Handbook of Set Theory

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I. Proper forcing

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1. Introduction

It is sometimes the case that new concepts not only widen our horizons, but also bring difficult old results into main stream and within common knowledge. Properness, introduced and developed by Shelah, is such a concept and the wealth of results, both old and new, that it provides justifies its early introduction into advanced set theory courses. My aim is to provide an introductory exposition of the theory of proper forcing which will also give some of its interesting applications, to the point where the reader can continue with research papers and with the more advanced material in Shelah's book [15]. I assume that the reader has some knowledge of axiomatic set theory and is familiar with the basics of the forcing method, including some iterated forcing (the consistency proof of Martin's Axiom is sufficient).

We deal here with countable support iteration. This type of iteration appeared in Jensen's consistency proof of the continuum hypothesis with the Souslin hypothesis, and it also appeared in Laver's work on Borel's conjecture (see [3] and [10]). (These two outstanding results will not be treated here. They have now simpler proofs in which the theory of proper forcing is used to concentrate on the single step of the iteration.)

The chapter contains four parts. First: preservation of properness in countable support iteration. Second: preservation of the ${}^{\omega}\omega$-bounding property, and an application concerning non-isomorphism of ultrapowers of elementarily equivalent structures. Third: preservation of unboundedness, and an application concerning two cardinal invariants, the bounding number and the splitting number. Fourth: Dee-completeness, forcings that add no countable sets of ordinals. These results are all due to Saharon Shelah, and most of them appear in Chapters V and VI of his Proper Forcing book ([15]), but are presented here at a more concrete level, and sometimes with simpler proofs. (Theorem 5.8, concerning the chromatic number of Hajnal–Máté graphs, is due to the author.)

Our notations follow a standard usage, and the introduction describes them and reviews some elementary facts about forcing. Note that $a \leq b$ in a forcing poset denotes here that b carries more information than a. We say then that b extends a.

A *preorder* is a transitive and reflexive relation on a set (its domain). If $\leq$ is a preorder on A and $a \leq b$ then we say that b *extends* a. We say that a , $b \in A$ are *compatible* if there exists some $x \in A$ that extends a and b. Otherwise, a and b are *incompatible*, and a set of pairwise incompatible elements is called an *antichain*. A set $D \subseteq A$ is *dense* iff

$$\forall a \in A \; \exists d \in D \; (a \leq d),$$

and D is predense iff

$$\forall a \in A \; \exists d \in D \; (a \text{ and } d \text{ are compatible}).$$

A preorder is *separative* iff whenever b does not extend a there is an extension of b that is incompatible with a. We say that $P = (A, \leq)$ is a *forcing poset* iff $\leq$ is a separative preorder on A with a minimal element 0_P. We often write $\leq_P$ for the preordering of P. In the context of iteration, preorderings are more convenient than (antisymmetric) orderings, because this is what one naturally obtains. (The reader who prefers orderings can remain with posets of course, but at the price of taking quotients with their notational burden.)

Let $P = (A, \leq)$ be a forcing poset. We say that $G \subseteq A$ is a *filter* iff G is downwards closed: $\forall x \leq y \; (y \in G \implies x \in G)$, and any two members of G are compatible in G. We say that G is (V, P)-generic iff G is a filter over P that meets (has a non-empty intersection with) every dense set of P that lies in V. It is convenient to employ (V, P)-generic filters and to be able to speak about actual generic extensions $V[G]$. I assume that the reader knows how to avoid such ontological commitments.

V^P is the class of all P forcing names in V. If $a \in V^P$ and G is (V, P)-generic, then $a[G]$ (or a^G) denotes the interpretation of a in $V[G]$. It is convenient to define interpretation of names in such a way that every set in V can be interpreted as a name: $a[G] = \{y[G] \mid \exists p \in G \; (\langle p, y\rangle \in a)\}$. Usually for a set $a \in V$, $\check{a}$ denotes the (canonical) name of a. However, often a rather than $\check{a}$ is written here in forcing formulas, for graphical clarity and since this is very rarely a source of confusion. If φ is a forcing formula then "φ holds in V^P" means that $0_P \Vdash_P \varphi$. Often $\Vdash_P \varphi$ is written instead of $0_P \Vdash_P \varphi$. Also, I seldom put quotation marks around forcing formulas. The canonical name of the (V, P)-generic filter is denoted $\utilde{G}$.

The reason for employing separative posets is the following characterization: P is separative iff for every p , $q \in P$ $(p \Vdash_P q \in \utilde{G})$ implies $q \leq p$. Notice that p , $q \in P$ is written rather than the more accurate p , $q \in A$. A poset may be used as a name for its own universe.

The relation $P \lhd Q$ on forcing posets P and Q means that there is a function (projection) $\pi : Q \to P$ such that

1. π is order preserving (that is $q_1 \leq_Q q_2$ implies $\pi(q_1) \leq_P \pi(q_2)$) π is onto P, and $\pi(0_Q) = 0_P$.

2. For every $q \in Q$ and $p' \in P$ such that $p' \geq \pi(q)$ there is a $q' \geq q$ in Q such that $\pi(q') = p'$.

If $P \lhd Q$ and $D \subseteq P$ is dense, then $\pi^{-1}D$ is dense in Q. Hence if H is a (V, Q)-generic filter, then its image π“H generates a (V, P)-generic filter.

We say that π is a *trivial* projection iff $\pi(q_1) = \pi(q_2)$ implies that q_1, q_2 are compatible in Q. It can be seen that π is trivial iff for every P generic filter G, $\pi^{-1}G$ is also a (generic) filter over Q.

In many applications the projection of $P \lhd Q$ satisfies a stronger property than 2: for every $p \in P$ and $q \in Q$ if $p \geq \pi(q)$ then q has an extension $q_1 \in Q$, denoted $p + q$, such that

1. $\pi(q_1) = p$ and

2. if $r \in Q$ is such that $r \geq q$ and $\pi(r) \geq p$, then $r \geq q_1$.

If this additional property holds then every $p \in P$ can be identified with $i(p) = p + 0_Q$. (That is, P can be assumed to be a subposet of Q.)

If $Q \in V^P$ is a forcing poset (that is, by our convention, forced by the zero condition to be a forcing poset), then $P * Q$, the two-step iteration, is the forcing poset defined as follows. First some sufficiently large set V_α is chosen so that if $b \in V^P$ is any name then there is already a name $a \in V_\alpha$ such that for every $p \in P$, if $p \Vdash_P b \in Q$, then $p \Vdash_P b = a$. (V_α here is the set of all sets of rank $< \alpha$.) Now form $P * Q$ as the set of all pairs (p, q) such that $p \in P$, $q \in V_\alpha \cap V^P$ and $p \Vdash_P q \in Q$. The preordering $\leq = \leq_{P*Q}$ is defined by

$$(p, q) \leq (p', q') \quad \text{iff} \quad p \leq_P p' \quad \text{and} \quad p' \Vdash_P q \leq_Q q'.$$

The map $\pi(\langle p, q \rangle) = p$ is a projection. Usually one does not bother to define the set V_α from which the names q are taken and $P * Q$ is presented as a class.

I will summarize (without proofs) some basic facts about two-step iterations. The two notions, projection and two-step iteration, are closely related. If $P \lhd R$ with projection $\pi : R \to P$, and if G_0 is a (V, P)-generic filter, then form in $V[G_0]$ the following set

$$Q = R/G_0 = \{r \in R \mid \pi(r) \in G_0\}$$

and define a partial, separative preorder $\leq = \leq_{R/G_0}$ on Q by

$$r_1 \leq r_2 \text{ iff every } \leq_R \text{ extension of } r_2 \text{ in } R/G_0 \text{ is} \\ \leq_R -\text{compatible with } r_1 \text{ in } R/G_0. \tag{I.1}$$

Equivalently,

$$r_1 \leq r_2 \text{ iff } r_1 \leq_R g + r_2 \text{ for some } g \in G_0 \text{ with } \pi(r_2) \leq_P g. \tag{I.2}$$

It follows that $r_1 \leq r_2$ iff for every $g_1, g_2 \in G_0$, $g_1 + r_1 \leq g_2 + r_2$. Sometimes, we write $R/\underset{\sim}{G}_0$, or even R/P, for the V^P name of $Q = R/G_0$.

1.1 Lemma. *Assume $P \lhd R$ as above and G_0 a (V, P)-generic filter. Suppose that in $V[G_0]$ we have sequence $\langle r_i \mid i \in \omega \rangle$ such that $r_i \in R/G_0$ and $r_i \leq_{R/G_0} r_{i+1}$. Then there is a sequence $\langle s_i \mid i \in \omega \rangle$ such that each s_i has the form $s_i = g_i + r_i$ for some $g_i \in G_0$ and $s_i \leq_R s_{i+1}$.*

Proof. Suppose that $g_i \in G_0$ is defined. Since $r_i \leq_{R/G_0} r_{i+1}$, there exists $g \in G_0$ such that $r_i \leq_R g + r_{i+1}$. We may take $g \geq g_i$. Since $\pi(g + r_{i+1}) = g$, we get $g_i + r_i \leq_R g + r_{i+1}$. So that $g_{i+1} = g$ works. ⊣

Now let $\underset{\sim}{Q}$ be the name of Q in V^P and form the two-step iteration $P * \underset{\sim}{Q}$. Then R with its original ordering $\leq_R$ is isomorphic to a dense subset of $P * \underset{\sim}{Q}$. Namely the map $r \mapsto (\pi(r), \check{r})$ is that embedding (where $\check{r}$ is the V^P name of r). Thus R itself can be viewed as a two step iteration: the projection P followed by the quotient poset whose name is denoted by $R/\underset{\sim}{G}_0$.

