ be such that for every $\alpha$, $f_\alpha \Vdash_P \langle a_\alpha, b_\alpha \rangle \in \tau$, and $\alpha \neq \beta \Rightarrow
\langle a_\alpha, b_\alpha \rangle \neq \langle a_\beta, b_\beta \rangle$. We will reach a contradiction if we find $\alpha$ and $\beta$ such that
$f_\alpha \cup f_\beta \in P$, but $(\langle a_\alpha, b_\alpha \rangle, \langle a_\beta, b_\beta \rangle)$ is not OP. We uniformize $(\langle f_\alpha, \langle a, b \rangle \rangle)_{\alpha < \kappa_1}$ as
in 9.1, hence we denote $f_\alpha = \langle (a(\alpha, 0), a(\alpha, 1)), \ldots, (a(\alpha, 2n-2), a(\alpha, 2n-1)) \rangle$, and we denote $a_\alpha = a(\alpha, 2n)$ and $b_\alpha = a(\alpha, 2n+1)$. W.l.o.g. all the $a(\alpha, 1)$'s are
distinct. Let $a(\alpha) = \langle (a(\alpha, 0), \ldots, a(\alpha, 2n+1)) \rangle$, $F = \langle a(\alpha) \mid \alpha < \kappa_1 \rangle$ and let $F$ be
the closure of $F_i$ in $((N_1, <))^2n+2$. We define $D$, $\gamma_0$, $a(t)$, $a$, $W$ etc. as in 9.1. In the duplication argument we distinguish between two cases.

Case 1. $R'(a(2n)) = R'(a(2n+1))$. Let $\nu$ be such that $R'(a(2n)) = R'(a(2n+1))$. We define $\varphi_\nu$ inductively as in 9.1, except in the case when $v = s$. Suppose $\varphi_{s+1}$ has been defined.

\[ M^s \models (\forall \alpha < \kappa) (\exists x_\alpha) (\text{Rng}(x_\alpha) > \alpha \land (x_{2n} \in G) \land (x_{2n+1+1} \in H) \land \varphi_{s+1}(a_0, \ldots, a_{s-1}, x_\alpha)). \]

Since $G \not\models H$,

\[ M^s \models (\exists x_\alpha, \ldots, x_{s+1}) (\text{Rng}(x_\alpha) \cap \text{Rng}(x_{s+1}) = \emptyset \land \bigwedge_{i=0}^{s} (x_{2n+i} \in G) \land \bigwedge_{i=0}^{s} (x_{2n+1+i} \in H) \land (\langle x_{2n}, x_{2n+1}, \ldots, x_{s+1} \rangle, \langle x_{2n}, x_{2n+1}, \ldots, x_{s+1} \rangle) \text{ is OR} \land \bigwedge_{i=0}^{s} \varphi_{s+1}(a_0, \ldots, a_{s-1}, x_\alpha)). \]

For every $i \in R_1$ and $l = 0, 1$, let $U^l_1$ be an interval definable from $D$ such that all the $U^l_1$s so far defined are pairwise disjoint and

\[ M^s \models (\exists x_\alpha, \ldots, x_{s+1}) \left( \bigwedge_{i=0}^{s} x^l_i \in \bigcup_{i \in R_1} U^l_1 \land (\langle x_{2n}, x_{2n+1}, \ldots, x_{s+1} \rangle) \text{ is OR} \land \bigwedge_{i=0}^{s} \varphi_{s+1}(a_0, \ldots, a_{s-1}, x_\alpha)). \]

Let $\varphi_\nu(x_0, \ldots, x_{s+1})$ be the formula obtained from the above formula by substituting $a_i$ by $x_i$ for every $s < \nu$.

As in 9.1, we can find for every $s < k$, $I(s) \in \{0, 1\}$ such that for every $l < n$: if $2i \in R_1$, $2i+1 \in R_2$, then $\langle U_{2n}^i \times U_{2n+1}^j \rangle$, $U_{2n}^{i+1} \times U_{2n+1}^{j+1}$ is OP. We continue as in 9.1 and find $\beta, \gamma$ such that for every $s < k$

\[ a(\beta) \upharpoonright R_1 \in \prod_{i \in R_1} U^i_1 \land a(\gamma) \upharpoonright R_2 \in \prod_{i \in R_2} U^{i-1}_1. \]

It follows that $f_\beta \cup f_\gamma \in P$ and that $\langle a_0, b_0 \rangle, \langle a_\nu, b_\nu \rangle$ is OR. A contradiction.

Case 2. $R'(a(2n)) \neq R'(a(2n+1))$. Let $R'(a(2n)) = R'(a(2n+1)) = R'w$.

Case 2.1: $R'w$ and $R''w$ are not in the same component of $G_f$. In this case we define $\varphi_\nu$, $s < k$, exactly as in 9.1. Let $S_0$ be a set such that $v, w \in S_0$, and for every component $L$ of $G_f$, $|S_0 \cap \{ s \mid R'(s) \in L \}| = 1$. We define $S_0$ as in 9.1. Next we define $I(s)$ for every $s \in S_0$. For every $s \in S_0 - \{ w \}$, let $I(s) = 0$. We define $I(w)$ to be equal to 0 or 1 according to whether $\langle U_{2n}^i \times U_{2n+1}^j \rangle$, $U_{2n}^{i+1} \times U_{2n+1}^{j+1}$ is OR or OP. We now define $I(s)$ for $s \in S_0$ by induction on $i$ as in 9.1. Let $\beta, \gamma < k$, be such that for every $s < k$, 

\[ a(\beta) \upharpoonright R_1 \in \prod_{i \in R_1} U^i_1 \land a(\gamma) \upharpoonright R_2 \in \prod_{i \in R_2} U^{i-1}_1. \]

It is easy to see that $f_\beta \cup f_\gamma \in P$ and that $\langle a_0, b_0 \rangle, \langle a_\nu, b_\nu \rangle$ is OR. A contradiction.

Case 2.2: $R''w$ and $R''w$ are in the same component of $G_f$. Let $v = v_0, v_1, \ldots, v_i = w$ be such that $B^w_0, \ldots, B^w_i$ is the unique path in $G_f$ connecting $E^w$
and \(E^x\). By the symmetry between the roles of \(A\) and \(B\) we can assume that \(E^v\) and \(E^w\) are connected by \(\langle a(2i), a(2j + 1) \rangle\) where \(a(2i) \in E^v\) and \(a(2j + 1) \in E^w\).

We define \(\varphi_s\) for \(s \leq k\) inductively. \(\varphi_0\) is defined as in 9.1. If \(s \neq 0\), then \(\varphi_s\) is defined from \(\varphi_{s+1}\) as in 9.1. Suppose \(\varphi_{v+1}\) has been defined, and we define \(\varphi_v\).

\[M^v = \{ \langle x, y \rangle, (a_0, \ldots, a_{s-1}, x) \wedge x_{2j} \in A \wedge x_{3j} \in G \}.\]

Since \(A \cup G\), for \(Z = P\) and \(Z = R\):

\[M^v = \exists x_{2j}, x_{3j} \left( \bigwedge_{0}^{1} \varphi_{v+1}(a_0, \ldots, a_{s-1}, x) \wedge ((x_{2j}, x_{3j}), (x_{2j}, x_{3j})) \in OZ \right).\]

Let \(U_1^{1,\varphi}\), \(I = 0, 1\), \(Z = P, R\) and \(i \in \mathbb{R}_s\) be pairwise disjoint open intervals disjoint from the previously defined \(I\)'s definable from \(D\) such that:

1. \((U_1^{1,\varphi}, U_2^{1,\varphi}, U_3^{1,\varphi}, U_4^{1,\varphi})\) is \(OZ\); and

2. \(M^v \wedge \bigwedge_{i \in \mathbb{R}_s} \exists x_{2j}, x_{3j} \left( x_{2j} \in \prod \cup U_1^{1,\varphi} \wedge \varphi_{v+1}(a_0, \ldots, a_{s-1}, x_{2j})\right).\)

Let \(\varphi_0\) be the formula obtained from the above formula by replacing each \(a_i\), \(s < v\), by \(x_i\). This concludes the definition of the \(\varphi_s\)'s.

Our next goal is to define \(l(s)\) for every \(s < k\). In fact we have also to decide whether to duplicate the \(U_1^{1,\varphi}\)'s or the \(U_1^{1,\varphi}\)'s. Let \(T = \{ s \mid E^s \text{ and } E^w \text{ are connected in } G_s \}\). We define \(l(s)\) for \(s \in k - T\) as in 9.1. So it remains to define \(l(s)\) for \(s \in T\). Let \(S_0 = \{ \omega \}\), and we define \(S_i\) inductively as in 9.1. Note that \(v \in S\). For \(t < r\) we define \(l(s)\) for \(s \in S_t\), as in 9.1. Let \(Z^0\) be defined as follows. If \((U_1^{1,\varphi}, U_2^{1,\varphi}, U_3^{1,\varphi}, U_4^{1,\varphi})\) is \(O\), then \(Z^0 = P\), and if the above pair of sets is \(OR\), then \(Z^0 = R\). We denote each \(U_1^{1,\varphi}\) by \(U_t\) and proceed in the definition of \(l(s)\) as in 9.1. Let \(\beta\) and \(\gamma\) be such that for every \(s < k\):

\[a(\beta) \upharpoonright R_s \in \prod_{i \in \mathbb{R}_s} U_1^{1,\varphi} \quad \text{and} \quad a(\gamma) \upharpoonright R_s \in \prod_{i \in \mathbb{R}_s} U_1^{1,\varphi}.\]

By the proof of 9.1, \(f_{a} \cup f_{b} \in P\). We check that \(\langle a_{\alpha}, b_{\alpha} \rangle, \langle a_{\beta}, b_{\beta} \rangle\) is \(OR\). By the construction of \(l(s)\), \((U_2^{1,\varphi}, U_3^{1,\varphi}, U_4^{1,\varphi}, U_2^{1,\varphi}, U_3^{1,\varphi}, U_4^{1,\varphi})\) is \(OR\). \((U_2^{1,\varphi}, U_3^{1,\varphi}, U_4^{1,\varphi}, U_2^{1,\varphi}, U_3^{1,\varphi}, U_4^{1,\varphi})\) was chosen to be \(OR\) or \(OP\) according to whether \((U_2^{1,\varphi}, U_3^{1,\varphi}, U_4^{1,\varphi}, U_2^{1,\varphi}, U_3^{1,\varphi}, U_4^{1,\varphi})\) was \(OP\) or \(OR\). Since the composition of an \(OP\) and an \(OR\) function is \(OR\) it follows that \((U_2^{1,\varphi}, U_3^{1,\varphi}, U_4^{1,\varphi})\) is \(OR\). Since \(\langle a_{\alpha}, b_{\alpha} \rangle \in U_2^{1,\varphi}, U_3^{1,\varphi}, U_4^{1,\varphi}\) and \(\langle a_{\beta}, b_{\beta} \rangle \in U_2^{1,\varphi}, U_3^{1,\varphi}, U_4^{1,\varphi}\) it follows that \(\langle a_{\alpha}, b_{\alpha} \rangle, \langle a_{\beta}, b_{\beta} \rangle\) is \(OR\). Hence we reach a contradiction again.

The proof that, if \(G_i\) is 2-entangled in \(V\), then it remains 2-entangled after forcing with \(P\), is very similar to the above proof. So is the proof that \(G_i\) remains an increasing set, if it was increasing in \(V\). □

**Lemma 9.7 (A1).** Let \(\lambda < 2^{\kappa}\); for every \(i < \lambda\) let \(G_i, H_i \in K\) be such that \(G_i \perp H_i\). Suppose \(Q\) is a c.c.c. forcing set such that \(\models Q - (G_0 \perp H_0)\). Then there is a c.c.c. forcing set \(R\) of power \(\kappa_1\) such that \(\models R (Q\) is not c.c.c. \(\wedge (\forall i < \lambda) (G_i \perp H_i)\).
Proof. This is an instance of the explicit contradiction method, and it is very similar to Claim 1 in 8.5. So we omit the details.

Let $M$ be a model encoding $G_0, H_0$ as in 9.6, and let $C$ be $M^n$-thin. Let $\tau$ be a $Q$-name such that $\Vdash_{Q} \neg \exists f$ is an uncountable OP function and $\tau \in M^n$ as in 9.6. Let

$$\{ (q_\alpha, a(\alpha, 0, 0), a(\alpha, 0, 1), a(\alpha, 1, 0), a(\alpha, 1, 1)) | \alpha < \aleph_1 \}$$

be a sequence with the following properties: denote

$$a(\alpha, l) = (a(\alpha, 0, 0), a(\alpha, 0, 1), a(\alpha, 1, 0), a(\alpha, 1, 1)) \quad \text{and} \quad a(\alpha) = a(\alpha, 0) \wedge a(\alpha, 1);$$

then (1) $q_\alpha \Vdash \bigwedge_{l=0}^{1} a(\alpha, l) \in \tau$, and (2) if $\langle \alpha, l \rangle \neq \langle \beta, m \rangle$, then

$$\{ E^n(a(\alpha, l, 0)), E^n(a(\alpha, l, 1)) \cap \{ E^n(a(\beta, m, 0)), E^n(a(\beta, m, 1)) \} = \emptyset. \}

Let $\alpha, \beta < \aleph_1$. We say that $q_\alpha, q_\beta$ are explicitly contradictory if there is $l(\alpha, \beta) = l \in \{ 0, 1 \}$ such that $(a(\alpha, l), a(\beta, l))$ is OR. Clearly if $q_\alpha$ and $q_\beta$ are explicitly contradictory, then they are incompatible in $Q$. Let $R = \{ \sigma \in P_{\aleph_0}(\aleph_1) \} \text{ for every distinct } \alpha, \beta < \aleph_1, q_\alpha \text{ and } q_\beta \text{ are explicitly contradictory} \} \sigma \leq \eta \text{ if } \sigma \subseteq \eta. \text{ The proof that } R \text{ is as required is very similar to the proof of claim 1 in 8.5, hence we leave it to the reader. }$}

In the next two lemmas we make the preparation for the use of the tail method under the assumption A1. Let $A \subseteq B$ denote that $|A - B| = \aleph_0$.

Lemma 9.8 (A1). Let $\lambda < 2^{\aleph_0}$; for every $i < \lambda$ let $C_i \subseteq \aleph_1$ be a club. Then there is a club $C \subseteq \aleph_1$ such that $C \subseteq C_i$ for every $i < \lambda$.

Proof. Let $M = (\lambda, <, R)$ where $R = \{ (\alpha, i) | \alpha \in C_i \}$. It is easy to see that if $C$ is $M$-thin, then for every $i < \lambda$, $C \subseteq C_i$. Q.E.D

Lemma 9.9 (A1). Let $\lambda < 2^{\aleph_0}$; for every $i < \lambda$ let $A_i = \{ a(\alpha, i) | \alpha < \aleph_1 \} \in K$, where $\{ a(\alpha, i) | \alpha < \aleph_1 \}$ is a $1$-$1$ enumeration of $A_i$, and let $A \in K$. Then there is $B \subseteq A$ and for every $i < \lambda$ a club $C_i \subseteq \aleph_1$ such that $B \in K$ and is dense in $A$, and if $B_i = \{ a(\alpha, i) | \alpha \in C_i \}$, then $B_i$ is dense in $A_i$, $B \cap B_i$, and $B$ is $2$-entangled.