Suppose now that some forcing poset $Q \in V^P$ is given and $R = P * Q$ is formed. Then $P \lhd R$, with projection $\pi(p, q) = p$. The stronger property of projections holds here, and we can identify $p \in P$ with $(p, \underset{\sim}{0}) \in P * Q$. Let G_0 be (V, P)-generic, and form R/G_0 as above. $R/G_0 = \{(p, q) \in P * Q \mid p \in G_0\}$ ordered as in (I.1). Then the following holds for $(p_1, q_1), (p_2, q_2) \in R/G_0$

$$(p_1, q_1) \leq_{R/G_0} (p_2, q_2) \text{ iff } \exists p \in G_0 \text{ such that } (p_1, q_1) \leq_{P*Q} (p, q_2).$$

In $V[G_0]$, both R/G_0 and $Q[G_0]$ can be formed ($Q[G_0]$ is the interpretation of Q in $V[G_0]$.) These are essentially the same poset. That is, the map

$$i : (p, q) \mapsto q[G_0]$$

taking $(p, q) \in R/G_0$ into the interpretation of q is a trivial projection.

Let P and R be any posets such that $P \lhd R$. We have said that if H is a (V, R)-generic filter, then $G_0 = \pi``H$ is (V, P)-generic, and clearly $H \subseteq R/G_0$. In fact, H is a $(V[G_0], R/G_0)$-generic filter.

On the other hand, if G_0 is (V, P)-generic, and if G_1 is $(V[G_0], R/G_0)$-generic, then $G_1 \subseteq R$ is a (V, R) generic filter and $\pi``G_1 = G_0$.

Now, given a poset P, suppose that $Q \in V^P$ is a poset and $R = P * Q$ is formed. Let G_0 be (V, P)-generic, and G_1 be $(V[G_0], Q[G_0])$-generic, where $Q[G_0]$ is the interpretation of Q in $V[G_0]$. Define in $V[G_0]$ the trivial projection $i : R/G_0 \to Q[G_0]$ as above, and define $H = i^{-1}G_1$ in $V[G_0][G_1]$. That is

$$H = \{(p, q) \in P * Q \mid p \in G_0 \wedge q[G_0] \in G_1\}.$$

Then H is an $(R/G_0, V[G_0])$-generic filter (the pre-image of a trivial projection). Hence (by the preceding paragraph) H is also $(V, P * Q)$-generic, and $\pi``H = G_0$. Abusing the product notation we write $H = G_0 * G_1$.

Let $R = P * Q$. I want to explain the equation $V^R = (V^P)^Q$. We shall define a function

$$\rho : V^R \to V^P$$

from the R-names into the P-names such that the following holds for every (V, P)-generic filter H and $G_0 = \pi``H$ its projection.

1. As we have seen $G_0 = \pi``H = \{p \in P \mid (p, q) \in R \text{ for some } q\}$ is (V, P)-generic, and H is $(V[G_0], R/G_0)$-generic.

2. For every $a \in V^R$, $a[H] = (\rho(a)[G_0])[H]$. That is, the interpretation $a[H]$ of a in $V[H]$ can be obtained in two steps: first interpret $\rho(a)$ in $V[G_0]$, as a name, and then interpret this name in the remaining forcing over R/G_0.

Usually, we write a/G_0 instead of $\rho(a)[G_0]$ (without mentioning ρ) and then $a/\underset{\sim}{G}_0$ is written for $\rho(a)$.

To define ρ let *pair* $: V^P \times V^P \to V^P$ be a (definable) map such that for any $\tau_1, \tau_2 \in V^P$, $pair(\tau_1, \tau_2)[G_0] = \langle \tau_1[G_0], \tau_2[G_0] \rangle$. In plain words, $pair(\tau_1, \tau_2)$ is a canonical name for the pair formed from (the interpretations of) τ_1 and τ_2. Define now ρ by rank induction so that for any $a \in V^R$

$$\rho(a) = \{\langle \pi(r), pair(\check{r}, \rho(y)) \rangle \mid r \in R \text{ and } \langle r, y \rangle \in a\}.$$

Where $\check{r}$ is the canonical P-name for r. Suppose now generic filters $H \subset R$ and $G_0 \subset P$ defined as the projection $G_0 = \pi``H$. For any $a \in V^R$

$$\begin{aligned} \rho(a)[G_0] &= \{pair(\check{r}, \rho(y))[G_0] \mid (r, y) \in a \wedge \pi(r) \in G_0\} \\ &= \{\langle r, \rho(y)[G_0] \rangle \mid (r, y) \in a \wedge r \in R/G_0\}. \end{aligned}$$

Now the required equality

$$\rho(a)[G_0][H] = a[H]$$

is proved by the following equalities:

$$\begin{aligned}\rho(a)[G_0][H] &= \{\rho(y)[G_0][H] \mid \exists r \in H\ ((r,y) \in a \wedge r \in R/G_0)\}\\ &= \{y[H] \mid \exists r \in H\ (r,y) \in a\}\\ &= a[H].\end{aligned}$$

The function ρ can be defined as above whenever $P \lhd R$.

The following lemma is often used.

1.2 Lemma. *Assume that $P \lhd R$, and $D \subseteq R$ is dense. Let G be a (V, P)-generic filter. Then the following holds in $V[G]$: Every $q \in R$ with $\pi(q) \in G$ has an extension $d \in D$ with $\pi(d) \in G$. In other words, $D \cap R/G$ is dense in R/G.*

Proof. Suppose that $p \in P$ forces that $\pi(q) \in \underset{\sim}{G}$. Then $\pi(q) \leq_P p$, and hence there is an extension q_1 of q with $\pi(q_1) = p$. Now extend further q_1 to a condition $d \in D$, and then $\pi(d)$ forces $d \in D \cap R/G$ as required. $\dashv$

We will encounter the following situation in Section 3.

1.3 Lemma. *$Q_0 \lhd Q_1 \lhd Q_2$ are posets with projections $\pi_{i,j} : Q_i \to Q_j$ for $i > j$. The projections commute: $\pi_{2,0} = \pi_{1,0} \circ \pi_{2,1}$. Suppose G_0 is a (V, Q_0) generic filter, and form $Q'_1 = Q_1/G_0$ and $Q'_2 = Q_2/G_0$. Then $\pi_{2,1} : Q'_2 \to Q'_1$ is a projection and the quotient Q'_2/Q'_1 can be seen to be exactly Q_2/Q_1.*

Let $\underset{\sim}{G}_{(Q_1/G_0)}$ be the canonical name in $V[G_0]$ of the Q_1/G_0 generic filter. Then $(Q_2/G_0)/\underset{\sim}{G}_{(Q_1/G_0)} \in V[G_0]^{Q_1/G_0}$. If G_1 is a $(V[G_0], Q_1/G_0)$ generic filter, then the interpretation of that name, $(Q_2/G_0)/G_1$, is equal to Q_2/G_1. In addition, G_1 is (V, Q_1) generic and if $\underset{\sim}{f} \in V^{Q_2}$ then $(\underset{\sim}{f}/G_0) \in V[G_0]^{Q_1/G_0}$ and $(\underset{\sim}{f}/G_0)/G_1$ is $\underset{\sim}{f}/G_1$.

Countable support iterations

We deal here only with countable support iterations. An iteration of length γ (an ordinal) is defined by induction. For this, one needs a scheme to produce the next poset in the iteration. Suppose that this scheme is given by some function F (a formula that defines a function) such that for every forcing poset P, $F(P) = Q$ is in V^P a forcing poset. Then the iteration $\langle P_\alpha \mid \alpha \leq \gamma\rangle$ is defined so that:

1. Members of P_α are functions defined on α.
2. If $\alpha < \gamma$ then $P_\alpha = \{f\restriction\alpha \mid f \in P_\gamma\}$.

3. Every $f \in P_\gamma$ has a countable support, which is a set $S_f = S \subseteq \gamma$ such that f is trivial outside of S, that is $f(\xi)$ is the P_ξ name of the zeroth condition for $\xi \in \gamma \setminus S$.

The definition of the iteration is as follows.

1. P_0 is the trivial poset consisting of the minimal condition $\emptyset$ alone, and V^{P_0} is (or is isomorphic to) V. If P_α is already defined and $\alpha < \gamma$, then $P_{\alpha+1}$ is defined as the set of all functions f defined on $\alpha+1$ such that $f\restriction\alpha \in P_\alpha$, $f(\alpha) \in V^{P_\alpha}$ and
$$f\restriction\alpha \Vdash_{P_\alpha} f(\alpha) \in Q,$$
where $Q = F(P_\alpha)$. We define $f_1 \leq f_2$ iff $f_1\restriction\alpha \leq f_2\restriction\alpha$ and $f_2\restriction\alpha \Vdash_{P_\alpha}$ $f_1(\alpha) \leq f_2(\alpha)$. It is evident that $P_{\alpha+1}$ is defined to be isomorphic to $P_\alpha * Q$. (Since we want $P_{\alpha+1}$ to be a set, we must limit the possible values of $f(\alpha)$.)