Proof. Let $\{ a(\alpha, i) | \alpha < \aleph_1 \}$ be a $1$-$1$ enumeration of $A$; let $h : (\lambda + 1) \times \aleph_1 \rightarrow \aleph_1$ be defined as follows: $h(i, \alpha) = a(i, \alpha)$. Let $R = \{ (i, \alpha, \beta) | a(i, \alpha) < a(i, \beta) \}$. Let $M = (\lambda + 1, <, h, R)$, and let $C$ be $M^n$-thin. Let $\{ E_\alpha | \alpha < \aleph_1 \}$ be an enumeration of $\mathcal{C}$ in an increasing order. Let $D \subseteq \aleph_1$ be a club such that $|\aleph_1 - D| = \aleph_1$. For every $i < \lambda$ let $C_i = \bigcup \{ E_\alpha | \alpha \in D \}$ and $B_i = \{ a(i, \alpha) | \alpha \in C_i \}$. Clearly $C_i$ is a club, $B_i \in K$ and is dense in $A_i$. It is easy to find $B \subseteq A$ with the following properties:

1. if $a(\alpha, i), a(\alpha, \beta) \in B$ and are distinct, then $E^n(a(\alpha, i)) \neq E^n(a(\alpha, \beta))$ and $\alpha \notin \bigcup \{ E_\gamma | \gamma \in D \}$; and

2. $B \subseteq K$ is and dense in $A$.

Suppose by contradiction $f \subseteq B \times B_i$ is an uncountable monotonic function. Let
The consistency of some partition theorems

\[ f' = \{ (\alpha, \beta) \mid (h(\lambda, \alpha), h(i, \beta)) \in f \}, \quad F = \text{cl}(f) \text{ and } F' = \{ (\alpha, \beta) \mid (h(\lambda, \alpha), h(i, \beta)) \in F \}. \]

There is \( d \in \mathcal{M}^c \) such that \( F \) is definable from \( d \). Let \( \gamma_0 < \kappa_1 \) be such that for every \( \alpha \geq \gamma_0 \), if \( \alpha \in C \), then there is \( N < \mathcal{M}^c \) such that \( d \in \mathcal{N} \) and \( \mathcal{N} \cap \kappa_1 = \alpha \). Let \( \langle \alpha, \beta \rangle \in f' \) and \( \gamma_0 \leq \alpha, \beta \). Without loss of generality, \( \alpha < \beta \). By the definition of \( B \) and \( C \), there is \( \gamma \in C \) such that \( \alpha < \gamma \leq \beta \). Let \( N < \mathcal{M}^c \) be such that \( d \in \mathcal{N} \) and \( \mathcal{N} \cap \kappa_1 = \gamma \). Hence \( \alpha \in \mathcal{N} \backslash \beta \). However, \( |F'(\alpha)| < 2 \) hence \( \beta \) is definable from \( \alpha \) and \( d \), and thus \( \beta \in \mathcal{N} \), a contradiction.

The 2-entangledness of \( B \) is proved similarly.

Let \( A \in K \). \( A \) is into rigid (I-rigid) if there is no monotonic \( f: A \rightarrow A \) other than the identity. Let \( \text{RHA} \) be the following axiom:

**Axiom RHA.** For every \( A \in K \) there are \( B, C \subseteq A \) such that \( B, C \in K \) and are dense in \( A \), \( B \) is I-rigid and \( C \) is homogeneous.

Note that \( \text{RHA} \Rightarrow \neg \text{CH} \).

**Theorem 9.10.** Let \( V \models \text{CH} \), and \( \lambda \) be a regular cardinal in \( V \) satisfying \( \lambda^{\aleph_1} = \lambda \) and \( \sum_{\alpha < \lambda} 2^\alpha = \lambda \). Then there is a forcing set \( P \in V \) such that \( V^P (2^{2^\lambda} = \lambda) \land \text{MA} \land \text{RHA} \land \text{NA} \).

**Remark.** The assumption that \( \sum_{\alpha < \lambda} 2^\alpha = \lambda \) is needed just for \( \text{MA} \) and not for \( \text{RHA} \) or \( \text{NA} \).

**Proof.** Let \( V_0 \models V \) and \( V_0 \models (2^{2^\lambda} = \lambda) \land A1 \). The construction, of such \( V_0 \) is done in 5.4. Let \( \{ \tau_i \mid i < \lambda \} \) be an enumeration of \( H(\lambda) \) such that for every \( \tau \in H(\lambda) \), \( |\{ i \mid \tau_i = \tau \}| = \lambda. \) We regard each \( \tau_i \) as a task of one of the following types: if \( \tau_i \) is a name of an element of \( K \), we shall find two subsets \( H \) and \( R \) of \( \tau \) belonging to \( K \); we shall make \( H \) homogeneous, and will make some obligations which will assure that \( R \) will be I-rigid. If \( \tau_i \) is a name of a pair of members of \( K \), then we will define \( \tau_i \) to make these two sets near to one another. If \( \tau_i \) is a name of a c.c.c. forcing set, then either we make \( \tau_i \) the next step in the iteration or by the explicit contradiction method we destroy the c.c.c-ness of \( \tau_i \).

We define a finite support iteration \( \langle P_i \mid i < \lambda \rangle \) \( \{ \pi_i \mid i < \lambda \} \). Along with the construction of the \( \pi_i \)'s we also define some obligations. An obligation which is added in the \( i \)-th stage of the iteration is a \( P_i \)-name of an object of the following form \( (H, \{ d_\alpha \mid \alpha < \kappa_1 \} \) where \( H, \{ d_\alpha \mid \alpha < \kappa_1 \} \) is a 1-1 enumeration and \( \mathcal{V}_R (H \uplus \{ d_\alpha \mid \alpha < \kappa_1 \}) \) is an obligation, then \( i(s) \) denotes the stage in which it was defined; \( H(s) \) denotes \( H \); \( D(s) \) denotes the name of the set \( \{ d_\alpha \mid \alpha < \kappa_1 \} \) and \( \delta(s, \alpha) \) denotes \( \delta_\alpha \). If \( s \) is an obligation, then for every \( j \geq i(s) \) we will have a club \( C(s, j) \subseteq \kappa_1 \) such that \( \mathcal{V}_R (H(s) \uplus \{ \delta(s, \alpha) \mid \alpha \in C(s, j) \}) \) by \( D(s, j) \).

Suppose \( \delta < \lambda \) is a limit ordinal and for every \( i < \delta \), \( P_i \) has been defined, \( P_\delta \) is
defined automatically. Let \( s \) be an obligation with \( i(s) < \delta \). By Lemma 9.8 there is a club \( C \subseteq \Theta \) such that \( C \subseteq C(s, i) \) for every \( i < j < \delta \). Let \( C(s, \delta) = C \).

Suppose \( P_i \) has been defined. In order to simplify the explanation we define the value of \( \pi_i \) in a \( P_i \)-generic extension of \( V_\theta \) instead of defining \( \pi_i \) itself. Hence let \( G \) be \( P_i \)-generic and \( W = V[G] \). Suppose first that \( Q \Vdash \nu_\theta(\pi_i) \) is a c.c.c. forcing set. If for every obligation \( s \), \( \Vdash Q \upharpoonright H(s) \upharpoonright D(s, i) \), we define \( \nu_\theta(\pi_i) = Q \); for every obligation \( s \) we define \( C(s, i+1) = C(s, i) \); and we do not add new obligations. Otherwise, by 9.7 there is a c.c.c. forcing set \( R \) of power \( \Theta \) such that \( \Vdash Q \upharpoonright (Q \text{ is not c.c.c.}) \wedge \forall s (H(s) \upharpoonright D(s, i)) \). Let \( \nu_\theta(\pi_i) = R \), for every \( s \) let \( C(s, i+1) = C(s, i) \), and we do not add new obligations.

Suppose that \( \nu_\theta(\pi_i) \in K \) and denote \( \nu_\theta(\pi_i) = A \). W.l.o.g. \( A \) is dense in \( R \). By Lemma 9.9 there is \( B \subseteq A \) and for every \( s \) a club \( C(s, i+1) \) such that \( B \subseteq K \) is dense in \( A \) and is 2-entangled, and for every \( s \), \( B \upharpoonright D(s, i+1) \). Let \( H, R \subseteq B \) be disjoint dense subsets of \( B \) such that \( H, R \subseteq K \).

By repeated application of Lemma 9.6 there is a c.c.c. forcing set \( P \) of power \( \Theta \) such that

\[
\Vdash_p (H \text{ is homogeneous}) \wedge (R \text{ is 2-entangled}) \wedge \forall s (H(s) \upharpoonright D(s, i+1)) \]

Let \( \nu_\theta(\pi_i) = P \). For every two disjoint rational intervals of \( R, I \) and \( J \), we define a new obligation \( s(R, I, J) \):

\[
s(R, I, J) = (R \cap [0, 1], (d_\alpha \mid \alpha < \Theta))
\]

where \( (d_\alpha \mid \alpha < \Theta) \) is a 1-1 enumeration of \( R \cap J \). For every new obligation \( s \) we define \( C(s, i+1) \) to be \( \Theta \). It is easy to check that the induction hypotheses hold.

If \( \nu_\theta(\pi_i) = (A, B) \) where \( A, B \in K \), then as in the previous case we find \( A' \subseteq A \), \( B' \subseteq B \) and for every obligation \( s \), \( C(s, i+1) \), such that \( A', B' \subseteq K \) and for every \( s \), \( A', B' \subseteq D(s, i+1) \). By 9.6 there is a c.c.c. forcing set \( P \) of power \( \Theta \) such that

\[
\Vdash_p A = B' \wedge \forall s (H(s) \upharpoonright D(s, i+1)) \]

Let \( P = \nu_\theta(\pi_i) \).

If \( \nu_\theta(\pi_i) \) is none of the above, we define \( \nu_\theta(\pi_i) \) to be a trivial forcing set. This concludes the definition of \( P_i \) and \( \pi_i \).

Let \( P = P_\alpha \), let \( G \) be \( P \)-generic and \( W = V[G] \). Let \( A \in K^W \). It is easy to see that \( A \) contains a homogeneous member of \( K \) which is dense in \( A \). For some \( i \), \( A = \nu_{\theta\cap B}(\pi_i) \). Let \( R \subseteq A \) be as defined in the \( i \)th stage of the construction. We show that \( R \) is \( I \)-rigid in \( W \). Suppose by contradiction \( f : R \to R \) is monotonic and is not the identity. For some disjoint rational intervals \( I \) and \( J \), \( f(I \cap R) \subseteq I \cap R \), and let \( s = (R, I, J) \). Let \( j \geq i \) be such that \( f \in V[G \cap P_j] \). But \( D(s, j) \subseteq I \cap R \) and \( V[G \cap P_j] \Vdash D(s, j) \upharpoonright I \cap R \). A contradiction, hence \( R \) is \( I \)-rigid.

The proof that \( \text{WFA} \) is well known. \( \square \)

**Question 9.11.** Is \( \text{RHA} + (VB \in K) (\exists C, D \in K) (C, D \subseteq B \wedge (C \upharpoonright D) \) consistent?

**Question 9.12 (Baumgartner).** Is it consistent with \( \text{ZFC} \) that \( 2^\omega > \Theta \) and every two \( \Theta \)-dense sets of reals are isomorphic?
10. The structure of $\mathcal{K}$ and $\mathcal{K}^H$ when $\mathcal{K}^H$ is finite

In Section 6 we started the investigation of the possible structure of $\mathcal{K}$ and $\mathcal{K}^H$
under the assumption of $\text{MA}_\omega$. In this section we continue the investigation in
this direction. Our main goal is to characterize the structure of $\mathcal{K}^H$ under the
assumption $\text{MA}_\omega + (\mathcal{K}^H/\approx$ is finite). In this case we obtain a full description of the
possible structure of $\mathcal{K}^H$, and we also obtain quite a good description of how $\mathcal{K}$
is built from $\mathcal{K}^H$.

We did not pursue an analogous result for the general case, still we know to
construct a large variety of different $\mathcal{K}^H$’s. There is still a shortcoming in our
proof — we do not know how to enlarge $\aleph_1$ beyond $\aleph_2$.

We make an abuse of notation and denote $\mathcal{K}^H/\approx$ by $\mathcal{K}^H$. By 6.1(d), $\prec$ induces a
partial ordering on $\mathcal{K}^H/\approx$, hence we regard $\prec$ as a partial ordering of $\mathcal{K}^H/\approx$. Since
$(A =_B) \Rightarrow (A^* = B^*)$ too can be regarded as an operation on $\mathcal{K}^H/\approx$. Clearly $*$ is
an automorphism of order 2 of $\langle \mathcal{K}^H/\approx, \prec \rangle$; we call such an automorphism an
involution. The main theorem in this section is the following.

Theorem 10.1. Let $(L, \leq, *)$ be a finite poset with an involution. Then
$\text{CON} (\text{MA}_\omega + (\langle \mathcal{K}^H \cup \{\emptyset\}/\approx, \leq, * \rangle = (L, \leq, *))$ is a distributive lattice
with an involution.

We start with the easy direction of Theorem 10.1.

Definition 10.2. Let $\mathcal{A} \subseteq \mathcal{K}^H$; $\mathcal{A}$ generates $\mathcal{K}^H$ if every element of $\mathcal{K}^H$ is a shuffle
of a countable subset of $\mathcal{A}$. $\mathcal{K}^H$ is countably generated if there is $\mathcal{A} \subseteq \mathcal{K}^H$ such
that $|\mathcal{A}| \leq \aleph_0$ and $\mathcal{A}$ generates $\mathcal{K}^H$.

Lemma 10.3 (MA$_\omega$). (a) $\mathcal{K}^H$ is a $\sigma$-complete upper-semilattice, that is, every
countable subset of $\mathcal{K}^H$ has a least upper bound.
(b) If $\mathcal{K}^H$ is countably generated, then $\mathcal{K}^H \cup \{\emptyset\}$ is a distributive complete lattice.
(c) If $\langle \mathcal{K}^H, \prec \rangle$ is well founded, then for $A \in \mathcal{K}$ there is a nwd subset $B$ of $A$
such that $A - B$ is the ordered sum of members of $\mathcal{K}^H$.