2. If $\delta \leq \gamma$ is a limit ordinal, and P_i is defined for every $i < \delta$ then P_δ is the set of all countably supported functions f defined on δ and such that for every $\alpha < \delta$, $f\restriction\alpha \in P_\alpha$. Thus for every $\alpha < \delta$ $f(\alpha) \in V^{P_\alpha}$ and f has a countable support. Define $f_1 \leq f_2$ iff for every $\alpha < \delta$ $f_1\restriction\alpha \leq_{P_\alpha} f_2\restriction\alpha$.

We assume that the reader knows the basic properties of these countable support iterations. First that they form forcing posets, and then that if $\gamma_0 < \gamma$ then $\pi : P_\gamma \to P_{\gamma_0}$ defined by $\pi(p) = p\restriction\gamma_0$ is a projection of P_γ onto P_{γ_0}. If $q \in P_{\gamma_0}$ and $q \geq \pi(p)$, then $p_1 = q + p$ is defined in P_γ by the requirement that
$$p_1(\xi) = q(\xi) \text{ for } \xi < \gamma_0, \text{ and } p_1(\xi) = p(\xi) \text{ for } \xi \geq \gamma_0. \tag{I.3}$$
Strictly speaking P_{γ_0} is not a subset of P_γ, but in practice we identify $f \in P_{\gamma_0}$ with $f + 0_\gamma$ which is the trivial extension of f on γ. The ordering on P_γ is denoted $\leq_{P_\gamma}$ or just $\leq_\gamma$ for clarity. We shall often write $\Vdash_\gamma$ instead of $\Vdash_{P_\gamma}$.

The poset name $F(P_\alpha) = Q_\alpha$ is called "the αth iterand". In complex consistency proofs, the exact definition of F, and even its existence, is often passed over in silence. The term "bookkeeping device" is often invoked to refer to that part of the construction which is omitted.

2. Properness and its iteration

In this section we define *properness* and prove that the countable support iteration of proper forcing is proper. A slightly different proof can be read in [4].

In discussing proper forcing, the phrase "let λ be a sufficiently large cardinal, and H_λ the collection of sets of cardinality hereditarily $< \lambda$" appears so often that it deserves a remark. The role of H_λ is to encapsulate enough of the universe of sets V to reflect the statements in which we are interested. So the exact meaning of "sufficiently large" depends on the circumstances, and other reflecting sets such as V_λ can replace H_λ (but less naturally). When dealing with a forcing poset of cardinality κ, any cardinal $\lambda > 2^\kappa$ is sufficiently large for our purposes. We will be interested in countable elementary substructures of $\langle H_\lambda, \in, <, etc\rangle$, where $<$ is some fixed well-ordering of H_λ, and *etc* may include the poset P, the forcing relation, and other relevant parameters. The role of the well-ordering $<$ is to allow for inductive constructions. For notational clarity we just write $M \prec H_\lambda$, and omit the $\in$ relation, the well-order and the other parameters. We often say "$M \prec H_\lambda$ is as usual" to indicate that H_λ refers to a richer structure and the reader may have to include in M all those parameters that are relevant.

So let P be a poset, $M \prec H_\lambda$ be countable, with $P \in M$, as usual, and let G be (V, P)-generic. Define $M[G] = \{a[G] \mid a \in M\}$. So $M[G]$ is the set of all interpretations of names that lie in M. Since the forcing relation is (definable) in M (by virtue of the largeness of λ) the Forcing Theorem implies that $M[G] \prec H_\lambda[G]$ (for details, see Theorem 2.11 of [15]). However $M[G]$ is not necessarily a generic extension of M. What does it mean that $M[G]$ is a generic extension of M? This is a delicate question because of the special status of the members of M: on one account they are just points with no other meaning than that provided by the structure M itself, and on the other hand, they are bona fide members of H_λ and carry information of which M is not aware. By collapsing M onto a transitive structure only the local properties remain and we are no longer confused by this double role. So let $\pi : M \to \bar{M}$ be the transitive collapsing of M, and let $\pi``G = \bar{G}$ be the image of G. Then genericity of G over M means that $\bar{G}$ is $\bar{M}$-generic over $\pi(P)$. Namely that $\bar{G}$ has a non empty intersection with every dense subset of $\pi(P)$ that lies in $\bar{M}$. I find that collapsing is illuminating, but of course one can give a more direct definition: G is (M, P) generic iff for any $D \subseteq P$ dense in P such that $D \in M$, $G \cap D \cap M \neq \emptyset$.

Properness of P, as we shall see in a moment, ensures that this is the normal situation. A good example for a non-proper forcing is the forcing P that collapses ω_1 to ω. (The "conditions", that is members of P, are finite functions from ω into ω_1 and the ordering is extension.) Here it is obvious that the generic function $g : \omega \to \omega_1$ onto ω_1 is not $\bar{M}$-generic, since it involves ordinals not in $\bar{M}$.

To define properness, we need the concept of an (M, P)-generic condition. Let P be a poset and $M \prec H_\lambda$, with $P \in M$, be an elementary substructure. A condition $q \in P$ is said to be (M, P)-generic iff for every dense subset $D \subseteq P$ such that D is in M, $D \cap M$ is predense above q, i.e., for every $q_1 \geq q$ there is a $q_2 \geq q_1$, extending some $d \in D \cap M$. Sometimes, when the identity of P is clear from the context, we just say that q is an "M-generic" condition.

In the proof of the properness preservation theorem we shall employ the following

2.1 Lemma. *A condition q is (M, P)-generic iff for every $D \in M$ dense in P there is a name $\underset{\sim}{p} \in V^P$ such that*

$$q \Vdash_P \underset{\sim}{p} \in M \cap D \cap \underset{\sim}{G}.$$

2.2 Definition. A poset P is called *proper* iff for every $\lambda > 2^{|P|}$ and countable $M \prec H_\lambda$ with $P \in M$, every $p \in P \cap M$ has an extension $q \geq p$ that is an M-generic condition.

Properness, and genericity of a condition have the following equivalent property which is often used. A condition q is (M, P)-generic iff

$$q \Vdash_P M[\underset{\sim}{G}] \cap \mathrm{Ord} = M \cap \mathrm{Ord}.$$

Assuming a predicate V that denotes the ground model, we can replace this by

$$q \Vdash_P M[\underset{\sim}{G}] \cap V = M \cap V.$$

Thus P is proper iff for every $\lambda > 2^{|P|}$ and countable $M \prec H_\lambda$ with $P \in M$, every $p \in P \cap M$ has an extension $q \geq p$ such that for every $\tau \in M \cap V^P$ and every $q' \geq q$, if $q' \Vdash_P \tau \in V$, then $q' \Vdash_P \tau \in M$.

We will prove that in the definition of properness, the quantification "for every $\lambda > 2^{|P|}$ and every countable $M \prec H_\lambda$" can be weakened to "for $\lambda = (2^{|P|})^+$ (or, for some $\lambda > 2^{|P|}$) and for a closed unbounded set of $M \prec H_\lambda \ldots$", and the resulting definition is equivalent to the original. (See also Chapter III of [15].)

One of the first consequences of properness is the following. If P is proper, and G a (V, P)-generic filter, then $\aleph_1$ is not collapsed in $V[G]$. Moreover, every countable set of ordinals in $V[G]$ is included in an old countable set (from V). Indeed if q is M–generic and $\tau \in M$ is a name for an ordinal, then q forces τ to be in M. (By the alternative definition, or argue as follows. The set D of conditions that determine the value of τ is dense in P and is in M, and hence $D \cap M$ is dense above q which implies that every extension of q can be further extended to force $\tau = \alpha$ for some $\alpha \in M$.) It follows from this observation that if P is proper and forcing with P introduces no new subsets of ω, then forcing with P adds no new countable sets of ordinals.

The simplest examples of proper forcing posets are the countably closed posets and the c.c.c posets. It is illuminating to realize that despite the obvious difference between these two families of posets, they have (at some level of abstraction) the same reason for not collapsing ω_1—namely their properness. If P is c.c.c, then *any* condition is (M, P)-generic, and if P is countably closed then any upper bound of an (M, P)-complete sequence is (M, P)-generic. For an arbitrary proper poset, finding generic conditions is usually the main burden of the proof.

A large family of posets was defined by Baumgartner [1], and called Axiom A posets. It turns out that they are all proper.

2.3 Definition. A poset $(P, \leq)$ satisfies Axiom A iff there are partial orders $\langle \leq_i \mid i < \omega \rangle$ on P, with $\leq_0 = \leq$, such that:

1. For $i < j$, $\quad \leq_j \subseteq \leq_i$.
2. For every $p \in P$ and dense $D \subseteq P$, for every $n < \omega$ there are $p' \in P$ and countable $D_0 \subseteq D$ such that $p \leq_n p'$ and D_0 is predense above p' (i.e., if $p'' \geq p'$ then p'' is compatible with some condition in D_0).
3. If $\langle p_i \in P \mid i \in \omega \rangle$ is a sequence such that $p_i \leq_i p_{i+1}$, then there is $p \in P$ (called the fusion of the sequence) such that for every i, $p_i \leq_i p$.