Proof. (a) is just a reformulation of 6.1(f).
(b) Suppose $\mathcal{K}^H$ is countably generated, then by (a), $\mathcal{K}^H$ is a complete
upper-semilattice, but then $\mathcal{K}^H \cup \{\emptyset\}$ is also a complete lattice. Let $\mathcal{K}^H \cup \{\emptyset\}$ be
denoted by $\mathcal{K}^{HZ}$. In order to show that $\mathcal{K}^{HZ}$ is distributive it suffices to show one of the
distributive laws. We show that $(a_1 \vee a_2) \wedge b = (a_1 \wedge b) \vee (a_2 \wedge b)$. In fact we
can show somewhat more: $(\bigvee_{i \in \omega} a_i) \wedge b = \bigvee_{i \in \omega} (a_i \wedge b)$. We do not know however
whether the dual identity holds.
Let us denote the operations in $\mathcal{K}^{HZ}$ by $\wedge$ and $\vee$. Let $A \in \mathcal{K}^{HZ}$ and for every
$i \in \omega$, let $B_i \in \mathcal{K}^{HZ}$ and suppose that $A \lessdot \bigvee_{i \in \omega} B_i$. We prove the following claim:

(*) There are $A_i \in \mathcal{K}^{HZ}$ such that $A_i \lessdot B_i$ and $A = \bigvee_{i \in \omega} A_i$. 

W.l.o.g. each $B_i$ is dense in $R$, hence $B \equiv \bigcup_{i \in \omega} B_i = \bigvee_{i \in \omega} B_i$. Let $f : A \to B$ be OP. Recall that $C^m$ denotes a mixing of $C$. If $|C| = n$ and $C^m = \emptyset$, then $C^m = \emptyset$. For every $i \in \omega$, let $C_i = f^{-1}(B_i)$ and $A_i = C_i^*$. $C_i \leq A_i$, $B_i$ hence by 6.1(c), $A_i \leq A_i \leq B_i$. Hence $\bigvee_{i \in \omega} A_i \leq A_i$. On the other hand by picking appropriate copies of $C_i^*$ as $A_n$, one can assume that $A \leq \bigcup_{i \in \omega} A_i$, and since $\bigcup_{i \in \omega} A_i = \bigvee_{i \in \omega} A_i$ it follows that $A \leq \bigvee_{i \in \omega} A_i$. Hence $A = \bigvee_{i \in \omega} A_i$. We have thus proved (a).

Let $A \in K^{HF}$ and for every $i \in \omega$, $B_i \in K^{HF}$. $A \wedge \bigvee_{i \in \omega} B_i \leq \bigvee_{i \in \omega} B_i$, hence there are $A_i \leq B_i$ such that $\bigvee_{i \in \omega} A_i = A \wedge \bigvee_{i \in \omega} B_i$. $A_i \leq A_i \leq B_i$. Thus $A \wedge \bigvee_{i \in \omega} B_i = \bigvee_{i \in \omega} A_i \leq \bigvee_{i \in \omega} (A \wedge B_i)$. The inequality $\bigvee_{i \in \omega} (A \wedge B_i) \leq A \wedge \bigvee_{i \in \omega} B_i$ holds in every lattice. (c) Suppose $\langle K, \leq \rangle$ is well founded, and let $A \in K$. It suffices to show that every non-empty open interval of $A$ contains a homogeneous subinterval. If $A_1$, $A_2$ are intervals of $A$ and $A_1 \subseteq A_2$, then $A_1^w \leq A_2^w$. Let $A_1$ be an interval of $A$, since $K^{HF}$ is well founded, $A_1$ has a non-empty open subinterval $A_2$ such that for every subinterval $A_3$ of $A_2$, $A_3^w = A_1^w$. We show that $A_3^w \leq A_2$. For every subinterval $I$ of $A_2$, there is a family of functions $\{g_i \mid i \in \omega\}$ of OP functions such that $\text{Rng}(g_i) = I$ and $\bigcup_{i \in \omega} \text{Dom}(g_i) = A_3^w$. Let $\mathcal{F}$ be a countable dense family of subintervals of $A_2$. Then $A_3^w, A_2, \{g_i \mid i \in \mathcal{F}, i \in \omega\}$ satisfy the conditions of 6.1(b), hence $A_3^w \leq A_2$. $A_2 \leq A_2^w$, hence by 6.1(g), $A_2 = A_2^w$. □

We next turn to the proof of the other direction of Theorem 10.1. Let $\langle I, \vee, \wedge, * \rangle$ be a finite distributive lattice with an involution. $a \in L$ is indecomposable if for no $b, c < a$, $b \vee c = a$. Let $I(L)$ denote the set of indecomposable elements of $L$. Clearly $I(L)$ is closed under $*$, and every element of $L$ is a sum of elements in $I(L)$. The following proposition shows that $I(L)$ determines $L$ uniquely, and will guide us in the construction of a universe in which $K^{HF} \cong L$.

**Proposition 10.4.** (a) Let $\langle A, \leq, * \rangle$ be a finite partially ordered set with an involution. Then there is a unique distributive lattice with an involution $L$ such that $\langle I(L), \leq, * \rangle \cong \langle A, \leq, * \rangle$.

(b) (MA) Let $\{A_i \mid i < n\} \subseteq K^{HF}$ be such that: for no $i < n$, $A_i$ is a shuffle of other members of $\{A_i \mid i < n\}$, and for every $A \in K^{HF}$, $A$ is a shuffle of some members of $\{A_i \mid i < n\}$. Then $K^{HF}$ is finite, and $I(K^{HF}) = \{A_i \mid i < n\}$.

**Proof.** Easy. □

By the above proposition it is clear what has to be done in order to construct a universe in which $K^{HF} \cong L$. We start with a universe $V$ satisfying CH, and with a family $\{A_a \mid a \in I(L)\} \subseteq K^{HF}$ such that no $A_a$ is a shuffle of other $A_a$'s and such that $a \mapsto A_a$ is an isomorphism between $\langle I(L), \leq, * \rangle$ and $\langle \{A_a \mid a \in I(L)\}, \leq, * \rangle$. We then construct $W \supseteq V$ which satisfies MA, and in which every element of $K^{HF}$ is a shuffle of some members of $\{A_a \mid a \in I(L)\}$, and no $A_a$ is a shuffle of other $A_a$'s. In such a universe $K^{HF} \cong L$. 
For the rest of this section \( L \) is a fixed finite distributive lattice with an involution. W.l.o.g. \( I(L) = \{0, \ldots, n - 1\} \), and we denote the partial ordering on \( I(L) \) by \( \leq \). Recall that \( * \) is an involution of \( \langle 0, \ldots, n - 1 \rangle \).

The method of proof of the following lemma is well known and is due to Sierpinski [8]. We take the liberty to present a proof here since the technical details are not completely obvious.

**Lemma 10.5 (CH).** There are \( \{A_i \mid i < n\} \subseteq K^H \) such that:

1. If \( i \lesssim j \), then \( A_i \subseteq A_j \).
2. If \( i = j^* \), then \( A_i = A_j^* \).
3. Let \( B_i = A_i - \bigsqcup_{i=1} A_i \). Then, if \( i \lesssim j \), then \( B_i \perp A_j \).

**Question.** Does the above lemma follow from A1?

**Proof.** Let \( f \subseteq \mathbb{R} \times \mathbb{R} \). \( f \) is a maximal OP function if \( f \) is an OP function and there is no OP function \( g \) such that \( f \preceq g \subseteq \mathbb{R} \times \mathbb{R} \). Let \( \{f_\alpha \mid \alpha \in \aleph_1\} \) be an enumeration of all maximal OP functions. Let \( \bar{g} \) denote the function such that for every \( x \in \mathbb{R} \), \( \bar{g}(x) = -x \); for every \( f \) let \( f^* = \bar{g} \circ f \circ \bar{g}^{-1} \); and if \( F \) is a set of functions, let \( F^* = \{f^* \mid f \in F\} \).

We define by induction on \( \alpha < \aleph_1 \) a family of pairwise disjoint countable dense subsets of \( \mathbb{N} \), \( \{F(i, \alpha) \mid i < n\} \), and families of OP functions \( \{F(i, \alpha) \mid i < n\} \). Let \( A(i, \alpha) = \bigcup_{i<\alpha} B(i, \alpha) \). Our induction hypotheses are: (1) If \( i = j^* \), then \( B(i, \alpha) = B(j, \alpha)^* \) and \( F(i, \alpha) = F(j, \alpha)^* \); and (2) for every \( i < j < n \), if \( f \in F(i, \alpha) \), then \( f: \mathbb{R} \to \mathbb{R} \), and \( f(A(i, \alpha)) = A(i, \alpha) \), and for every \( x, y \in A(i, \alpha) \) there is an \( f \in F(i, \alpha) \) such that \( f(x) = y \).

It is easy to define \( \{B(i, 0) \mid i < n\} \) and \( \{F(i, 0) \mid i < n\} \). If \( \delta \) is a limit ordinal, let \( B(i, \delta) = \bigcup_{\alpha < \delta} B(i, \alpha) \) and \( F(i, \delta) = \bigcup_{\alpha < \delta} F(i, \alpha) \).

Suppose \( \{B(i, \alpha) \mid i < n\}, \{F(i, \alpha) \mid i < n\} \) have been defined, and we wish to define \( \{B(i, \alpha + 1) \mid i < n\} \). Let \( B(i, \alpha + 1) = B(i, \alpha) \cup \bigcup_{\beta < \alpha} F\beta(B(i, \alpha)) \). Let \( U \subseteq \{0, \ldots, n - 1\} \) be such that for every \( i < n \), \( |U \setminus \{i, i^*\}| = 1 \). For \( x \in \mathbb{R} \) and a set of functions \( F \), let \( c(x, F) \) denote the closure of \( x \) under \( F \cup f^{-1} \). It is easy to construct a set \( \{x_i \mid i \in U\} \) such that: (1) for every \( i \in U \), \( c(x_i, F(i, \alpha)) \cap c(-x_i, F(i, \alpha)) = \emptyset \); and (2) for every \( i \in U \), \( c(x_i, F(i, \alpha)) \cap \bigcup_{i < \alpha} B(j, \alpha) = \emptyset \).

Let \( i < n \); if \( i \in U \) and \( i \neq i^* \) let \( B(i, \alpha + 1) = B(i, \alpha) \cup c(x_i, F(i, \alpha)) \); if \( i = i^* \) let \( B(i, \alpha + 1) = B(i, \alpha) \cup c(x_i, F(i, \alpha)) \cup c(-x_i, F(i, \alpha)) \); and if \( x_i \notin U \) let \( B(i, \alpha + 1) = B(i, \alpha) \cup c(-x_i, F(i, \alpha)) \).

Since for every \( i \), \( F(i, \alpha + 1)^* = F(i^*, \alpha) \), it follows that \( B(i, \alpha + 1)^* = B(i^*, \alpha + 1) \). By the choice of the \( x_i \)'s, \( B(i, \alpha + 1) \cap B(j, \alpha + 1) = \emptyset \) whenever \( i \neq j \). It is easy to define for every \( i < n \), \( F(i, \alpha + 1) \geq F(i, \alpha) \) so that the induction hypotheses will hold.

Let \( A_i = \bigcup_{\alpha < \aleph_1} A(i, \alpha) \). It is easy to see that each \( A_i \) belongs to \( K^H \), and (1) and (2) of 10.5 hold. Suppose by contradiction \( i \lesssim j \) but \( N(B_i, A_j) \). Hence for some \( k \neq i \), \( N(B_i, B_k) \). Let \( f \) be a maximal OP function such that \( f \cap B_i \times B_k = K_1 \).
some $\alpha$, $f = f_\alpha$. By the construction, for every $\beta > \alpha$, $f_\alpha(B(\alpha, \beta)) \cap (B(k, \beta) \cup \{B(k, \gamma) \mid \gamma < \beta\}) = \emptyset$, hence $f_\alpha(B_i) \cap B_k = \emptyset$, a contradiction. \qed

Let $\{A_i \mid i < n\}$ and $\{B_i \mid i < n\}$ be as assured by Lemma 10.5. For every $i < n$ and every rational interval $I$ let $\{b(\alpha, i, I) \mid \alpha < K_i\}$ be a 1-1 enumeration of $B_i \cap I$. Let $F_i$ be the following filter: $F_i = \{B \subseteq B_i \mid \text{for every rational interval } I, \{\alpha \mid b(\alpha, i, I) \in B\} \text{ contains a club}\}$. For simplicity we assume that if $i = j^*$, then for every $I$ and $\alpha$, $b(\alpha, i, I) = -b(\alpha, j, I^*)$; this ascertains that $\hat{g}$ is an isomorphism between $F_i$ and $F_j$. Note that if $B \in F_i$, then $B \in K$ and $B$ is dense in $B_i$.

For subsets of $\mathbb{R}$, $C_0, \ldots, C_{n-1}$, let $\bigwedge_{i=0}^{n-1} C_i = 0$ denote the following fact: there is no $C \in K$ such that $C \leq C_i$ for every $i < k$. Note that for $C_i \in K$, there is no meaning to $\bigwedge_{i=0}^{n-1} C_i$ since there are not meets in $K$.

Let $\{x_i \mid i < n\}, \{y_i \mid i \leq n\}$ be sets of variables. Let

$$\varphi_0 = \bigwedge_{i<\omega} (x_i \land x_i = 0) \land \bigwedge_{i<\omega} (x_i \land x_i = 0).$$

Let $z^*$ denote $z$ if $e = 0$, and $z^*$ if $e = 1$. A farness formula ($F$-formula) is a formula of the form $\bigwedge_{i<\omega} z_i^{e(i)} = 0$ where $\{z_i \mid i \in I\}$ is any set of variables and $e(i) \in \{0, 1\}$. Let $\chi$ be an $F$-formula with variables belonging to $\{x_i \mid i < n\} \cup \{y_i \mid i \leq n\}$. We say that $\varphi_0$ implies $\chi$ ($\varphi_0 \Rightarrow \chi$) if for every distributive lattice with an involution $L$ and for every assignment $s$ such that for every $\tau$, $s(y_\tau) = \bigvee \{s(x_i) \mid i \in \tau\}$; then $L \vdash \chi[s]$. More explicitly,

$$\varphi_0 \Rightarrow \bigwedge_{i<\omega} x_i^{e(i)} \land \bigwedge_{i<\omega} y_i^{e(i)} = 0,$$

if there is $i_0 < n$ such that for every $i < I$, $i_0 = i^{e(i)}$, and for every $\tau \in J$, $i_0 \in \{i^{e(i)} \mid i \in \tau\}$.

Let $\varphi_1 = \{\chi \mid \chi$ is an $F$-formula and $\varphi_0 \Rightarrow \chi\}$. Let $C_0, \ldots, C_{n-1} \in K$ and suppose $K \vdash \varphi_1[C_0, C_0, \ldots, C_{n-1}]$. It is obvious that $K \vdash \varphi_1[s]$ where $s$ is the assignment which maps each $x_i$ to $C_i$ and each $y_i$ to $\bigcup_{\tau \in \omega} C_i$. Let $s$ be an assignment such that Dom($s$) $\subseteq \{x_i \mid i < n\} \cup \{y_i \mid i \leq n\}$. We say that $\varphi_1[s]$ holds ($\varphi_1[s]$), if $\chi[s]$ holds for every conjunct $\chi$ of $\varphi_1$ whose variables belong to Dom($s$).

For every $i < n$, let $\eta_i = \{j \mid i < j\}$; let $s$ be the assignment which maps each $i$ to $B_i$ and each $\eta_i$ to $A_i$; by the above discussion $K \vdash \varphi_1[s]$.