The Sacks-Spector conditions (subtrees of $2^{<\omega}$ with arbitrarily high splitting) satisfy a stronger version (we call Axiom A^*), in which D_0 is required to be finite.

It is easy to prove that any Axiom A forcing is proper (see Baumgartner [1]). In fact, if $M \prec H_\lambda$ is countable with $P \in M$ and $p_0 \in P \cap M$, then for any i there is an (M, P)-generic condition p such that $p_0 \leq_i p$.

The projection of a proper poset is also proper. That is, if $P \lhd Q$ and Q is proper, then P is proper. In fact, if $M \prec H_\lambda$ and $q \in Q$ is (M, Q)-generic, then $\pi(q)$ is (M, P)-generic.

Another equivalent definition of when a condition is (M, P) generic can be obtained from the following lemma.

2.4 Lemma. *Let P be a poset, $M \prec H_\lambda$ countable with $P \in M$ (where $\lambda > 2^{|P|}$ so that $\mathcal{P}(P) \in H_\lambda$) and suppose that $p \in P$ is some (M, P)-generic condition. If $x, y \in M \cap V^P$ are such that $p \Vdash x \in y$, then for some $p_1 \geq p$ and $(a, b) \in y \cap M$, where $a \in P$ and $b \in V^P$, $a \leq p_1$ and $p_1 \Vdash x = b$ hold.*

Proof. In order to illustrate in a simple setting two possible approaches, we give two proofs for this lemma. It follows immediately from the definition of forcing that there is an extension of p (denoted p_1) and a pair $(a, b) \in y$ such that $p_1 \geq a$ and $p_1 \Vdash x = b$. The point of the lemma, however, is to get such a pair (a, b) in M whenever p is (M, P)-generic. Consider the set

$$E = \{p_0 \in P \mid \exists (a, b) \in y \; (a \leq p_0 \text{ and } p_0 \Vdash x = b\}.$$

Note that $E \in M$ is an open subset of P that is dense above p. Let $F \subseteq P$ be the set of all conditions that are in E or else are incompatible with every condition in E. Then $F \in M$ is dense (open) and so p is compatible with some $p_0 \in F \cap M$. Since E is dense above p, $p_0 \in E \cap M$ and hence the defining clause applies to p_0. But M is an elementary substructure, and hence there is a pair $(a, b) \in y \cap M$ with $a \leq p_0$ and such that $p_0 \Vdash x = b$. Which proves the lemma.

For the second proof, let $\pi_0 : M \to \bar{M}$ be the transitive collapse. Let G be an arbitrary (V, P)-generic filter containing p. Define $G_0 = \pi_0'' G \cap M$. Then G_0 is $(\bar{M}, \pi_0(P))$-generic filter and π_0 can be extended to $\pi_1 : M[G] \to \bar{M}[G_0]$. For every $m \in M \cap V^P$, $\pi_1(m[G]) = \pi_1(m)[G_0]$. We have $M[G] \prec H_\lambda[G]$, so that π_1^{-1} is an elementary embedding of $\bar{M}[G_0]$ into $H_\lambda[G]$.

Since $p \Vdash x \in y$, $x[G] \in y[G]$. Hence $\pi_1(x[G]) \in \pi_1(y[G]) = \pi_1(y)[G_0]$. So there are $(a_0, b_0) \in \pi_1(y)$ such that $a_0 \in G_0$ and $\pi_1(x[G]) = b_0[G_0]$. Hence, for $(a, b) = \pi_1^{-1}(a_0, b_0)$, $(a, b) \in y \cap M$ and we have $a \in G$ and $x[G] = \pi_1^{-1}(b_0[G_0]) = b[G]$.

Back in V, let $p_1 \geq p$ be an extension that forces these facts about (a, b). Namely $p_1 \Vdash a \in \utilde{G}$ (so that $p_1 \geq a$) and $p_1 \Vdash x = b$, as required. $\dashv$

The next lemma is used in the proof that the two-step iteration of proper posets is proper.

2.5 Lemma. *Let P be a forcing poset, and $Q \in V^P$ a forcing poset in V^P. Let $M \prec H_\lambda$ be countable with P, $Q \in M$, and suppose that λ is sufficiently large. Then $(p, q) \in P * Q$ is $(M, P * Q)$-generic iff*

$$p \text{ is } (M, P)\text{-generic}$$

and

$$p \Vdash_P q \text{ is } (M[\utilde{G}_0], Q)\text{-generic},$$

where $\utilde{G}_0$ is the canonical name for the P-generic filter.

Now we prove that the iteration of two proper posets is again proper, and in fact the following stronger claim holds which we prepare for later use.

2.6 Lemma. *Suppose that P_0 is proper, and P_1 is proper in V^{P_0} (that is, P_1 is a P_0-name and 0_{P_0} forces it to be proper). Let $R = P_0 * P_1$ be the two step iteration, and let $\pi : R \longrightarrow P_0$ be the projection defined by $\pi(p, \utilde{q}) = p$. Suppose that $M \prec H_\lambda$ is countable with $R \in M$. Not only has every $r \in R \cap M$ an M-generic extension, but the following hold in addition. Suppose that $p_0 \in P_0$ is an (M, P_0)-generic condition. For any name $\utilde{r} \in V^{P_0}$ if*

$$p_0 \Vdash_{P_0} \utilde{r} \in M \cap R \text{ and } \pi(\utilde{r}) \in \utilde{G}_0 \tag{I.4}$$

($\utilde{G}_0$ is the canonical name of the generic filter over P_0) then there is some $\utilde{p}_1 \in V^{P_0}$ such that $(p_0, \utilde{p}_1)$ is an M-generic condition and

$$(p_0, \utilde{p}_1) \Vdash_R \utilde{r} \in \utilde{G}.$$

(Being a P_0-name, $\underset{\sim}{r}$ is also an R-name and it may appear in R–forcing formulas. $\underset{\sim}{G}$ is the canonical name of the R-generic filter.)

Proof. Let G_0 be any (V, P_0)-generic filter containing p_0. The name $\underset{\sim}{r}$ is not necessarily in M, but by (I.4) it is interpreted as some condition r in $M \cap R$ such that $\pi(r) \in G_0$. Say $r = (r_0, \underset{\sim}{r}_1)$ where $\underset{\sim}{r}_1$ is a P_0 name for a condition in P_1. In $M[G_0]$, $\underset{\sim}{r}_1$ is interpreted as a condition r_1 in (the G_0 interpretation of) P_1. Since P_1 is proper, there is an extension p_1 of r_1 that is $M[G_0]$-generic. Let $\underset{\sim}{p}_1$ be a name of p_1 forced to have all of these properties. In particular

$$p_0 \Vdash_{P_0} \underset{\sim}{p}_1 \text{ extends the second component of } \underset{\sim}{r} \text{ in } P_1.$$

Then $u = (p_0, \underset{\sim}{p}_1)$ is as required. Firstly Lemma 2.5 gives that

$$u \text{ is } (M, R)\text{-generic.}$$

Secondly,

$$u \Vdash_R \underset{\sim}{r} \in \underset{\sim}{G} \tag{I.5}$$

is proved as follows. Observe that $\underset{\sim}{r}$ is not a condition, but a V^{P_0} name and hence a V^R name of a condition in R. However, any condition above p_0 can be extended to decide the value of $\underset{\sim}{r}$ as a condition in R. Suppose any $u' \in R$ that extends u and determines for some $r \in R$ that

$$u' \Vdash \underset{\sim}{r} = r,$$

where $r \in R$ is of the form $r = (r_0, \underset{\sim}{r}_1)$. To prove (I.5) we will show that $u' \Vdash r \in \underset{\sim}{G}$. Assume that $u' = (u'_0, \underset{\sim}{u}'_1)$. Since $u'_0 \Vdash_{P_0} \pi(\underset{\sim}{r}) \in \underset{\sim}{G}_0$, and as P_0 is separative, $r_0 \leq u'_0$. But

$$u'_0 \Vdash \text{“}\underset{\sim}{p}_1 \text{ extends } \underset{\sim}{r}_1 \text{ (the second component of } \underset{\sim}{r})\text{”},$$

and this implies $r \leq u'$ in R. Thus $u' \Vdash r \in \underset{\sim}{G}$. Since u' is an arbitrary extension of u that identifies $\underset{\sim}{r}$, $u \Vdash_R \underset{\sim}{r} \in \underset{\sim}{G}$. ⊣

2.1. Preservation of properness

We prove here that the countable support iteration of proper forcing posets is proper. The expression “$\langle P_\alpha \mid \alpha \leq \gamma \rangle$ is an iteration of posets that satisfy property X” (such as properness) means that each successor stage $P_{\alpha+1}$ (which is isomorphic to $P_\alpha * Q_\alpha$) is formed with an iterand $Q_\alpha \in V^{P_\alpha}$ that satisfies property X in V^{P_α}.