We now outline in more detail the proof of the second half of Theorem 10.1. We start with a universe $V$ satisfying $\mathcal{CH}$, and with $\{A_i \mid i < n\}, \{B_i \mid i < n\},$ and as described above. We define by induction $\nu < n_2$ a finite support iteration of c.c.c. forcing sets $\langle P_\nu \mid \nu < n_2 \rangle$, and a sequence $\langle \hat{B}(\nu, 0), \ldots, \hat{B}(\nu, n-1) \mid \nu > n_2 \rangle$ such that for every $\nu$ and $i$, $\hat{B}(\nu, i)$ is a $P_\nu$-name, and

$$\overline{P}(\forall i < n)(\hat{B}(\nu, i) \in F_i \land \hat{B}(\nu, i)^* = \hat{B}(\nu, i^*) \land (K \vdash \varphi_1[\hat{B}(\nu, 0), \ldots, \hat{B}(\nu, n-1), A_0, \ldots, A_{n-1}])$$

where each $\hat{B}(\nu, i)$ replaces $x_i$ and each $A_i$ is replacing $y_i$. 


We prepare in advance a list of tasks. There are two kinds of tasks: the first one is designed in order to take care that \( \mathbb{P}_n \) MA, and the second is to assure that

\[ \mathbb{P}_\Gamma \mathbb{F}_n \mathbb{G}_n \approx \mathbb{F}_n \mathbb{G}_n \] is finite.

If \( \nu \) is a limit ordinal, then \( \mathbb{P}_\nu \) is defined automatically and \( \{ \mathbb{B}(\nu, i) \mid i < n \} \) is defined as in 4.1 or 9.10.

Suppose \( \mathbb{P}_\nu \{ \mathbb{B}(\nu, i) \mid i < n \} \) have been defined, and we wish to define \( \pi_\nu \) and \( \{ \mathbb{B}(\nu + 1, i) \mid i < n \} \).

Case 1: Suppose the \( \nu \)'s task is a \( \mathbb{P}_\mu \)-name \( \pi \) such that \( \mathbb{P}_\nu \) "\( \pi \) is c.c.c.". If \( \mathbb{P}_\nu \{ \mathbb{B}(\nu, 0), \ldots, \mathbb{B}(\nu, n - 1), A_0, \ldots, A_{n-1} \} \), then we define \( \pi_{\nu + 1} \) to be \( \pi \) and \( \mathbb{B}(\nu + 1, i) \) to be \( \mathbb{B}(\nu, i) \). Otherwise we define \( \pi_{\nu + 1} \) to be the \( \mathbb{P}_\nu \)-name of the forcing set \( R \) which is constructed in the following lemma.

**Lemma 10.6 (CH).** Let \( s \) be an assignment such that \( K \vDash \varphi_1[s] \). Suppose \( Q \) is a c.c.c. forcing set such that \( K \vDash (\neg \varphi_1[s]) \). Then there is a c.c.c. forcing set \( R \) of power \( K \) such that \( R \vDash (K \vDash \varphi_1[s]) \land (Q \text{ is not c.c.c.}) \).

**Proof.** \( R \) is constructed by the method of explicit contradiction. The details of the proof are similar to claim 1 in Lemma 8.5. Thus we give the definition of \( R \) but omit the proof that \( R \) satisfies the requirements of the lemma.

Since \( K \vDash \varphi_1[s] \), there are \( C_0, \ldots, C_{n-1} \in K^{\mathcal{V}} \) such that in \( V \), \( K \vDash \bigwedge_{i < n} C_i = 0 \), but \( Q \) forces that \( K \vDash (\bigwedge_{i < n} C_i = 0) \). Let \( M \) be a model which encodes all the relevant information. Recall that a set of two \( k \)-tuples \( \{(a_0, \ldots, a_{n-1}), (b_0, \ldots, b_{n-1})\} \) is called OP, if for every \( i < j < k : a_i < b_i \) iff \( a_j < b_j \). Let \( \tau \) be a \( Q \)-name such that \( K \vDash \tau \in \prod_{i < n} C_i \) \( \land (\tau \vDash \mathbb{K}_\nu) \). The number of variables in \( \varphi_1 \) is \( n + 2^2 \), accordingly let \( m = 2(n + 2^2) + 1 \). Let \( \{a_{\alpha, i} \mid \alpha < n, i < m\} \) be such that: (1) for every \( \alpha < \mathbb{K}_n \) and \( i < m \), \( a_{\alpha, i} \vDash \tau \in \tau \); (2) for every \( \alpha < \mathbb{K}_n \) and \( i < j < m \), there is \( \gamma \in C_M \) such that \( a_{\alpha, i} < \gamma \vDash a_{\alpha, j} \); and (3) for every \( \alpha < \beta < \mathbb{K}_n \), there is \( \gamma \in C_M \) such that \( a_{\alpha, m - 1} < \gamma \vDash a_{\beta, 0} \).

We say that \( x_\alpha \) and \( y_\beta \) are explicitly contradictory if for some \( i < m \), \( \{a_{\alpha, i}, a_{\beta, i}\} \) is not OP. Let \( R = \{ \sigma \in \mathbb{P}_n(K) \mid \) for every distinct \( \alpha, \beta \vDash \sigma, p_\alpha \) and \( p_\beta \) are explicitly contradictory \}. \( \sigma_1 \vDash \sigma_2 \) if \( \sigma_1 \vDash \sigma_2 \). As in 8.5 it can be proved that \( R \) satisfies the requirements of the lemma. \( \square \)

Case 2: Suppose \( \nu \)'s task is a \( \mathbb{P}_\mu \)-name of a member \( B \) of \( K \). In this case we define \( \pi_\nu \) and \( \mathbb{B}(\nu + 1, i) \) according to the following lemma.

**Lemma 10.7 (CH).** Let \( A_0, B_0, F_1 \) be as above, let \( B(\nu, i) \in F_1 \) be such that \( K \vDash \varphi_1(B(\nu, 0), \ldots, B(\nu, n - 1), A_0, \ldots, A_{n-1}) \), and let \( B \in K \). Then there is an interval \( A \) of \( B \) such that \( \nu \subseteq n, B_\nu \in F_1, i < n, \) and a c.c.c. forcing set \( P \) of power \( K \), such that for every \( i < n \), \( B^{\cap \nu} = B_\nu^{\cap i} \) and

\[ \mathbb{P}_\nu (K \vDash \varphi_1[B_0, \ldots, B_{n-1}, A_0, \ldots, A_{n-1}]) \land (A \vDash \bigcup_{i \in \nu} A_i) \].
The remainder of this section is devoted to the proof of the above lemma. But first we show how to define $\pi_{n+1}$ and $B(n+1, i)$, and how Theorem 10.1 follows from what has been described so far. Let $\pi_{n+1}$ be the $P_n$-name of the forcing set $P$ of Lemma 10.7 and $\bar{B}(n+1, i)$ be the $P_n$-name of $B_i$ of Lemma 10.7.

Let $P = P_{\pi_n}$. Clearly $\models P \vdash M A_i$ and $\models P (\forall A \in K^P) (\exists x \leq n) (A = V_{\langle x, A_i \rangle})$. It remains to show that if $i \neq j$, then $\models P A_i \neq A_j$. Suppose the contrary. Let $G$ be a $P$-generic and $W \models V[G]$, and let $f : A_i \to A_j$ be an isomorphism between $A_i$ and $A_j$ belonging to $W$. For some $v < \kappa_2$, $f \in V[G \cap P_v] \models W$. W.l.o.g. $i \neq j$. Let $B(n, i) = q_{G, \bar{B}}(\bar{B}(n, i))$. Hence in $W'$, $B(n, i) \vdash A_j$, but $f(B(n, i)) \not\subseteq A_j$, a contradiction.

The proof of Theorem 10.1 will be concluded if we prove Lemma 10.7.

Let $\{x_i^l \mid i < n, l \in \{0, 1\}\}$ be sets of variables, $x_i^0$ are called copies of $x_i$ and $y_i^0, y_i^1$ are called copies of $y_i$. A formula $\phi$ is called a copy of $\phi_0$ if it is gotten from $\phi_0$ by replacing every occurrence of a variable in $\phi_0$ by one of its copies. Note that two occurrences of the same variable need not be replaced by the same copy of that variable. A copy of $\phi_2$ is defined similarly. Let $\psi_0$ be the conjunction of all copies of $\phi_0$, and $\psi_1$ be the conjunction of all copies of $\phi_1$. We again make the convention that for an assignment $s$ with

$$\text{Dom}(s) = \{x_i^l \mid i < n, l \in \{0, 1\}\} \cup \{y_i^l \mid i < n, l \in \{0, 1\}\},$$

$K \models \psi_0[s]$ means that all conjuncts of $\psi_0$ whose variables belong to $\text{Dom}(s)$ are satisfied.

**Proof of 10.7.** Let $A_0, B_0, F_0, B(n, i)$ and $B$ be as in 10.7. We denote $B(n, i)$ as $B(i)$, and let $F(i)$ be the restriction of $F_i$ to $p(B(i))$. Note that $F(i)$ is defined from some enumerations of $B(i) \cap I$ in the same way that $F_i$ was defined. Hence for the rest of the section we ignore $B_i$ and $F_i$, and have to remember just the properties of $B(i)$ and $F(i)$.

For further reference let us recall the properties of $A_0, B(i)$ and $F(i)$. (1) $A_0$ and $B(i)$ are dense in $R$, $B(i) \in K$; (2) $B_i \models (A_i \models \forall i < n \models K^P)$; (3) $B_0 \models A_i \models \forall i < n \models A_i$; (4) $g(B(i)) = B(i)^g$; (5) for every interval $I$ there is a $1$-$1$ enumeration $\{b(a, i) \mid a < \kappa_1\}$ of $B(i) \cap I$ such that $B(i) = \{B' \subseteq B(i)\}$ for every rational interval $I$ $\{a \subseteq b(a, i, I) \in B(i)\}$ contains a club. $b(a, i, I)^g = -b(a, i, I)$; (5) for every $i < n$ let $\tau = (j \mid j < i)$ and $\tau = A_i$; then $K \models \psi_0[B(0), \ldots, B(n-1), A_{\tau_0}^{\prime}, \ldots, A_{\tau_n}^{\prime}]$.

For $D_1, \ldots, D_{k-1} \subseteq K$, let $D \subseteq \bigwedge_{i \leq k} D_i$ mean that $(\forall i < k)(D \subseteq D_i)$. Let $D \subseteq K$. We define

$$\eta(D) = \{\sigma \subseteq n \mid (\exists D' \subseteq K) ((D' \subseteq D \land \bigwedge_{i \leq k} A_i) \land (\forall i \neq \sigma)(D' \land A_i = 0))\}.$$

Note that if $D_1 \subseteq D_2$, then $\eta(D_2) \subseteq \eta(D_2)$. Hence there is an interval $A$ of $B$ such that for every interval $A' \subseteq A$, $\eta(A') = \eta(A)$.

10.7 follows from the following three lemmas.

**Lemma 10.8 (CH).** Let $A_0, F(i), B(i)$ be as above, and let $A \in K$ be such that for
every interval $A'$ of $A$, $\eta(A') = \eta(A)$. Then there are $\emptyset \neq \tau \subseteq n$, \{\(B_j(i) \mid i \in \tau\}\} and \{\(B_0(i) \mid i < n\}\ such that:

1. If $j < i \in \tau$, then $j \in \tau$.
2. $B_1(i) \in K$, $B_1(i)$ is a dense subset of $A$, and for every $i \neq j$, $B_1(i) \cap B_1(j) = \emptyset$.
3. $B_0(i) \in P(i)$.
4. Let $\tau = \{\{j \mid j \leq i\}, A_0^0 = A_0, A_0^1 = A_0, B_0^0 = B_0^0(i)\ \text{and}\ B_0^1 = B_0(i)$. Then $K \models \exists \psi_1[B_0^0, \ldots, B_{n-1}^0; A_0^1]$.\]

**Lemma 10.9** (CH). For $l = 0, 1$ let $n_l \equiv n$, $B_l(0), \ldots, B_l(n_l - 1) \in K$, $\Gamma_l \subseteq P(n_l)$ and \{\(A_\tau^l \mid \tau \in \Gamma_l\}\} \subseteq K. Suppose the following conditions hold:

1. $B_l(i)$ and $A_\tau^l$ are dense in $\mathbb{R}$.
2. If $i \in \tau \subseteq \Gamma_l$, then $b_l(i) \subseteq A_\tau^l$; and if $i < j < n_l$, then $B_l(i) \cap B_l(j) = \emptyset$.
3. If $\tau_1, \tau_2 \subseteq \Gamma_l$ and $\tau_1 \subseteq \tau_2$, then $A_\tau_1^l \subseteq A_\tau_2^l$.
4. $K \models \psi_1[B_0(0) \mid i < 2 \wedge i < n_l; A_\tau^0 \mid i < 2 \wedge \tau \subseteq \Gamma_l]$.\]

Let $i \in \{0, 1\}$. Then there are pairwise disjoint \{\(B_i^l \mid i < n_l\}\} such that:

1. $K \models \exists \psi_1[B_i^l \mid i < n_l; B_{n_l}(i) \mid i < n_{l-1}; A_\tau^l \mid \tau \subseteq \Gamma_{l-1}]$.
2. For every $\tau \subseteq \Gamma_l$, $A_\tau^l \subseteq \bigcup_{i < n_l} B_i^l$.
3. $K \models \psi_1[B_i^l \mid i < n_l; B_{n_l}(i) \mid i < n_{l-1}; A_\tau^l \mid \tau \subseteq \Gamma_{l-1}]$.\]

**Lemma 10.10** (CH). Let $n_1 \equiv n_0 \equiv n$, let \{\(B_i^l \mid i < 2 \wedge i < n_l\), \(D_i^l \mid i < 2 \wedge i < n_l\)\} be such that all the $B_i^l$s and $D_i^l$s belong to $K$ and are dense in $\mathbb{R}$, for every $i < 2$ and $i < j < n_l$, $B_i \cap B_j = \emptyset$, $D_i \subseteq B_i^1$ and $K \models \psi_1[B_0^0, \ldots, B_{n_l-1}^0, B_0^1, \ldots, B_{n_l-1}^1]$. Then there is a c.c.c. forcing set $P$ of power $N_1$ such that

\[
\text{ir}_P \left( \bigcup_{i < n_l} D_i^0 \uplus \bigcup_{i < n_l} D_i^1 \right) \wedge (K \models \psi_1[B_0^0, \ldots, B_{n_l-1}^0, B_0^1, \ldots, B_{n_l-1}^1]).
\]

**Remark.** Lemma 10.8 and 10.10 can be proved assuming $\text{A}_1$; we do not know how to prove 10.9 without assuming CH; this is the reason why in 10.1 we cannot enlarge $2^\alpha$ beyond $N_2$.

We first conclude the proof of 10.7 assuming 10.8–10.10. Let $A$ be an interval of $B$ such that for every interval $A'$ of $A$, $\eta(A') = \eta(A)$. From 10.8 we obtain $\tau \subseteq n$ and $B_0(i)$'s. By renaming \{0, \ldots, n - 1\} we can assume that $\tau = \{0, \ldots, n_1 - 1\}$. Let us denote $\tau = \{i \mid i < l\}, A_0^0 = A_0, A_0^1 = A_0, \Gamma_0 = \{\tau_0, \ldots, \tau_{n_1 - 1}\}, \Gamma_{\tau} = \{\tau\}, n_0 = n$ and $r = 1$. (n$_1$ has already been defined.) The conditions of 10.9 are satisfied by the $B_0(i)$'s, $A_0^i$s etc., hence from 10.9 we obtain \{\(B_i^l \mid i \in \tau\}\}. By intersecting each $B^l_i$ with $A_0^i$ we can assume that $\bigcup_{i \in \tau} B_i^l = A_0^l$, the other properties of the $B_i^l$s are not destroyed. Obviously

\[
K \models \psi_1[B_0(i) \mid i < n_0; B_i^l \mid i < n_l; A_0^l \mid \tau \subseteq \Gamma_0; A_0^l].
\]

We can now apply Lemma 10.9 with $\tau = 0$ to the $B_0(i)$'s, $B_i^l$s, $A_0^i$s and $A_0^l$. We thus obtain from 10.9 the $B_0^i$s. From 10.9 we know that $K \models \psi_0[B_0^0, \ldots, B_{n-1}^0, B_0^1, \ldots, B_{n-1}^1]$. For $i \in \tau$ let $D_i^l = B_i^l$ and $D_0^l = B_0^l \cup \bigcup_{i \in \tau} A_i$.\]
The conditions of Lemma 10.10 hold, hence let $P$ be the forcing set obtained in 10.10. Let $j \in \tau$; then $\bigcup_{i < j} B_{\sigma}^i \models A_{\sigma} \models A_j$. Since $i < j \in \tau \Rightarrow i \in \tau$, $\bigcup_{i < j} \tau_j = \tau$, hence $\bigcup_{i < j} D^i_j = \bigcup_{i < j} A_i$. Recall that $\bigcup_{i \in \tau_j} D^i_j = A_j$, hence $\forces P, A \equiv \bigcup_{i \in \tau_j} A_i$, $\forces P, (K \models \varphi \models (B_0^i, \ldots, B_{i-1}^i))$; for $\sigma \in \tau$ let $A^i_{\sigma} = \bigcup_{i \in \tau_j} B^i_j$, hence for every $i$, $A^i_{\sigma} \models A_{\sigma}$. Clearly $\forces P, (K \models \varphi \models (B_0^i | i < n; A^i_{\sigma} | i < n))$. Recall that for every $i$, $B^i_0 \models B_0(i)$, hence $\forces P, (K \models \varphi | B_0(i) | i < n; A^i_{\sigma} | i < n)$. But according to 10.8, $B_0(i) \in \mathcal{F}(i) \subseteq F_j$. For every $i < n$, let $B^i = B_0(i) \cap g (B_0(i))$, hence $B^i | F_j \models A^i_{\sigma} | F_j$, $\forces P, A \equiv \bigcup_{i \in \tau_j} A_i$, and $\forces P, (K \models \varphi | B_0, \ldots, B^i_{n-1}, A_0, \ldots, A_{n-1})$. This concludes the proof of 10.7. □

Proof of 10.8. W.l.o.g. $A$ is dense in $\mathcal{R}$. Let $\tau = \{ j | (\exists \sigma \in \eta(A)) (j \in \bigcap_{i \in \tau} \tau_i) \}$. Using the fact that $K \models \varphi, \{ A_{\sigma_0}, \ldots, A_{\sigma_n} \}$, it is easy to see that for every $\sigma \in \eta(A)$, $\bigcap_{i \in \sigma} \tau_i \neq \emptyset$ (we denote $\bigcap_{i \in \sigma} \tau_i = \eta$). Since obviously $\eta(\varphi, \{ A_{\sigma_0}, \ldots, A_{\sigma_n} \}) \neq \emptyset$, it follows that $\tau \neq \emptyset$. It is obvious that if $j < i \in \tau$, then $j \in \tau$. For every $\sigma \in \eta(A)$, let $D_\sigma \in K$ be a dense subset of $\mathcal{R}$ exemplifying that fact that $\sigma \in \eta(A)$. For every $i \in \tau$, let $D^j_i = \bigcup_{j \in \tau_i} A_j$. By an argument similar to Lemma 9.9, it is easy to find $\{ B_0(i) | i \in \tau \}$ and $B_1(i) | i \in \tau$ such that for every $i \in \tau$, $B_1(i) \in K$ and $B_1(i)$ is a dense subset of $D^j_i$; if $i \neq j$, then $B_1(i) \cap B_1(j) = \emptyset$, $B_0(i) \models B_0(j)$, $B_0(i) \in \mathcal{F}(i)$ and for every $i \neq j$

$$B_1(i) \land (B_1(i)^* \cup B_1(j) \cap B_1(j)^* \cup B_0(i) \cup B_0(j)^*) = 0.$$  
(Here we assume that $B_1(j) = \emptyset$ if $i \neq j$.) Clearly (1)-(3) of 10.8 hold. Recall that $A_i$ is denoted by $A_{\tau_i}^0$ and that $A_i^+ = A_i$. Let $s$ be the assignment such that $s(x^i_\tau) = B_1(i)$ and $s(y^i_\tau) = A_i^+$. Recall that an F-formula is a formula of the form $\Lambda_{i \in \tau} z_i = 0$. Let $\chi$ be an F-formula, let $\chi^*$ be the formula obtained from $\chi$ be replacing every variable $z$ of $\chi$ by $z^*$. Let $\chi^+$ be the formula obtained from $\chi$ by replacing every occurrence of $(x^0)^* \land (y^0)^*$ in $\chi$ by $x^0_\tau$ and $y^0_\tau$, respectively where $s^0 = \{ i^* | i \in \sigma \}$. Clearly $\chi$ is a conjunct of $\psi_1$ if $\chi^+$ is, and the same holds for $\chi^*$. Also $K \models \chi^*[s]$ iff $K \models \chi^+[s]$ iff $K \models \chi^*[s]$. Let $\chi$ be an F-formula and suppose $K \models \chi(s)$. We show that $\chi$ is not a conjunct of $\psi_1$. By the definition of the $B_1(i)$'s it is clear that there is at most one occurrence of a variable of the form $x^i_\tau$ in $\chi$. Replacing, if necessary, $\chi$ by $\chi^+$, $\chi^*$ or $\chi^{++}$ it can be assumed that

$$\chi = \bigwedge_{i \in \tau} (x^0_\tau \land A_i^0 \land \{ r | i \in T \}) = 0,$$

where $T$ is a subset of $\{ x^i_\tau | j < n \} \cup \{ x^i_\tau | j \in \tau \} \cup \{ x^i_\tau | j \in \tau \} \cup \{ y^0_\tau, (y^0_\tau)^* \}$, and $T$ intersects the union of the first three sets in at most one element. The case $T = \{ x^i_\tau \}$ follows trivially from the fact that $B_1(i) \subseteq B(i)$ and from property (6) in 10.7. Suppose $T = \{ x^i_\tau \}$, hence $\Lambda_{i \in \tau} A_{\tau_i}^0 \land B_1(i) \neq 0$. By the definition of $B_1(i)$ there is $\sigma' \in \eta(A)$ such that $j \in \bigcap_{i \in \tau} \tau_i$ and $\Lambda_{i \in \tau} A_{\tau_i}^0 \land D_{\sigma} \neq 0$. It follows from the definition of the $D_{\sigma}$'s that $\sigma \subseteq \sigma'$, and hence $j \in \bigcap_{i \in \tau} \tau_i$. This means that $\chi$ is not a conjunct of $\psi_1$. We next check that if $\{ B^i_\tau \}^* \land \Lambda_{i \in \tau} A_{\tau_i}^0 \neq 0$, then $j^0 \in \bigcap_{i \in \tau} \tau_i$. If the above holds,
then for some \( \sigma \in \eta(A) \), \( j \in \bigcap_{i \in \sigma} \tau_i \) and \((D_{\sigma^k})^* \wedge \Delta \eta_0 A_{\sigma^k}^0 \neq 0\). \((A_{\sigma^k}^0)^* = A_{\sigma^k}^0\), hence \(D_{\sigma^k} \wedge \Delta \eta_0 A_{\sigma^k}^0 \neq 0\). Let \( \sigma^k = \{ x^k_i \mid i \in \sigma \} \), hence \(D_{\sigma^k} \wedge \Delta \eta_0 A_{\sigma^k}^0 \neq 0\). By the definition of \( D_{\sigma^k} \), \( \sigma^k \neq \sigma \); hence \( j \in \bigcap_{i \in \sigma^k} \tau_i \) and hence

\[
j^* \in \left( \bigcap_{i \in \sigma^k} \tau_i \right)^* = \bigcap_{i \in \sigma^k} \tau_i^* = \bigcap_{i \in \sigma^k} \tau_i = \tau_i.
\]

We next check that if \((B_i)^{\sigma^k} \wedge A_i^0 \neq 0\), then \(j^* \in \tau_i\). Suppose the above happens, and let \( \sigma \in \eta(A) \) be such that \( j \in \bigcap_{i \in \sigma} \tau_i \) and \((D_j)^{\sigma^k} \wedge A_j^0 \neq 0\). Let \( K \ni D \wedge \Delta \eta_0 A_i^0\) since \(D_i \wedge A_i = 0\) for every \(i \notin \sigma\), and since \(D^* \wedge \Delta \eta_0 A_i^0 = 0\) for every \(i \notin \sigma^k\), hence \(D \wedge A_i = 0\) for every \(i \notin \sigma^k\). Obviously \(D^* \wedge \Delta \eta_0 A_i^0 \). Since \( \sigma^k \in \eta(A) \). Since \( j \in \bigcap_{i \in \sigma^k} \tau_i \) hence \( j^* \in \tau_i \). This implies that \( \chi \) is not a conjunct of \( \psi_i \).

Suppose \( T = \{ (y_j^* \eta_j)^\psi \} \). Then \((B_j)^{\sigma^k} \wedge \Delta \eta_0 A_j^0 \neq 0\), hence \( j^* \in \bigcap_{i \in \sigma} \tau_i \) and \( j^* \in \bigcap_{i \in \sigma} \tau_i \) hence \( \chi \) is not a conjunct of \( \psi_i \).

The last case that we check is \( T = \{ (y_j^* \eta_j)^\psi \} \). Hence \( \Delta \eta_0 A \wedge A \wedge \Delta \eta_0 A \neq 0\), so there is \( D \in K \) such that \( D^* \wedge \Delta \eta_0 A \wedge A \neq 0\). Let \( \sigma \leq \eta \) and \( \sigma' \leq \eta \) be such that \( K \ni \Delta \eta_0 A \wedge A \wedge \Delta \eta_0 A \wedge A \neq 0\), and for every \(\sigma' \leq \eta\), \( \sigma' \neq \sigma \). Hence \( \sigma \in \eta(A) \), and since \( \Delta \eta_0 A \wedge A \neq 0\). Clearly, since \( K \ni \Delta \eta_0 A \wedge A \wedge \Delta \eta_0 A \wedge A \neq 0\), and hence \( j \in \bigcap_{i \in \sigma} \tau_i \) since \( \sigma \leq \eta \) and \( j \in \bigcap_{i \in \sigma} \tau_i \). Moreover \( j \neq \in \bigcap_{i \in \sigma} \tau_i \). These facts imply that \( \chi \) is not a conjunct of \( \psi_i \).

We leave the (easy) remaining cases to the reader. \( \square \)

**Proof of Lemma 10.9.** Let \( M \) be a model with universe \( K \) encoding all the information mentioned in the lemma. Suppose w.l.o.g. \( i = 0 \). For every \( C_{\lambda} \)-slice \( E \) we decide how to divide the elements of \( E \) among the various \( B_i \)'s. This is done independently of the elements of other \( C_{\lambda} \)-slices are divided. Let \( \{ a_m \mid m \in \omega \} = E \). Let \( F \) be the set of all real monotonic functions definable from ordinals \( \alpha < \min(E) \). Note that \( F \) is countable and is closed under composition. We decide by induction on \( m \) to which \( B_i \), \( a_m \) will belong. Hence at stage \( m \) we have sets \( \{ B_i(\alpha) \mid i < \eta \} \). We denote \( B_i(j) \) by \( B_i(\alpha) \). We assume by induction:

\[
(*) \quad \chi = \bigwedge_{j \in \omega} \left( x_j^{D^*(\alpha) \wedge \Delta} \wedge \bigwedge_{j \in \omega} (y_j^* \wedge \Delta \eta_0) \wedge \bigwedge_{j \in \omega} (y_j^* \wedge \Delta \eta_0) = 0 \right)
\]

is a conjunct of \( \psi_i \), then there is no \( a \in E \), \( \{ f_\lambda \mid \lambda \in (0,1), j \in \sigma \} \subseteq F \) and \( \{ f_\lambda \mid \lambda \in (0,1), \sigma \in \eta \} \subseteq F \) such that \( f_i^* \) and \( f_i^* \) are OP or OR according to whether \( e(i, j) \) and \( e(i, \tau) \) are \( 0 \) or \( 1 \), \( f_i(a) \) \( \in B_i(\alpha) \) and \( f_i(a) \) \( \in A \).
Let $B_0^0(0) = B_0(i)$. The induction hypotheses holds since

$$K \vdash \psi_i[B_0(i) \mid i < n_0; B_0(i) \mid i < n_1; A^m_i \mid \tau \in \Gamma_0; A^m_i \mid \tau \in \Gamma_1],$$

and since $M$ encodes the above sets.

Suppose $B_0^0(m), \ldots, B_0^0(m)$ have been defined. Let

$$\tau_1 = \cap \{ \tau \mid a_m \in A^m_0 \},$$

$$\tau_2 = \cap \{ \tau \mid (\exists f \in F) (f \text{ is OP and } f(a_m) \in A^m_0) \}, \quad l = 0, 1,$$

$$\tau_3 = \cap \{ \tau \mid (\exists f \in F) (f \text{ is OR and } f(a_m) \in A^m_0) \}, \quad l = 0, 1,$$

$$\tau_4 = \{ i \mid (\exists f \in F) (f \text{ is OP and } f(a_m) \in B_0^0(m)) \}, \quad l = 0, 1,$$

$$\tau_5 = \{ i \mid (\exists f \in F) (f \text{ is OR and } f(a_m) \in B_0^0(m)) \}, \quad l = 0, 1.$$

By the induction hypothesis each of the sets $\tau_2 \cup \tau_3$, $\tau_2 \cup \tau_4$ contains at most one element and if $i \in \tau_2 \cup \tau_3$, $i = 4, 5$, and $i_0 = (i_0)^e$. By the induction hypothesis

$$\tau \overset{\text{def}}{=} \tau_1 \cap \bigcap_{l=0}^1 \tau_2 \cup \bigcap_{l=0}^1 (\tau_4)^e \neq 0,$$

and if $i \in \tau_2 \cup \tau_4$, then $i \in \tau$, and if $i \in \tau_2 \cup \tau_4$, then $i \in \tau$. Let $i_0 \in \tau_2 \cup \tau_4$ if $\tau_2 \cup \tau_4 \neq 0$, $i_0 \in (\tau_2 \cup \tau_4)^e$ if $\tau_2 \cup \tau_4 \neq 0$, and otherwise let $i_0$ be any member of $\tau$. Let $B_i^0(m + 1) = B_i^0(m)$ for every $i \neq i_0$ and $B_i^0(m + 1) = B_i^0(m) \cup \{ a_m \}$. It is easy to check that the induction hypothesis holds, and that the construction yields $B_i^0$ as required. \(\square\)

**Proof of 10.1.** The construction of $P$ resembles the forcing set constructed to prove Theorem 9.2. The proof that $\Gamma_P (K \vdash [B_0^0, \ldots, B_0^0, B_0^0, \ldots, B_0^0])$ resembles the proof of 9.6. We thus leave the details of the proof to the reader. \(\square\)

This concludes the proof of 10.1.