2.7 Theorem. *Let δ be a limit ordinal. Suppose that $\langle P_\alpha \mid \alpha \leq \delta \rangle$ is a countable support iteration of proper forcings. Then P_δ is proper.*

We assume that δ is a limit ordinal, since the successor case was handled by Lemma 2.6. The inductive proof of the theorem is a paradigm for all preservation theorems given here, but first an intuitive (yet incorrect) overview of the proof is given. Let be given a countable structure $M \prec H_\lambda$, with $P_\delta \in M$, and a specified condition $p_0 \in P_\delta \cap M$. We are required to extend p_0 to an (M, P_δ)-generic condition. This is done in ω steps. Fix an increasing ω-sequence $\gamma_i \in \delta \cap M$, unbounded in $\delta \cap M$. (The sequence itself is not assumed to be in M, only its members.) At the n-th step we want to define $q_n \in P_{\gamma_n}$ that is (M, P_{γ_n})-generic, and is an extension of $p_0 \restriction \gamma_n$. We also require that $q_{n+1} \restriction \gamma_n = q_n$. The final condition $q = \bigcup_{n<\omega} q_n$ is in P_δ, and it extends the given condition p_0 since each initial condition does. Now at the n-th step we must also take care of D_n, the n-th dense set of P_δ in M in some pre-fixed enumeration of all the dense subsets of P_δ that are in M. It follows that we need at this step an auxiliary condition $p_n \in P_\delta \cap M \cap D_{n-1}$, that extend p_{n-1} and such that q_n extends $p_n \restriction \gamma_n$. We will first extend p_n to some $p_{n+1} \in D_n$ and then commit all the following q_m's to extend $p_{n+1} \restriction \gamma_m$ as well. Surely we cannot succeed in such a construction, for if we do, then $q \in \bigcap_{n\in\omega} D_n$ and this is too much (unless no reals are added, but this is a different story). So where did we go astray? When we claimed that p_{n+1} with $p_{n+1} \restriction \gamma_n \leq q_n$ can be found in D_n. We could do that only in case $\{r \in P_{\gamma_n} \cap M \mid r \leq q_n\}$ is a generic filter over M. Otherwise we may only have a *name* of such p_{n+1}. It turns out that this is enough for the proof, but we must formulate a slightly more involved inductive assumption.

2.8 Lemma. *(The Properness Extension Lemma). Let $\langle P_\alpha \mid \alpha \leq \gamma \rangle$ be a countable support iteration of proper forcing posets. Let λ be a sufficiently large cardinal. Let $M \prec H_\lambda$ be countable, with $\gamma, P_\gamma \in M$ etc. For any $\gamma_0 \in \gamma \cap M$, and $q_0 \in P_{\gamma_0}$ that is (M, P_{γ_0})-generic the following holds. If $\underset{\sim}{p}_0 \in V^{P_{\gamma_0}}$ is such that*

$$q_0 \Vdash_{P_{\gamma_0}} \underset{\sim}{p}_0 \in P_\gamma \cap M \quad \text{and} \quad \underset{\sim}{p}_0 \restriction \gamma_0 \in \underset{\sim}{G}_0$$

(where $\underset{\sim}{G}_0$ is the canonical name for the generic filter over P_{γ_0}), then there is an (M, P_γ)-generic condition q such that $q \restriction \gamma_0 = q_0$ and

$$q \Vdash_{P_\gamma} \underset{\sim}{p}_0 \in \underset{\sim}{G} \tag{I.6}$$

(where $\underset{\sim}{G}$ is the canonical name of the generic filter over P_γ, and we view $\underset{\sim}{p}_0$ as a name in V^{P_γ}).

An equivalent formulation of (I.6) is that for every $q' \geq q$, if q' identifies $\underset{\sim}{p}_0$ (that is $q' \Vdash_{P_\gamma} \underset{\sim}{p}_0 = p_0$, for some $p_0 \in P_\gamma$) then $p_0 \leq q'$.

We emphasize that $\underset{\sim}{p}_0$ is not necessarily in M, but it is forced by q_0 to be a condition in $P_\gamma \cap M$.

Proof of Theorem 2.7. Observe first that the lemma gives Theorem 2.7. Given $M \prec H_\lambda$ and $p_0 \in P_\delta$, we apply the lemma with $\lambda_0 = 0$ and p_0 viewed as a name in the trivial poset $P_0 = \{\emptyset\}$. ⊣

Proof of Lemma 2.8. The proof of the lemma is by induction on γ. For γ successor, the lemma was essentially stated as Lemma 2.6. (There are two subcases here. $\gamma_0 + 1 = \gamma$, and $\gamma_0 + 1 < \gamma$. The first subcase is essentially stated as Lemma 2.6, and the second subcase is brought to the first by the inductive hypothesis.)

So assume that γ is a limit ordinal. Pick an increasing sequence $\langle \gamma_i \mid i \in \omega\rangle$ cofinal in $\gamma \cap M$, with $\gamma_i \in M$ and where γ_0 is the given ordinal. (Note that γ may well be uncountable but $\gamma \cap M$ is a countable set of ordinals.) Fix an enumeration $\{D_i \mid i \in \omega\}$ of all the dense subsets of P_γ that are in M.

We will define by induction on $n < \omega$ a condition $q_n \in P_{\gamma_n}$, and a name $\underset{\sim}{p}_n$ in $V^{P_{\gamma_n}}$ such that:

1. $q_0 \in P_{\gamma_0}$ is the given condition. $q_n \in P_{\gamma_n}$ is (M, P_{γ_n})-generic.
$$q_{n+1} \restriction \gamma_n = q_n.$$
2. $\underset{\sim}{p}_0$ is given. $\underset{\sim}{p}_n$ is a P_{γ_n}-name such that
$$q_n \Vdash_{P_{\gamma_n}} \quad \underset{\sim}{p}_n \text{ is a condition in } P_\gamma \cap M \text{ such that:}$$
$$\text{(a) } \underset{\sim}{p}_n \restriction \gamma_n \in \underset{\sim}{G}_{\gamma_n},$$
$$\text{(b) } \underset{\sim}{p}_{n-1} \leq_\gamma \underset{\sim}{p}_n,$$
$$\text{(c) } \underset{\sim}{p}_n \text{ is in } D_{n-1} \text{ (for } n > 0).$$

(Here and subsequently $\Vdash_\gamma$ may be written instead of $\Vdash_{P_\gamma}$ and $\leq_\gamma$ instead of $\leq_{P_\gamma}$.) To see where we are going, suppose that q_n, $\underset{\sim}{p}_n$ have already been constructed for all $n \in \omega$. Then let $q = \bigcup_n q_n$. We claim that for every n:
$$q \Vdash_\gamma \underset{\sim}{p}_n \in \underset{\sim}{G}_\gamma. \tag{I.7}$$

It can be seen that this claim implies that q is (M, P_γ)-generic (because $\underset{\sim}{p}_n$ is forced to be in $D_{n-1} \cap M$, and by Lemma 2.1).

To prove the claim in (I.7), we first demonstrate for every two indices $n < m$ that
$$q \Vdash_\gamma \underset{\sim}{p}_n \leq_\gamma \underset{\sim}{p}_m.$$
This follows from 2(b). It follows from 2(a) that for every m
$$q \Vdash_\gamma \underset{\sim}{p}_m \restriction \gamma_m \in \underset{\sim}{G}_{\gamma_m}.$$

Hence, for every m and n such that $m \geq n$

$$q \Vdash_\gamma \underset{\sim}{p}_n \restriction \gamma_m \in \underset{\sim}{G}_{\gamma_m} .$$

This implies that for every n

$$q \Vdash_\gamma \underset{\sim}{p}_n \in \underset{\sim}{G}_\gamma .$$

Indeed, for any q' extending q in P_γ, if $q' \Vdash \underset{\sim}{p}_n = p$, for some $p \in P_\gamma$, then

$$q' \Vdash_\gamma p \in P_\gamma \cap M \text{ and } p \restriction \gamma_m \in \underset{\sim}{G}_{\gamma_m}, \text{ for } m \geq n.$$

Since the posets P_{γ_n} are separative, it follows that $p \restriction \gamma_m \leq q'$ for every m. Hence $p \leq q'$: because $p \in M$ and thus $\mathrm{dom}(p) \subseteq \gamma \cap M$, so that $\mathrm{dom}(p) \subseteq \sup\{\gamma_m \mid m < \omega\}$. So $q' \Vdash p \in \underset{\sim}{G}_\gamma$. This holds for any extension q' of q that determines $\underset{\sim}{p}_n$, and hence $q \Vdash \underset{\sim}{p}_n \in \underset{\sim}{G}_\gamma$.

Returning now to the inductive construction, assume that q_n and $\underset{\sim}{p}_n$ have been constructed. We will first define $\underset{\sim}{p}_{n+1}$ in $V^{P_{\gamma_n}}$, and then q_{n+1}. Imagine a generic extension $V[G_n]$, made via P_{γ_n}, such that $q_n \in G_n$. Then $\underset{\sim}{p}_n[G_n]$, the realization of $\underset{\sim}{p}_n$, is some condition, p_n, in $P_\gamma \cap M$, such that $p_n \restriction \gamma_n \in G_n$. Since q_n is (M, P_{γ_n})-generic, $G_n \cap M$ intersects every dense subset of P_{γ_n} that lies in M. An easy density argument now gives a condition $p_{n+1} \in D_n \cap M$, extending p_n, such that $p_{n+1} \restriction \gamma_n \in G_n$. (Argue in the collapsed structure $\bar{M}$ and use Lemma 1.2.) We let $\underset{\sim}{p}_{n+1}$ be a P_{γ_n}-name of p_{n+1}, forced by q_n to satisfy all of these properties of p_{n+1}. That is,

$$q_n \Vdash_{\gamma_n} \underset{\sim}{p}_{n+1} \in D_n \cap M, \ \underset{\sim}{p}_n \leq_\gamma \underset{\sim}{p}_{n+1}, \text{and } \underset{\sim}{p}_{n+1} \restriction \gamma_n \in \underset{\sim}{G}_{\gamma_n}$$

Now that $\underset{\sim}{p}_{n+1}$ is defined, apply the inductive assumption of this lemma to $\gamma_n < \gamma_{n+1}$, to q_n, and to $\underset{\sim}{p}_{n+1} \restriction \gamma_{n+1}$. This gives $q_{n+1} \in P_{\gamma_{n+1}}$ that satisfies the required inductive assumptions. $\dashv$

We draw two additional conclusions from the Properness Extension Lemma.

2.9 Corollary. *Let $\langle P_i \mid i \leq \delta \rangle$ be a countable support iteration of proper forcings.*

1. *Suppose that $\mathrm{cf}(\delta) > \omega$. Then any real (or countable set of ordinals) in V^{P_δ} already appears in V^{P_i} for some $i < \delta$.*
2. *Suppose $cf(\delta) = \omega$, and $\langle \gamma_i \mid i \in \omega \rangle$, is increasing and cofinal in δ. Then for every name $\underset{\sim}{f} \in V^{P_\delta}$ of a countable sequence of ordinals, for every condition $p_0 \in P_\delta$, there is an extension $p \geq p_0$ such that for every $n \in w$ there is a P_{γ_n} name c_n such that $p \Vdash_\delta \underset{\sim}{f}(n) = c_n$.*

Proof. Let $\underset{\sim}{f} \in V^{P_\delta}$ be a real. Find a countable $M \prec H_\lambda$ with $\underset{\sim}{f} \in M$, and let $q \in P_\delta$ be some (M, P_δ)-generic condition (given by the lemma). If $i = \sup(M \cap \delta)$ then $i < \delta$. The support of any condition in $P_\delta \cap M$ is included in i and $\underset{\sim}{f} \cap M$ can be viewed as a V^{P_i} name forced by q to be equal to $\underset{\sim}{f}$.
Proof of item 2: Let $M \prec H_\lambda$ be countable, containing all relevant parameters (including the name of the real). Repeat the proof of the Extension Lemma with $\langle \gamma_i \mid i \in \omega \rangle$ as the cofinal sequence and instead of dealing with all dense sets let D_{n-1} be the set of conditions that determine $\underset{\sim}{f}(n-1)$. Let p be the resulting (M, P_δ)-generic condition. $\dashv$

2.2. The $\aleph_2$ chain condition

In almost all applications given here, we assume the continuum hypothesis (CH) in the ground model, and iterate ω_2 proper forcings, each of size $\aleph_1$. To conclude that $\aleph_2$ and higher cardinals are not collapsed, the following theorem can be invoked. More general chain condition theorems (which deal with bigger posets) can be found in Shelah's book ([15] Chapter VIII on the p.i.c. condition, for example) and in Section 5.4.

2.10 Theorem. *Assume CH. Let $\langle P_i \mid i \leq \delta \rangle$ be a countable support iteration of length $\delta \leq \omega_2$ of proper forcings of size $\aleph_1$. Then P_δ satisfies the $\aleph_2$-c.c.*

Proof. The assumption is that each iterand Q_α has size $\aleph_1$ in V^{P_α}, but the posets P_i themselves may be large ($2^{\aleph_1}$, because of the names involved). In any family $\{r_\xi \mid \xi \in \omega_2\} \subseteq P_\delta$, we must find two compatible conditions. Fixing a large λ, pick for every $\xi \in \omega_2$ a countable $M_\xi \prec H_\lambda$ such that $r_\xi \in M_\xi$. Look at the isomorphism types of the countable structures M_ξ. Since CH holds, there is a set $I \subseteq \omega_2$ of size $\aleph_2$ such that all M_ξ for $\xi \in I$ are pairwise isomorphic. But the transitive collapse is determined by the isomorphism type, and hence there is a single transitive structure $\bar{M}$ to which all M_ξ for $\xi \in I$ are collapsed. In addition, we may form a Δ-system for the countable sets $M_\xi \cap \omega_2$ (again by CH). This leads to the following assumptions on I.

1. For some fixed transitive structure $\bar{M}$, $h_\xi : M_\xi \to \bar{M}$, where h_ξ are the collapsing functions for $\xi \in I$.

2. The countable sets $M_\xi \cap \omega_2$ form a Δ-system: For some countable $C \subseteq \omega_2$,
$$M_{\zeta_1} \cap M_{\zeta_2} \cap \omega_2 = C \text{ for every } \zeta_1 \neq \zeta_2 \text{ in } I.$$
Moreover, C is an initial segment of $M_\xi \cap \omega_2$, and there is no interleaving of the $M_\xi \cap \omega_2 \setminus C$ parts. That is, if we define $\mu_\xi = \min(M_\xi \cap \omega_2 \setminus C)$

then for $\xi_1 < \xi_2$ in I, $\sup(C) < \mu_{\xi_1}$, and $\sup(M_{\xi_1} \cap \omega_2) < \mu_{\xi_2}$. We also assume that $h_\xi(r_\xi)$ does not depend on ξ and is a fixed member of $\bar{M}$.

We now claim that any two conditions with indices in I are compatible. Given ξ_1, $\xi_2 \in I$, it suffices to show that $r_1 = r_{\xi_1} \restriction \mu_{\xi_1} = r_{\xi_1}\restriction\sup(C)$ and $r_2 = r_{\xi_2} \restriction \mu_{\xi_2}$ are compatible. Because then, if $r \in P_{\sup(C)}$ extends r_1 and r_2, then $p = r \cup r_{\xi_1}\restriction(\omega_2 \setminus C) \cup r_{\xi_2}\restriction(\omega_2 \setminus C)$ extends the two given conditions. The fact that C is an initial segment of $M_\xi \cap \delta$ and hence that p is a function in P_δ is used in the proof (if the iteration were of length $\delta > \omega_2$ then $\aleph_2$ may indeed be collapsed).

Let $\mu = \mu_{\xi_1}$, and let $h : M_{\xi_1} \to M_{\xi_2}$ be the isomorphism of the two structures. Then h is the identity on $\mu \cap M_{\xi_1} = h(\mu) \cap M_{\xi_2}$, and $h(r_1) = r_2$. We shall prove that if p is any (M_{ξ_1}, P_μ)-generic condition extending r_1, then p extends r_2, and hence r_1 and r_2 are compatible as required. The iterands are of power $\aleph_1$ and we may assume that their universe is always ω_1. Then for every α in its domain, $r_1(\alpha)$ is forced by $r_1\restriction\alpha$ to be a countable ordinal. A similar statement holds for r_2. However, $r_1(\alpha)$ is not necessarily the same name as $r_2(\alpha)$.

The following lemma therefore suffices.

2.11 Lemma. *Let M_1 and M_2 be two isomorphic countable elementary substructures of H_λ. Let $h : M_1 \to M_2$ be the isomorphism, and $\mu \in M_1 \cap \omega_2$ be such that h is the identity on $\mu \cap M_1$. (We do not require that $h(\mu) \neq \mu$, but this is possible.) If $p \in P_\mu$ is any (M_1, P_μ)-generic condition then for any condition $r \in P_\mu \cap M_1$, $p \geq r$ implies $p \geq h(r)$. (Hence any (M_1, P_μ)-generic condition is also $(M_2, P_{h(\mu)})$-generic.)*

Proof. The proof is by induction on μ.

Case 1 μ is a limit ordinal. Remark that, for any $r \in P_\mu \cap M_1$, r and $h(r)$ have the same support, since h is the identity on μ and the support is countable. Assume that $p \in P_\mu$ is (M_1, P_μ)-generic and $r \in P_\mu \cap M_1$, $p \geq r$ as in the lemma. In order to prove that $p \geq h(r)$, it suffices to prove for every $\mu' \in \mu \cap M_1$ that $p\restriction\mu' \geq h(r)\restriction\mu'$. This follows from the inductive assumption, because $p\restriction\mu'$ is $(M_1, P_{\mu'})$-generic, and $p\restriction\mu' \geq r\restriction\mu'$ implies that $p\restriction\mu' \geq h(r \restriction \mu') = h(r) \restriction \mu'$.