**On the possible infinite $K^H$'s**

We did not pursue a characterization of all possible $K^H$'s, and not even all countably generated $K^H$'s. However the construction of 10.1 can be applied to yield some new infinite $K^H$'s. Also, some additional information about the structure of $K$ and $K^H$ can be derived from MA$_\omega$. In the remainder of this section we first present some additional facts about the case of an infinite $K^H$, we then discuss some open problems, and finally we prove the theorems stated before.

**Definition 10.11.** (a) $A \subseteq K$ is quasi-homogeneous (QH), if there is a family $\{ A_i \mid i \in \omega \} \subseteq K^H$ such that for every $i \in \omega$, $A_i \subseteq A$ and $|A - \bigcup_{i \in \omega} A_i| = \aleph_0$.

(b) Let $L$ be a $\sigma$-complete upper-semi-lattice, $a \in L$ is countably indecomposable (CIP), if $a \neq 0$, and whenever $a \leq \bigvee_{i \in \omega} a_i$ there is $i \in \omega$ such that $a \leq a_i$. Let $L^C$ denote the set of CIP elements of $L$. 
The consistency of some partition theorems

(c) Let \((M, \leq, 0)\) be a poset with a smallest element 0. \(A \subseteq M\) is dense in \(M\) if for every \(b \in M - \{0\}\) there is \(a \in A - \{0\}\) such that \(a \leq b\).

(d) A poset \(M\) is scattered if \(\{0, <\}\) is not embeddable in \(M\).

**Theorem 10.12** (MA\(_{\aleph_0}\)). (a) Let \(A \preceq B\) mean that \(A \preceq B\) and \(A \not\preceq B\). If \(\mathcal{A}\) generates \(K^H\), \(\mathcal{A}\) is countable and \((\mathcal{A}, <)\) is well-founded, then \(K^{HC}\) generates \(K^H\).

(b) If \(K^{HC}\) is dense in \(K^{HE}\), then \(K^{HC}\) is dense in \(K \cup \{0\}\).

(c) Suppose \(K^{HC}\) is countable and scattered, and \(K^{HC}\) generates \(K^{HE}\), then every member of \(K\) is \(OH\).

**Lemma 10.13.** Let \((M, \leq, \ast)\) be a countable poset. Then up to isomorphism there is a unique complete lattice \(L\) with the following properties: \(M = L^C\) and \(M\) generates \(L\). In this unique lattice \(L\) the distributive law \(b \land \bigvee \alpha : a = \bigvee (b \land a_i)\) holds.

**Theorem 10.14.** (a) Let \((M, \leq, \ast)\) be a countable poset with an involution with the following property: \((*)\) "For every \(A \subseteq M\): if for every \(\tau \in P_M(A)\) there is \(b \in M\) such that for every \(a \in \tau b \leq a\), then there is \(b \in M\) such that for every \(a \in A b \leq a\)." Then

\[\text{CON(MA}_{\aleph_0} + \langle(K^{HC}, <, \ast) = (M, \leq, \ast)\rangle + (K^{HC} \text{ generates } K^H) + (\text{every member of } K \text{ is } QH).]\]

(b) Let \(V \models \text{CH} \land \lambda \leq \kappa\) be cardinals in \(V\). Then there is an extension \(W\) of \(V\) which has the same cardinals as \(V\) such that \(W \models MA_{\aleph_0} + "\text{There is a family } \{A_i \mid i < \lambda\} \subseteq K \text{ such that for every } i \neq j, A_i \cup A_j \text{ and for every } A \in K, \text{ there is } i < \lambda \text{ such that } A_i \leq A^{\langle 2 \rangle} + 2^{\kappa} \geq \kappa."}\)

(c) It is consistent that \(MA_{\aleph_0}\) holds and \((K^{HC}, >) = (P_{\alpha_1}(\aleph_1) \cup \{\aleph_1\}, \subseteq)\). It is consistent that \(MA_{\aleph_0}\) holds and \((K^{HC}, <) = (\aleph_1 + 1, <)\).

Let us now explain what seem to be the main open questions.

**Question 10.15.** It is easy to construct a universe satisfying

\[\text{MA} + \forall A \in K (\exists B \in K (B \subseteq A \text{ and } B \text{ is } 2\text{-entangled}).]\]

In such a universe every \(A \in K^H\) contains \(B \in K^H\) such that every member of \(K^H\) contained in \(B\) is decomposable.

Construct a universe \(W\) satisfying \(MA_{\aleph_0}\) in which \(K^H\) is countably generated but \(K^{HC}\) does not generate \(K^H\). Moreover, can \(W\) be constructed so that \(K^{HC} = \emptyset\)?

**Question 10.16.** Does the first or the second part of 2.14(c) remain true when \(\kappa_1\) is replaced by some \(\lambda > \kappa_1\)?

**Question 10.17.** Is \(MA_{\aleph_0} + (K^{HC} \text{ is countable}) + (K^{HC}\text{ generates } K^H) + (\exists A \in K (A \text{ is not } QH))\) consistent?
Question 10.18. Is the consistency result of Theorem 10.14(a) true when \( \langle M, \leq \rangle \) is any countable poset?

**Proof of Theorem 10.12.** (a) The proof is easy.
(b) The following claim follows easily from 6.1(b).

**Claim 1.** If \( A \in K, B \in K^H \) and for every interval \( I \) of \( A, B \leq I \), then \( B \leq A \).

Let us next prove the following claim.

**Claim 2.** If \( A \in K, B \in K^{HC} \) and \( B \leq A \), then \( B \leq A \).

**Proof.** Let \( A_1 = \bigcup \{ \{ I \} \mid I \text{ is an interval of } A \} \) and \( B \leq A \). By the countable indecomposability of \( B, B \notin A \). Let \( A_2 = A - A_1 \), hence \( A_2 \notin \emptyset \). Moreover, since \( B \leq A \), for every interval \( J \) of \( A_2 \), \( B \leq J \). By claim 1, \( B \leq A_2 \subseteq A \). □

(b) follows easily from claim 2.

(c) **Claim 3.** Suppose \( K^{HC} \) is countable and it generates \( K^H \). Let \( A \in K, B \in K^{HC} \) and \( B \leq A \). Then there is \( A_1 \subseteq A \) such that \( A - A_1 = B \) and for every interval \( I \) of \( A_1 \), if \( B \subseteq I \), then there is \( C \in K^{HC} \) such that \( C \leq I \).

**Proof.** Let \( A_2 = \bigcup \{ \{ I \} \mid I \text{ is an interval of } A \} \), \( A_3 = A - A_2 \) and \( A_4 = \bigcup \{ \{ I \} \mid I \text{ is an interval of } A \} \), and for no \( C \in K^{HC} \), \( B \subseteq C \). Clearly there is \( B' \subseteq B \) such that \( B' \) is a dense subset of \( A_3 \). If \( A_4 \subseteq \mathbb{N} \), let \( A_1 = A - B' \); it is easy to see that \( A_1 \) is as required. Otherwise, it is easy to see that for some \( B_i \mid i < \alpha \leq \omega \) \( \subseteq K^{HC} \), \( A_5 = \bigcup_{i<\alpha} B_i \), \( B_0 \subseteq B \) and for every \( 0 < i < \alpha \), \( B \neq B_i \). Hence there is a countable family of OP functions \( \mathcal{G} \) such that for every \( g \in \mathcal{G} \), there is \( i(g) < \alpha \) such that \( g \in B_{i(g)} \times A_6 \) and \( \bigcup \{ \text{Rng}(g) \mid g \in \mathcal{G} \} = A_4 \). Let \( B'' = B' \cup \bigcup \{ \text{Rng}(g) \mid g \in \mathcal{G} \} \) and \( i(g) = 0 \) and \( A_1 = A - B'' \). Since \( B' \) is dense in \( B'' \), \( B'' \subseteq B \). Let \( I \) be an interval of \( A \) and suppose that \( B \subseteq I \). Clearly \( B \subseteq I - A_2 \), that is, \( B \subseteq A_3 \cap I \). By contradiction there is no \( C \in K^{HC} \) such that \( B < C \leq I \). Let \( I' \) be the convex hull of \( A_3 \cap I \) in \( A_3 \). Since \( I' - I < B \) and \( I - A_3 \nless B \) there is no \( C \in K^{HC} \) such that \( B < C \leq I' \). Hence \( I' \leq A_4 \). We will reach a contradiction if we show that \( B \nless I' - B'' \). Since \( I' - B'' \subseteq \bigcup \{ \text{Rng}(g) \mid g \in \mathcal{G} \} \) and \( (\text{Dom}(g))^\# \nless B \) it follows that \( B \nless I' - B'' \). Hence the claim is proved. □

**Claim 4.** Suppose that \( A \in K, B \in K^{HC} \), \( B \leq A \) and for no \( C \in K^{HC} \), \( B < C \leq A \), then there is \( B' \subseteq B \) such that \( B \nless A - B' \).

**Proof.** This is a special case of claim 3. □

**Definition.** Let \( \langle M, \leq \rangle \) be a poset and let \( \{ M_i \mid i \in \omega \} \) be a family of subsets of \( M \).
$M$ is an $\omega$-sum of $\{M_i \mid i \in \omega\}$ if $M = \bigcup_{i \in \omega} M_i$, and for every $i \neq j \in \omega$, $a \in M_i$ and $b \in M_j$, $a \neq b$. We define $\omega^*\text{-s}\text{ums}$ analogously.

Let $\mathcal{M}_0$ be the class of all posets with exactly zero or one element. For a limit ordinal $\delta$ let $\mathcal{M}_\delta = \bigcup_{\alpha < \delta} \mathcal{M}_\alpha$. Let $\mathcal{M}_\omega$ be the class of all posets that can be represented as $\omega$-sums or $\omega^*\text{-s}\text{ums}$ of members of $\mathcal{M}_\omega$. Let $\mathcal{M} = \mathcal{M}_\omega$.

Claim 5. Let $M$ be a countable poset. Then $M$ is scattered iff $M \in \mathcal{M}$.

Proof. It is easy to show by induction on $\alpha$ that every member of $\mathcal{M}_\alpha$ is scattered.

Let $(M, \leq)$ be a scattered poset. By a theorem of Bonnet and Pouzet [5], there is a scattered linear ordering $\leq'$ on $M$ extending $\leq$. Suppose $M$ is countable, hence by a theorem of Hausdorff there is $\alpha < \aleph_1$ such that $(M, \leq') \in \mathcal{M}_\alpha$. It is easy to see that $(M, \leq)$ also belongs to $\mathcal{M}_\alpha$. □

For $A \in K$ let $A^{HC} = \{B \in K^{HC} \mid B \subseteq A\}$. We assume that $K^{HC}$ generates $K^H$.

(c) follows from the following chain which is proved by induction on $\alpha < \aleph_1$.

Claim 6. Let $A \in K$ and $A^{HC}$ is a sum of $K_1$ and $K_2$, that is, $A^{HC} \subseteq K_1 \cup K_2$ and for every $B_1 \in K_1$ and $B_2 \in K_2$, $B_1 \neq B_2$. Then, if $K_2 \in \mathcal{M}_\omega$, then there is a QH set $A_1 \subseteq A$ such that $(A - A_1)^{HC} \subseteq K_1$.

Proof. The case $\alpha = 0$ is just a reformulation of claim 4. There is nothing to prove for a limit ordinal $\alpha$. Suppose the claim is true for $\alpha$ and we prove it for $\alpha + 1$. Let $A^{HC}$ be the sum of $K_1$ and $K_2$, and $K_2 \in \mathcal{M}_\omega$. Let us first deal with the case that $K_2$ is the $\omega^*\text{-s}\text{um}$ of $\{M_i \mid i \in \omega\}$ where each $M_i \in \mathcal{M}_\omega$. Using the induction hypothesis we can define inductively $\{C_i \mid i \in \omega\}$ such that for every $i$, $C_i$ is QH,

$$C_i \subseteq A - \bigcup_{j < i} C_j \text{ and } \left(A - \bigcup_{j < i} C_j\right)^{HC} \subseteq K_1 \cup \bigcup_{j \geq i} M_j.$$  

Let $A_2 = \bigcup_{i \in \omega} C_i$; it is easy to see that $A_2$ is as required.

Let us assume that $K_2$ is an $\omega$-sum of $\{M_i \mid i \in \omega\}$. $A^{HC} = \bigcup \{A + q \mid r, q \in Q\}$. Let $f : \bigcup \{B \mid B \in K_1 \cup K_2\} \rightarrow A^{HC}$ be an isomorphism. (We assume that each $B \in K_1 \cup K_2$ is dense in R. For every $r, q \in Q$ let $\delta_{r,q}(x) = (1/r)(x - q)$. For every $B \in K_1 \cup K_2$, let $\delta_{r,q} : B \rightarrow A$, and $f_{\delta_{r,q}} = \delta_{r,q} \circ f \upharpoonright B$. Clearly, for every $B \in K_1 \cup K_2$, $f_{\delta_{r,q}} \subseteq B \times A$, and $f_{\delta_{r,q}}(B)$ is dense in $A$.) For every $i \in \omega$, let $C_i = \bigcup \{\delta_{r,q}(B) \mid B \in K_i \cup M_i, r, q \in Q\}$. Hence $\bigcup_{i \in \omega} C_i = A$, and for every $i \in \omega$, $(C_i)^{HC} \subseteq K_1 \cup \bigcup_{j \geq i} M_j$. Since $\bigcup_{i \in \omega} M_i$ can be regarded as an $\omega^*\text{-s}\text{um}$ (where some of the summands are empty), by the previous case there is a QH set $D_i \subseteq C_i$ such that $(C_i - D_i)^{HC} \subseteq K_1$.

Let $A_2 = \bigcup_{i \in \omega} D_i$. It is easy to see that $A_2$ is as required. □

Proof of Lemma 10.13. Existence. Let $\sim$ be the following equivalence relation on $p(M) : M_1 \sim M_2$ if for every $m_1 \in M_1$ there is $m_2 \in M_2$ such that $m_1 \leq m_2$ and for
every $m_2 \in M_2$ there is $m_1 \in M_1$ such that $m_2 \leq m_1$. Let $L = P(M)/\sim$. $M_1/\sim \leq M/\sim$
if for every $m_1 \in M_1$ there is $m_2 \in M_2$ such that $m_1 \leq m_2$. Clearly the definition of
$\leq$ does not depend on the choice of representatives. It is easy to check that $(L, \leq)$
is as required.

**Uniqueness.** For a lattice $L_1$ such that $L_1/\leq = M$ and $L_1/\leq$ generates $L_1$ let $\varphi : L_1 \to L$ be defined as follows: $\psi(a) = (m \in M \mid m \leq a)/\sim$. It is easy to check that $\varphi$ is an isomorphism between $L_1$ and $L$. □

**Proof of Theorem 10.14(a).** The proof of 10.14(a) resembles the proof of 10.1. However, some modifications have to be made. We skip those parts of the proof which are straight-forward generalizations of claims proved in 10.1. In order to simplify the technical details we deal with the special case in which the involution $\sim$ is the identity function. However, the proof is easily extended to the general case.

**Lemma 10.19 (CH).** Let $\langle M, \leq \rangle$ be a countable poset. Then there is a family
$\{A(m) \mid m \in M\} \subseteq K^d$ such that (1) if $m \leq n$, then $A(m) \subseteq A(n)$; (2) $A(m) = A(m)^a$; and (3) let $B_m = A(m) \setminus \cup_{n \leq m} A(n)$; then, if $m \neq n$, then $B_m \uplus A(n)$.

**Proof.** As in 10.5. □

For the rest of the proof $\langle M, \leq \rangle$ denotes a fixed countable poset with the property: (a) "If $A \subseteq M$ and every finite subset of $A$ has a lower bound, then $A$ has a lower bound". We also fix some family $\{A(m) \mid m \in M\}$ as constructed in 10.19; $B_m = A(m) \setminus \cup_{n \leq m} A(n)$.

**Proposition 10.20.** If $M$ is countable and has the property (a), then $M$ satisfies (a) in every generic extension.

**Proof.** Easy. □

We define $F_\varphi, \varphi_0$ and $\varphi_1$ as in 10.1. The induction hypotheses of the iteration are as in 10.1. So as in 10.7 at stage $n$ of the iteration we have sets $\{A(m) \mid m \in M\}$, $\{B(m) \mid m \in M\}$ and filters $\{F(m) \mid m \in M\}$, and we assume (1)–(6) of 10.7 hold.

Lemma 10.6 remains unchanged.

**Definition 10.21.** Let $\mathcal{A} \subseteq K^d$ and $D \in K$; $D$ is $\mathcal{A}$-QH if there is $\mathcal{B} \subseteq \mathcal{A}$ such that $|\mathcal{B}| \leq \aleph_0$, and for every $B \in \mathcal{B}$ there is $C(B) \subseteq D$ such that $C(B) \subseteq D$ and $|D \setminus \bigcup \{C(B) \mid B \in \mathcal{B}\}| \leq \aleph_0$.\}
Proposition 10.22 (MA). If $\mathfrak{d} \subseteq K^\kappa$ and every $D \in K$ is $\mathfrak{d}$-QH, then $\mathfrak{d}$ generates $K^\kappa$.

**Proof.** Trivial. □

For every $m \in M$, let $\tau_m = \{n \mid n \leq m\}$ and $A(\tau_m) = A(m)$. The following lemma is the counterpart of 10.7.

**Lemma 10.23 (CH).** Let $A(m), B(m), F(m), m \in M$, satisfy (1)–(6), and let $A \in K$. Then there are $B'(m) \subseteq F(m)$ and a c.c.c. forcing set $P$ of power $\kappa$, such that

$$\vdash_p (K \vDash \varphi_\varnothing(B'(m) \mid m \in M; A(\tau_m) \mid m \in M) \wedge (A \text{ is } \{A(m) \mid m \in M\}-\text{QH}).$$

It is obvious that 10.14(a) follows from 10.19, the analogue of 10.6, 10.20 and 10.21. We now formulate the analogues of 10.8–10.10. Let $\psi_0$ and $\psi_1$ be as in 10.1.

**Lemma 10.24 (CH).** Let $A(m), B(m), F(m), m \in M$, satisfy (1)–(6), and let $A \in K$. Then there is $\tau \leq M, \{B_i(m) \mid m \in \tau\}$ and $\{B_i(m) \mid m \in M\}$ such that:

1. If $n \leq m \in \tau$, then $n \in \tau$.
2. For every $m \in \tau$, $K \vDash B_i(m) \subseteq A$.
3. For every $m \in M$, $B_0(m) \subseteq F(m)$.
4. Let $\tau_m = \{n \mid n \leq m\}$, $A_0(\tau_m) \equiv A(m)$ and $A_1(\tau_m) \equiv A$; then $K \vDash \psi_0(B_0(m) \mid m \in M; B_1(m) \mid m \in \tau; A_0(\tau_m) \mid m \in M; A_1(\tau_m))$.
5. For every $n \in \tau$ and $B \in K$, if $B_i(n) \subseteq B \subseteq A$ and $K \vDash \psi_0(B_0(m) \mid m \in M; B \subseteq A \subseteq A_0(\tau_m); \{A_i(\tau) \mid m \in M; A_1(\tau_m)\}$, then $B_i(n)$ is dense in $B$.
6. If $n \leq m \in \tau$ then $B_i(n)$ is dense in $B_i(n) \cup B_1(m)$.

**Lemma 10.25 (CH).** For $l = 0, 1$, let $M_l \subseteq M$, $\Gamma_l \subseteq P(M_l), \{B_i(m) \mid m \in M_l\} \subseteq K$ and $\{A_i(\tau) \mid \tau \in \Gamma_l\} \subseteq K$. Suppose the following conditions hold:

1. If $m, n \in M_0$ are distinct, then $B_i(m) \cap B_i(n) = \emptyset$, and if $m \in \tau \in \Gamma_0$, then $B_i(m) \subseteq A^\kappa_i$.
2. If $\tau_1, \tau_2 \in \Gamma_1$ and $\tau_1 \subseteq \tau_2$, then $A_i(\tau_1) \subseteq A_i(\tau_2)$.
3. If $\Gamma \subseteq \Gamma_1$, and $\bigcap \Gamma = \emptyset$, then for some finite $\Gamma' \subseteq \Gamma, \bigcap \Gamma' = \emptyset$.
4. $K \vDash \psi_0(B_i(m) \mid l \in \{0, 1\}, m \in M_0, A_i(\tau) \mid \tau \in \Gamma_0)$.

Let $l \in \{0, 1\}$. Then there are pairwise disjoint $\{B_i^l(m) \mid m \in M_l\}$ such that:

1. $K \vDash B_i^l(m) \subseteq B_i(m)$.
2. For every $\tau \in \Gamma_0$, $A_i(\tau) \equiv \bigcup_{m \in \tau} B_i(m)$.
3. $K \vDash \psi_0(B_i^l(m) \mid m \in M_l, B_{1-l}(m) \mid m \in M_{1-l}; A_{1-l}(\tau) \mid \tau \in \Gamma_{1-l\varnothing})$.

**Lemma 10.26 (CH).** Let $M_0, M_1 \subseteq M$, $L \subseteq M_0 \subseteq M_1$, $\{B_i^l(m) \mid l < 2, m \in M_l\} \subseteq K$ and $\{D^l(m) \mid l < 2, m \in L\} \subseteq K$ be such that:

1. $m \neq n$ implies $B_i^l(m) \cap B_i^l(n) = \emptyset$.
2. $D^l(m) \subseteq B_i^l(m)$.
(3) For every \( m \in L \), \( D'(m) \) is dense in \( \bigcup \{ D'(n) \mid n \in L \} \).
(4) \( K \vdash \psi[l \in B'(m) \mid l < 2, m \in M_k] \).

Then there is a c.c.c. forcing set \( P \) of power \( \kappa_1 \) such that
\[
\dot{\mathcal{P}} = \left( \bigcup_{m \in L} D_0(m) \equiv \bigcup_{m \in L} D_1(m) \right) \wedge \left( K \vdash \psi[l \in B'(m) \mid l < 2, m \in M_k] \right).
\]

The argument why 10.23 follows from 10.24–10.26, resembles the analogous argument in 10.1. Also 10.26 is the same as 10.10. We also omit the proof of 10.25, since it involves no new difficulties.

**Proof of 10.24.** Let \( A(m), B(m), F(m) \) and \( A \) be as in 10.24. Let \( S(m) = \{ B \subseteq B(m) \mid (\forall B' \in F(m))(B \cap B' \neq \emptyset) \} \). Let \( K \equiv C \subseteq A \) and \( m \in M \). We say \( C \) is appropriate for \( m \) if: either \( m \) is a minimal element of \( M \), for every \( n \in M \) such that \( n \neq m \), \( C \equiv A(n) \), and for every \( n \neq m \) and \( B \in S(n) \), \( B \notin C \); or if for some \( B \in S(m) \), \( C \equiv B \). Note that if \( C \) is appropriate for \( m \), then there is \( B_0(m) \in F(m) \) such that \( K \vdash \psi[s] \) where \( s \) is the assignment in which for every \( n \neq m \), \( s(x^n_0) = B(n) \), \( s(x^n_0) = B_0(m) \), \( s(x^n_1) = C \), and for every \( m \in M \), \( s(y^n_0) = A(m) \).

Let \( \tau = \{ m \in M \mid (\exists C \subseteq A)(\exists m' \geq m)(C \text{ is appropriate for } m') \} \). W.l.o.g. \( A \) is dense in \( \mathbb{R} \). For every interval \( I \) of \( A \) with rational endpoints, let \( \tau_1 = \{ m \in M \mid (\exists C \subseteq I)(C \text{ is appropriate for } m) \} \) and for every \( m \in \tau_1 \), let \( C(I, m) \subseteq I \) be appropriate for \( m \). For every \( n \in \tau_1 \), let \( C(n) = \bigcup \{ C(I, m) \mid m \in \tau_1 \land n \leq m \} \). It is already standard to construct \( \{ B_0(m) \mid m \in M \} \), \( \{ B_1(m) \mid m \in \tau \} \subseteq K \) such that: (1) \( B_0(m) \in F(m) \); (2) \( B_1(m) \) is a dense subset of \( C(m) \); and (3) for every \( m \neq n \), \( B_1(n) \notin B_0(m) \). (Here we assume that \( B_0(n) = \emptyset \) if \( n \notin \tau \).)

We check that \( \tau_1 \), \( \{ B_0(m) \mid m \in M \} \) and \( \{ B_1(m) \mid m \in \tau \} \) satisfy the requirements of the lemma. Requirements (1)–(3) are automatically satisfied, (5) and (6) are easily checked. We deal with (4). As in the proof of 10.8, we have to prove the following claim. Let \( \chi = \bigwedge_{m \in M} \tau_0^{y_0} \wedge \bigwedge_{t \in T} \{ y_t \} \) where
\[
T = \{ x_0^m \mid m \in M_0 \} \cup \{ x_0^n \mid m \in M \} \cup \{ (x_0^n)^{y_0} \mid m \in \tau \} \cup \{ (x_1^n)^{y_1} \mid m \in \tau \}
\]
and \( T \) intersects the union of the first three sets in at most one element; and let \( s \) be the assignment such that \( s(x_0^m) = B'(m) \), \( s(y_0^n) = A(m) \) and \( s(y_1^n) = A(m) \). Then if \( K \vdash \neg \chi[s] \), then \( \chi \) is not a conjunct of \( \psi_t \).

In fact, the above claim has to be proved just for \( \chi \)'s in which \( \tau \) is finite. This follows easily from the fact: (a) “For every \( L \subseteq M \), if every finite \( L_0 \subseteq L \) has a lower bound, then \( L \) has a lower bound”.

Of the many cases in 10.8 we check only that case which calls for a new argument. Let \( \chi = \bigwedge_{m \in \sigma} \tau_0^{y_0} \wedge y_1^n \) and suppose that \( K \vdash \neg \chi[s] \). We have to prove that \( \chi \) is not a conjunct of \( \psi_t \), that is, we have to prove that there is \( n \in \tau \) such that for every \( m \in \sigma \), \( n \leq m \). \( K \vdash \neg \chi[s] \) means that \( A \wedge A^{y_0} \neq A(m) \), hence let \( C \in K \) be such that \( C \subseteq A \), and \( C \subseteq A(m) \) for every \( m \in \sigma \). If for some \( n \in M \) and some \( B \in S(m) \), \( B \subseteq C \), then \( n \in \tau \) and \( n \leq m \) for every \( m \in \sigma \), hence we are through. Suppose the above does not happen. We prove the following claim.
Claim. There is $K \equiv D \leq C$ such that every finite subset of $L = \{ m \mid \neg(A(m) \equiv D) \}$ has a lower bound.

Proof. Suppose the above claim is not true. We define a tree $\langle T, \leq \rangle \subseteq ^* \omega$, and for every $\nu \in T$ we define $m(\nu) \in M$ and $C_\nu \in K$ such that: (1) every member of $T$ has at least two successors; (2) for every $\nu \in T$, $C_\nu \leq C$; (3) if $\nu \leq \xi$, then $m(\xi) \leq m(\nu)$; and (4) if $\nu \in T$ and $\{ \xi_0, \ldots, \xi_{n-1} \}$ is the set of successors of $\nu$ in $T$, then $\langle m(\xi_0), \ldots, m(\xi_{n-1}) \rangle$ does not have a lower bound in $M$. The construction is done easily by induction.

We show that there is a branch $\{ \nu_i \mid i \in \omega \}$ of $T$ such that $\{ m(\nu_i) \mid i \in \omega \}$ does not have a lower bound. Let $\{ n_i \mid i \in \omega \}$ be an enumeration of $M$. Let $\nu_0 = \lambda$. Suppose $\nu_i$ has been defined. By (4) there is a successor $\xi$ of $\nu_i$ such that $m(\xi) \not\leq m(\nu_i)$. Let $\nu_{i+1} = \xi$. Hence $\{ m(\nu_i) \mid i \in \omega \}$ does not have a lower bound, however every finite subset of $\{ m(\nu_i) \mid i \in \omega \}$ has a lower bound. This contradicts property (a) of $M$. This concludes the proof of the claim. □

Let $D$ be as assured in the claim, and let $L$ be as defined in the claim, and let $n$ be a lower bound for $L$. We check that $D$ is appropriate for $n$. By our assumption for every $m \in M$ and $B \in S(m)$, $B \not\leq D$, and it follows from the properties of $L$ and $D$ that if $\neg(A(m) \equiv D)$, then $n \not\leq m$. Hence $n \in \tau$. For every $m \in \sigma$, $D \not\leq A(m)$, hence $m \in L$ and so $n \leq m$. We have thus found $n \in M$ such that $n \in \tau$ and for every $m \in \sigma$, $n \leq m$. This concludes the proof of 10.24 and the proof of 10.14(a). □

We leave the proofs of 10.14(b) and (c) to the reader, since they do not involve any new difficulties.

11. MA + OCA implies $2^\omega = \aleph_2$

In this short section we show that MA + OCA $\Rightarrow 2^\omega = \aleph_2$. This fact follows from the following theorem.