Case 2 $\mu = \mu' + 1$. As $p \geq r$ in $P_{\mu'+1}$ $(= P_{\mu'} * Q_{\mu'})$, $p \restriction \mu'$ extends $r \restriction \mu'$ and

$$p \restriction \mu' \Vdash p(\mu') \text{ extends } r(\mu') \text{ in } Q_{\mu'}. \tag{I.8}$$

By Lemma 2.5, $p \restriction \mu'$ is $(M_1, P_{\mu'})$-generic, and the inductive condition implies that $p\restriction\mu'$ extends $h(r)\restriction\mu'$. We want to prove that

$$p\restriction\mu' \Vdash p(\mu') \text{ extends } h(r)(\mu') \text{ in } Q_{\mu'}.$$

Consider any t in $P_{\mu'}$ that extends $p\restriction\mu'$, and we shall find an extension t' of t in $P_{\mu'}$ forcing $r(\mu') = h(r)(\mu')$. Thus

$$t' \Vdash\ p(\mu')\ \text{ extends } h(r)(\mu')$$

follows from (I.8) as required. The set of conditions that "know" the value of $r(\mu')$ (as an ordinal in ω_1) is dense above $r\restriction\mu'$ and is in M_1. Hence, by genericity of t, t is compatible with some $s \in P_{\mu'} \cap M_1,\ s \geq r \restriction \mu'$, such that

$$s \Vdash_{\mu'}\ r(\mu') = \alpha$$

for some $\alpha < \omega_1$. Since $h(\mu') = \mu'$, and $h(\alpha) = \alpha$, this implies that

$$h(s) \Vdash_{\mu'}\ h(r)(\mu') = \alpha.$$

Let $t' \in P_{\mu'}$ be a common extension of t and s. By the inductive assumption, t' extends $h(s)$ as well and so $t' \Vdash_{\mu'}\ r(\mu') = \alpha = h(r)(\mu')$.

$\dashv$

This ends the proof of Theorem 2.10. $\dashv$

Knowing the $\aleph_2$-c.c. we can prove by induction on $\delta \leq \omega_2$ that $|P_\delta| \leq 2^{\aleph_1}$. This helps in defining iterations of length ω_2 of proper forcing posets of size $\aleph_1$ each, when we want to ensure that every $A \subseteq \omega_1$ in $V^{P_{\omega_2}}$ has had its chance to be considered at some successor stage of the iteration.

A useful consequence of the previous lemma is the following

2.12 Theorem. *Assume CH. Let $\langle P_i \mid i \leq \delta\rangle$ be a countable support iteration of length $\delta < \omega_2$ of proper forcings of size $\aleph_1$. Then CH holds in V^{P_δ}.*

Proof. Write P for P_δ. Let $x \in V^P$ be a name forced by $p_0 \in P$ to be a function from ω_2 into $\mathcal{P}\omega$ (the power set of ω). We shall find an extension p_1 of p_0 and two indexes $\xi_0 \neq \xi_1$ such that $p_1 \Vdash x(\xi_0) = x(\xi_1)$.

As above, we define countable $M_\xi \prec H_\lambda$ with $P, x, \xi \in M_\xi$ for $\xi < \omega_2$. By CH, there are $\xi_1 \neq \xi_2$ with an isomorphism $h : M_{\xi_1} \to M_{\xi_1}$ taking ξ_1 to ξ_2 and such that $M_{\xi_1} \cap \delta = M_{\xi_2} \cap \delta$. Let p_2 be an (M_{ξ_1}, P)-generic condition extending p_1. Then p_2 is also (M_{ξ_2}, P) generic by the lemma, and hence

$$p_2 \Vdash x(\xi_1) = x(\xi_2)$$

follows from $h(\xi_1) = \xi_2$. $\dashv$

2.3. Properness: equivalent formulations

Suppose that Q is a non-proper poset and P is a proper poset. Then, in V^P, Q remains non-proper. To prove this basic result we need an equivalent formulation of properness in terms of preservation of stationary subsets of $\mathcal{P}_{\aleph_1}(A)$. Here, $\mathcal{P}_{\aleph_1}(A) = [A]^{<\aleph_1}$ is the collection of all countable subsets of A. We refer the reader to Jech's chapter on stationary sets for basic properties of the closed unbounded filter on $\mathcal{P}_{\aleph_1}(A)$. We shall rely on the following facts for any uncountable set A. (1) The collection of closed unbounded sets generates a countably closed and normal filter over $\mathcal{P}_{\aleph_1}(A)$. (2) For every closed unbounded set $C \subseteq \mathcal{P}_{\aleph_1}(A)$ there is a function $f : [A]^{<\aleph_0} \to A$ such that if $x \in \mathcal{P}_{\aleph_1}(A)$ is closed under f, then $x \in C$. (3) If $A_1 \subset A_2$ and $C \subseteq \mathcal{P}_{\aleph_1}(A_2)$ is closed unbounded, then $\{x \cap A_1 \mid x \in C\}$ contains a closed unbounded subset of $\mathcal{P}_{\aleph_1}(A_1)$. (4) A subset S of $\mathcal{P}_{\aleph_1}(A)$ is said to be stationary if it has non-empty intersection with every closed unbounded set. If S is stationary and f is a function such that $f(x) \in x$ for every non-empty $x \in S$, then for some $a \in A$, the set $\{x \in S \mid f(x) = a\}$ is stationary.

Let P be a poset and $p \in P$ a condition. We say that $D \subseteq P$ is *pre-dense* above p if $\forall p' \geq p \; \exists d \in D$ (p' and d are compatible in P). Equivalently, D is pre-dense above p if the set D' of all extensions of members of D is dense above p.

Given any poset P, form $A = P \cup \mathcal{P}(P)$, a disjoint union of the poset universe with its power-set. The "test-set" for P is the collection of all $a \in \mathcal{P}_{\aleph_1}(A)$ such that, for every $p_0 \in P \cap a$ there exists an extension $p \in P$ such that, for every $D \in a \cap \mathcal{P}(P)$, if D is dense in P then $D \cap a$ is pre-dense above p. We can say that p is "generic" for a, and then the test-set is the collection of all a, countable subsets of A, that have generic conditions extending each member of $P \cap a$.

If $T \subset \mathcal{P}_{\aleph_1}(A)$ is the test-set of P then its complement $\mathcal{P}_{\aleph_1}(A) \setminus T$ is called the "failure set" for P.

2.13 Theorem (Properness Equivalents). *For any poset P the following are equivalent.*

1. *P is proper (as in Definition 2.2).*
2. *For some $\lambda > 2^{|P|}$, for every countable $M \prec H_\lambda$ with $P \in M$, every $p_0 \in P \cap M$ has an extension $p \geq p_0$ that is an M-generic condition.*
3. *For every uncountable λ, P preserves stationary subsets of $\mathcal{P}_{\aleph_1}(\lambda)$. That is, if $S \subseteq \mathcal{P}_{\aleph_1}(\lambda)$ is stationary, then it remains so in any P-generic extension.*
4. *For $\lambda_0 = 2^{|P|}$, P preserves stationary subsets of $\mathcal{P}_{\aleph_1}(\lambda_0)$.*

5. *The test set for P, as defined above, contains a closed unbounded subset of $\mathcal{P}_{\aleph_1}(A)$.*

Proof. Clearly, $1 \Rightarrow 2$. We prove that $1 \Rightarrow 3$. So assume that P is proper, and let $S \subseteq \mathcal{P}_{\aleph_1}(\lambda)$ be a stationary set. To prove that S remains stationary in any extension via P, we take any $f \in V^P$ such that $p_0 \Vdash f : [\lambda]^{<\aleph_0} \to \lambda$, and we shall find an extension $p \in P$ so that some $x \in S$ is forced by p be closed under f. Pick a sufficiently large cardinal κ and a countable $M \prec H_\kappa$ with $\lambda, P, p_0, f \in M$ and such that $M \cap \lambda \in S$. We can find it since the collection of intersections $M \cap \lambda$ for structures M as above contains a closed unbounded subset of $\mathcal{P}_{\aleph_1}(\lambda)$. As P is assumed proper, there is an extension $p \geq p_0$ that is (M, P) generic. Genericity of p implies that $p \Vdash M \cap \lambda$ *is closed under* f.

$3 \Rightarrow 4$ is trivial, and $2 \Rightarrow 4$ is just like $1 \Rightarrow 3$. We prove now that $4 \Rightarrow 5$.

Assume that P preserves stationarity of subsets of $\mathcal{P}_{\aleph_1}(\lambda_0)$ for $\lambda_0 = 2^{|P|}$. Define $A = P \cup \mathcal{P}(P)$. Then $|A| = \lambda_0$. Suppose that S, the failure set for P as defined above, is stationary, and we shall derive a contradiction. By normality, we may assume that the failure is due to the same $p_0 \in a$ for $a \in S$. Let $G \subset P$ be a (V, P)-generic filter containing p_0. Then S is stationary in $V[G]$ since P preserves stationarity. Define a function $g : A \to A$ so that if D is dense in P then $g(D) \in G$. Since S is stationary, there is $x \in S$ closed under g. If $p \geq p_0$ forces this fact about x, then p shows that x is in fact in the test set for P.

Finally we prove $5 \Rightarrow 1$. Suppose that $\lambda > 2^{|P|}$, $M \prec H_\lambda$ is countable, and $P \in M$. Then $A = P \cup \mathcal{P}_{\aleph_1}(P) \in H_\lambda$ and $A \in M$. The test set for P is also in M. Assuming that this set contains a closed unbounded set, we may find such a closed unbounded set C in M. This implies that $M \cap A \in C$, and hence there exists an (M, P)-generic condition above any condition in $M \cap P$.