Theorem 11.1 (ZFC). There is a c.c.c. forcing set $P$ of power $\aleph_2$ and a family $\{ D_\nu \mid \nu < \aleph_2 \}$ of dense subsets of $P$ such that if $G \in V$ is a filter of $P$ which intersects every $D_\nu \nu < \aleph_2$, then $V$ contains a set $X \in \mathbb{R}$ and an open coloring $\mathcal{U} = \{ U_0, U_1 \}$ of $X$ such that $|X| = \aleph_1$ and $X$ cannot be partitioned into countably many $\mathcal{U}$-homogeneous sets.

Clearly, Theorem 11.1 implies that MA + OCA $\Rightarrow 2^\omega = \aleph_2$, for if $V \models MA + 2^\omega > \aleph_2$, then $V$ contains a filter $G \subseteq P$ which intersects the $D_\nu$'s, the thus $V$ contains a coloring refuting OCA.

Moreover, Theorem 11.1 shows that it is consistent that $V \models FA_1$, but still if
$W \supseteq V$ and $W \models \text{MA} + \text{OCA}$, then $\text{N}_0^{\omega_0}$ or $\text{N}_2^{\omega_0}$ are collapsed. This is true for the universe $V$ which is obtained in the following way. Let $V_0 \models \text{CH}$, and let $V = V_0 \{ P \in \text{Coll}(\theta_3) \} \mathbb{P}$ where $P$ is the forcing set of 11.1. The proof of 11.1 is divided into three claims.

**Lemma 11.2.** There is a symmetric function $F \in V$, $F : \text{N}_2 \times \text{N}_2 \rightarrow \text{N}_1$ such that for every universe $W \supseteq V$ and $A \in W$: if $\text{N}_1^W = \text{N}_1^1$, $\text{N}_2^W = \text{N}_2^2$, $A \subseteq \text{N}_2$ and $|A| = \text{N}_1$, then $F(A \times A)$ is unbounded in $\text{N}_1$.

**Proof.** For every $\xi \in \text{N}_1$, let $\{ \alpha(\nu, \alpha) \mid \alpha < \text{N}_1 \}$ be a 1-1 enumeration of $\nu$, and for $\xi < \text{N}_2$, let $F(\xi, \nu) = 0$ if $\xi = \nu$ or if $\nu < \text{N}_1$, and let $F(\xi, \nu) = \alpha$ if $\xi = \alpha(\nu, \alpha)$. It is easy to check that $F$ is as required. $\square$

We reserve the symbol $F$ to denote a function as in 11.2. Recall that for a set $A$, $D(A) = \bigwedge A - \{ (a, b) \mid a \in A \}$.

**Lemma 11.3.** Let $A = \{ a_\alpha \mid \alpha < \text{N}_1 \} \subseteq \omega_2$, $\mathcal{U} = \{ U_0, U_1 \}$ be a partition of $D(A)$ into symmetric open sets, and let $\{ H_\nu \mid \nu \in \text{N}_2 \}$ be such that: for every $\nu < \text{N}_2$ and $\nu < \text{N}_1$, $D(H_\nu) \subseteq U_0$ and for every $\nu, \xi < \text{N}_2$ there is a $\alpha(\nu, \xi) < \text{N}_1$ such that $\nu \in \alpha(\nu, \xi) \subseteq \{ a_\alpha \mid \alpha < \text{N}_1 \}$ and $\alpha(\nu, \xi) \in F(\nu, \xi)$. Then $A$ cannot be partitioned into countably many $\mathcal{U}$-homogeneous subsets.

**Proof.** Suppose by contradiction $\{ A_i \mid i \in \omega \}$ is a partition of $A$ into $\mathcal{U}$-homogeneous sets, and let $e(i)$ be such that $D(A_i) \subseteq U_e(i)$. For every $\nu < \text{N}_2$ let

$$\beta(\nu) = \text{Sup}(\alpha \mid (\exists i \in \{ 0, 1 \}) (e(i) = i \wedge a_\alpha \in A_i \wedge H_\nu \subseteq \{ a_\alpha \mid \alpha < \text{N}_1 \})).$$

$\beta(\nu)$ is a supremum of a countable set, hence $\beta(\nu) < \text{N}_1$. Let $\Gamma \subseteq \text{N}_2$ and $\beta_0 < \text{N}_1$ be such that $|\Gamma| = \text{N}_2$ and for every $\nu \in \Gamma$, $\beta(\nu) = \beta_0$. By the property of $F$ there are $\nu, \xi \in \Gamma$ such that $F(\nu, \xi) > \beta_0$. Hence $\alpha_0 = \alpha(\nu, \xi) > \beta_0$. Suppose $\alpha(\nu, \xi) \in A_0$. If $e(i) = 0$, then the fact that $a_\alpha \in A_0 \cap H_\nu$ implies that $\beta(\xi) \geq \alpha_0 > \beta_0$ and if $e(i) = 1$, then the fact that $a_\alpha \in A_1 \cap H_\nu$ implies that $\beta(\nu) > \alpha_0 > \beta_0$. In both cases we obtain that for some $\xi \in \Gamma$, $\beta(\xi) > \beta_0$, contradicting the choice of $\Gamma$. $\square$

**Lemma 11.4 (ZFC).** There is a c.c.c. forcing set $P$ of power $\text{N}_2$ and a family $\{ D_\nu \mid \nu < \text{N}_2 \}$ of dense subsets of $P$ such that if $V$ contains a filter of $P$ which intersects every $D_\nu$, $\nu < \text{N}_2$, then $V$ contains a system $A = \{ a_\alpha \mid \alpha < \text{N}_1 \}$, $\mathcal{U} = \{ U_0, U_1 \}$ and $\{ H_\nu \mid \nu = 0, 1 \}$ as in 11.3.

**Proof.** Let $F$ be as assured by Lemma 11.2. We first define $P$. Each element $p$ of $P$ is a triple $(\mathcal{U}(p), g(p), f(p))$ where:

1. $\mathcal{U}_p = (U(p, 0), U(p, 1))$ is a pair of disjoint symmetric clopen subsets of $\omega_2$ such that $U(p, 0), U(p, 1) \subseteq D(\omega_2)$. $\mathcal{U}_p$ is an approximation of the coloring $\mathcal{U}$ we wish to construct.
(2) \( g(p) \) is a function such that \( \text{Dom}(g(p)) = \sigma(p, 0) \times \sigma(p, 1) \) where the \( \sigma(p, l) \)'s are finite subsets of \( \mathbb{N} \), \( \text{Rng}(g(p)) \subseteq \mathbb{N} \), and for every \( (\alpha, \xi) \in \text{Dom}(g(p)), F(\nu, \xi) \leq g(p)(\nu, \xi) < F(\nu, \xi) + \omega \). We denote \( g(p)(\nu, \xi) \) by \( g(p, \nu, \xi) \). \( g(p) \) is a finite approximation of the function \( \alpha(\nu, \xi) \) of 11.3, that is, \( g(p, \nu, \xi) = \alpha \) will mean that \( \{\alpha_n\} \subseteq \mathbb{N}^\omega \cap \mathbb{N}^\omega \).

(3) \( f(p) \) is a function such that \( \sigma(p) = \text{Dom}(f(p)) \) is a finite subset of \( \text{Rng}(g(p)) \times \omega_n \) and \( \text{Rng}(f(p)) \subseteq \{0, 1\} \). \( f(p) \) is a finite function about the reals \( a_n \) where \( \alpha \in \text{Rng}(g(p)) \), that is \( f(p)(\alpha, n) = 0 \) will mean that for the real \( a_n \) of 11.3, \( a_n(n) = 0 \). We denote \( f(p)(\alpha, n) = f(p, \alpha, n) \).

A triple \( p = (\mathfrak{U}(\alpha), g(p), f(p)) \) as above belongs to \( P \) if:

(1) For every distinct \( \alpha, \beta \in \text{Rng}(g(p)) \), \( f(p) \) already determines the \( \mathfrak{U}(p) \)-color of \( \langle a_n, a_m \rangle \), that is, if for every \( \gamma < \mathbb{N} \) we denote by \( f(p, \gamma) \) the function such that for every \( n \in \alpha, f(p, \gamma)(n) = f(p, \gamma, n) \), then there is \( i \in \{0, 1\} \) such that

\[
U(p, \alpha, \beta) \equiv \{ (a, b) \in \omega^2 \mid a \equiv f(p, \alpha), b \equiv f(p, \beta) \} \subseteq U(p, l).
\]

(2) If \( \alpha_1, \alpha_2 \in \text{Rng}(g(p)) \), are distinct and for some \( \nu, \xi_1, \xi_2, \alpha_1 = g(p, \nu, \xi) \), then the coloring of \( \langle \alpha_1, \alpha_2 \rangle \) is determined to be 0, that is \( U(p, \alpha_1, \alpha_2) \subseteq U(p, 0) \), and if \( \alpha_1 = g(p, \xi_1, \nu) \), then the coloring of \( \langle \alpha_1, \alpha_2 \rangle \) is determined to be 1, that is \( U(p, \alpha_1, \alpha_2) \subseteq U(p, 1) \).

\( p \equiv q \) if \( U(p, l) \equiv U(q, l) \), \( l = 0, 1 \), \( g(p) \equiv g(q) \) and \( f(p) \equiv f(q) \). This concludes the definition of \( P \).

We leave it to the reader to check that by means of a family \( \{D_\nu \mid \nu < \mathbb{N}_2\} \) of dense subsets of \( P \) one can assure the existence of a system \( \mathfrak{U}, \{H_\nu \mid \nu \in \{0, 1\}, \nu < \mathbb{N}\} \) as required. We now turn to the proof that \( P \) is c.c.c.

Let \( \{ \mathfrak{U}_n \mid \alpha < \mathbb{N}_2 \} \subseteq P \). We uniformize \( \{ \mathfrak{U}_n \mid \alpha < \mathbb{N}_2 \} \) as much as possible. Hence we can assume that for some \( \mathfrak{U} = \langle U_1, U_2 \rangle \) for every \( \alpha < \mathbb{N}_2 \), \( \mathfrak{U}(p_\alpha) = \mathfrak{U}_\alpha \), and that \( \{\sigma(p_\alpha) \mid \alpha < \mathbb{N}_2\} \) are \( \Delta \)-systems for \( l = 0, 1 \). Let \( \sigma(p_\alpha) = \{ \nu(\alpha, l, 0), \ldots, \nu(\alpha, l, m_\alpha - 1) \} \), and \( m_\alpha \leq n \) be such that for every \( i < m_\alpha \) and \( \beta, \gamma < \mathbb{N}_2, \nu(\beta, l, i) = \nu(\gamma, l, i) \). We assume that for every \( i_0 < i_0 \) and \( i_1 < i_1 \), either all the \( g(\nu, \nu(\alpha, l, 0), \nu(\alpha, l, 1)) \) are pairwise distinct, or they are all equal. Finally we assume that letting \( \gamma(\alpha, i) \) denote \( g(\nu, \nu(\alpha, 0, l), \nu(\alpha, 1, i)) \), for every \( \alpha, \beta < \mathbb{N}_2 \), \( i_0 < i_0 \) and \( i_1 < i_1 \), \( f(\nu, \gamma(\alpha, i, 0, i)) = f(\nu, \gamma(\alpha, i, 1, i)) \).

We prove that for every \( \alpha, \beta, p_\alpha \) and \( p_\beta \) are compatible. Let \( q' = \langle \mathfrak{U}, g(p_\alpha), \bigcup f(p_\beta), f(p_\alpha) \cup f(p_\beta) \rangle \). \( q' \) is not a condition, but we prove that \( q' \) can be extended to a condition \( q \). First we check that \( q' \) does not contain contradictions. Since \( f(\nu, \gamma(\alpha, i_0, i_1)) = f(\nu, \gamma(\alpha, i_0, i_1)) \) for every \( i_0 > i_1 \), and since \( U_1 \) and \( U_2 \) do not intersect the diagonal of \( \times \omega \), \( \mathfrak{U} \) determines the color of \( \langle a_{\gamma(\alpha, i_0, i_1)}, a_{\gamma(\alpha, i_0, i_1)} \rangle \) iff \( i_0 \neq i_1 \), and if \( i_0 \neq i_1 \), then the color of the above pair is equal to the color of \( \langle a_{\gamma(\alpha, i_0, i_1)}, a_{\gamma(\alpha, i_0, i_1)} \rangle \). This implies that \( q' \) makes no mistakes in determining colors.

Let \( q = g(p_\alpha) \cup g(p_\beta) \), and \( \sigma_1 = \sigma(p_\alpha) \cup \sigma(p_\beta), l = 0, 1 \). \( \text{Dom}(q) \neq \sigma_0 \times \sigma_1 \), hence we have to define \( g \equiv g' \) such that \( \text{Dom}(g) = \sigma_0 \times \sigma_1 \). Since for every
ν, ξ < ω there are $N_0$ options of how to define $g(ν, ξ)$ and since $\text{Rng}(g')$ is finite for every $(ν, ξ) ∈ σ_0 σ_1 − \text{Dom}(g')$ we can find $g(ν, ξ) ∈ [F(ν, ξ), F(ν, ξ) + ω] − \text{Rng}(g')$ so that $g − g'$ is 1-1. We leave it to the reader to check that $\emptyset$ and $f(p_a) ∪ f(p_b)$ can be extended so that the color of every pair in $\{a, b| γ ∈ \text{Rng}(g)\}$ will be determined, and every such pair will have the right color. We thus constructed a condition extending $p_a$ and $p_b$, hence $p_a$ and $p_b$ are compatible. □

**Discussion.** Let A denote the axiom $\text{MA} + 2^{κ} > \aleph_2$. We found that A is not consistent with OCA, on the other hand it is consistent with BA, SOCA, NWD, RHA, and many other axioms whose consistency can be proved with the aid of the club method. There are two cases in which we do not know the answer to such a question.

**Question 11.5.** (a) Is A consistent with SOCA1?

(b) For which (finite) lattices with involution $L$ is $A + (K^H ≡ L)$ consistent?

Question 11.5(a) is related to the following questions:

**Question 11.6.** (a) Is $\neg \text{CH}$ consistent with the following axiom: “If $B$ is an uncountable set of reals and $\{A_i| i ∈ I\}$ is a family of nwd subsets of $B$, which contains all finite subsets of $B$, and such that $B$ is not contained in a union of countably many $A_i$’s, then there is an uncountable $B' ⊆ B$ such that $B'$ intersects each $A_i$ in at most $N_0$ points”?

(b) Is there a universe $V$ such that $V^+2^κ > \aleph_2$ and such that for every c.c.c. forcing set $P$ of power $<2^κ$, $V^P$ has the following property: “If $\{A_i| i ∈ I\}$ is a family of $<2^κ$ subsets of $\aleph_2$ such that $\aleph_1$ is not the union of any countably many $A_i$’s, then there is an uncountable $A ⊆ \aleph_1$ which intersects each $A_i$ in at most $N_0$ points”?

**Question 11.7.** Is OCA consistent with $2^κ > \aleph_2$?

**References**