$\dashv$

3. Preservation of ${}^\omega\omega$-boundedness

The set of functions from ω to ω is denoted ${}^\omega\omega$ (the "reals"). For $f,\ g \in {}^\omega\omega$ and $k < \omega$ define $f <_k g$ iff $\forall n \geq k \quad f(n) \leq g(n)$. $<^* = \bigcup_k <_k$ is the *bounding* (also called eventual bounding) relation: If $f <^* g$, then g *bounds* f, and if $f <_0 g$ then g *totally bounds* f. A basic fact is that any countable $F \subseteq {}^\omega\omega$ is bounded by some $g \in {}^\omega\omega$.

A forcing poset P is said to be ${}^\omega\omega$-bounding iff for every generic filter $G \subseteq P$, $V \cap {}^\omega\omega$ bounds $V[G] \cap {}^\omega\omega$, i.e., for every $g \in {}^\omega\omega \cap V[G]$ there is $h \in {}^\omega\omega \cap V$ with $g <^* h$ (we could equivalently require $g <_0 h$). Our aim is to prove that the countable support iteration of proper ${}^\omega\omega$-bounding posets is ${}^\omega\omega$-bounding.

Let P be a poset and $\underset{\sim}{f} \in V^P$ be a name of a real (i.e., a name forced by 0_P to be a real). We say that an increasing sequence $\bar{p} = \langle p_i \mid i \in \omega\rangle$ of conditions in P *interprets* $\underset{\sim}{f}$ as $f^* \in {}^\omega\omega$ iff for every $n < \omega$ p_n forces $\underset{\sim}{f} \restriction n = f^* \restriction n$. We write in this case $f^* = \mathrm{intp}(\bar{p}, \underset{\sim}{f})$.

3.1 Definition. Let P be a forcing poset and $\underset{\sim}{f} \in V^P$ a name of a real. Suppose that $\bar{p} = \langle p_i \mid i \in \omega\rangle$ is an increasing sequence of conditions in P that interprets $\underset{\sim}{f}$. We say that $\bar{p}$ *respects* $g \in {}^\omega\omega$ iff

$$\mathrm{intp}(\bar{p}, \underset{\sim}{f}) <_0 g. \tag{I.9}$$

The following surprising property turns out to be important for the preservation theorem.

3.2 Theorem. *If P is ${}^\omega\omega$-bounding, then the following stronger property of P holds. Let $\underset{\sim}{f} \in V^P$ be a name of a real and let $M \prec H_\kappa$ be countable, with $P, \underset{\sim}{f} \in M$. Suppose that $g \in {}^\omega\omega$ dominates all the reals of M, and $\bar{p} \in M$ is an increasing sequence of conditions in P that interprets $\underset{\sim}{f}$ and respects g. Then, for some $p \in P \cap M$ and $h \in M$, $h <_0 g$ and $p \Vdash_P \underset{\sim}{f} \leq_0 h$. So that*

$$p \Vdash_P \underset{\sim}{f} <_0 g.$$

Proof. The point of the theorem is this. As P is ${}^\omega\omega$-bounding, every condition in M can be extended to force that $\underset{\sim}{f}$ is bounded by some real in M and hence that $\underset{\sim}{f} <^* g$, but it takes the theorem to find $p \in M$ that forces $\underset{\sim}{f} <_0 g$.

Work in M. By assumption $\bar{p} = \langle p_i \mid i \in \omega\rangle$ is an increasing sequence of conditions in P that interprets $\underset{\sim}{f}$ as f^*, and $f^* <_0 g$. For every n pick an extension p'_n of p_n such that:

1. For some $h_n \in {}^\omega\omega \cap M$, $p'_n \Vdash \underset{\sim}{f} \leq_0 h_n$. (Such h_n exist because P is ${}^\omega\omega$-bounding.)
2. We can require that $h_n \restriction n = f^* \restriction n$ (because $p_n \Vdash \underset{\sim}{f} \restriction n = f^* \restriction n$).

Let $u \in {}^\omega\omega$ be defined by

$$u(m) = \max\{h_i(m) \mid i \leq m\}\ .$$

Then $u \in M$, and is hence bounded by g; say $u <_\ell g$. This implies that $h_\ell <_0 g$ by the following argument. For $k < \ell$, $h_\ell(k) = f^*(k) < g(k)$, and for $k \geq \ell$ $h_\ell(k) \leq u(k) < g(k)$. Now p'_ℓ is as required: it forces $\underset{\sim}{f} <_0 g$ since it forces $\underset{\sim}{f} \leq_0 h_\ell$.

Remark that we can require that p extends any given condition in the sequence $\bar{p}$. $\dashv$

A main tool in the preservation proof is the notion of a *derived sequence*. Let Q_1 and Q_2 be two forcing posets such that $Q_1 \lessdot Q_2$, and let $\pi : Q_2 \to Q_1$ be the related projection. Let $\underset{\sim}{f}$ be a Q_2-name forced by 0_{Q_2} to be a real, and let $\bar{r} = \langle r_i \mid i \in \omega \rangle$ be an increasing sequence of conditions in Q_2 that interprets $\underset{\sim}{f}$. Fix a well-ordering of Q_2. Suppose that G_1 is a (V, Q_1)-generic filter. Recall that $Q_2/G_1 = \{q \in Q_2 \mid \pi(q) \in G_1\}$. We shall define in $V[G_1]$ an increasing sequence $\bar{s} = \langle s_i \mid i \in \omega \rangle$ of conditions in Q_2/G_1 that interprets $\underset{\sim}{f}$ by the following induction.

1. If $\pi(r_i) \in G_1$ then $s_i = r_i$. (In this case $\pi(r_k) \in G_1$ for every $k < i$ and $s_k = r_k$.)

2. If $\pi(r_i) \notin G_1$, then let s_i be the first Q_2-extension of s_{i-1} that is in Q_2/G_1 and determines the value of $\underset{\sim}{f} \restriction i$.

Thus, if $\pi(r_i) \in G_1$ for all $i \in \omega_1$, then $s_i = r_i$ for all i, but if n is the first index such that $\pi(r_n) \notin G_1$, then $s_i = r_i$ for $i < n$, and for $i \geq n$ we define $s_i \in Q_2$ as the first extension of s_{i-1} with $\pi(s_i) \in G_1$ and such that s_i determines $\underset{\sim}{f} \restriction i$ in the forcing relation $\Vdash_{Q_2}$.

We say that the sequence $\bar{s}$ defined above in $V[G_1]$ is "derived" from $\bar{r}$, G_1, and $\underset{\sim}{f}$. We write $\delta_{G_1}(\bar{r}, \underset{\sim}{f})$ to denote this derived sequence in Q_2/G_1. The V^{Q_1}-name of the derived sequence is denoted $\underset{\sim}{\delta}_{Q_1}(\bar{r}, \underset{\sim}{f})$.

3.3 Lemma. *Let $Q_1 \lhd Q_2$ be posets with projection $\pi : Q_2 \to Q_1$, and suppose that Q_1 is ${}^\omega\omega$-bounding (no such assumption is made about Q_2). Suppose that:*

1. *$\underset{\sim}{f} \in V^{Q_2}$ is forced by every condition to be a real.*

2. *$\bar{r} = \langle r_i \mid i < \omega \rangle$ is an increasing sequence of conditions in Q_2 (above some given $p \in Q_2$) that interprets $\underset{\sim}{f}$.*

3. *$M \prec H_\kappa$ is countable with $Q_1, Q_2, \underset{\sim}{f}, \bar{r}, p \in M$.*

4. *$g \in {}^\omega\omega$ bounds all the reals of M, and $\mathrm{int}(\bar{r}, \underset{\sim}{f}) <_0 g$. That is, $\bar{r}$ respects g in its interpretation of $\underset{\sim}{f}$.*

Then some condition $s \in Q_1 \cap M$ extends $\pi(p)$ and forces that the derived sequence $\underset{\sim}{\delta}_{Q_1}(\bar{r}, \underset{\sim}{f})$ respects g in its interpretation of $\underset{\sim}{f}$ and is above p in the Q_2 ordering.

Proof. Let $\underset{\sim}{\delta} = \underset{\sim}{\delta}_{Q_1}(\bar{r}, \underset{\sim}{f})$ be the name of the derived sequence, $\underset{\sim}{\delta} \in V^{Q_1}$. Define a name of a real, $\underset{\sim}{h} \in V^{Q_1}$, by

$$\underset{\sim}{h} = \mathrm{int}(\underset{\sim}{\delta}, \underset{\sim}{f}).$$

That is, if G_1 is a (V, Q_1)-generic filter, let $\delta_{G_1}(\bar{r}, \underset{\sim}{f})$ be the resulting derived sequence, a Q_2 increasing sequence of conditions in Q_2/G_1 interpreting $\underset{\sim}{f}$, and let $\underset{\sim}{h}[G_1]$ be that interpretation.

Define $p_i = \pi(r_i) \in Q_1$ for $i \in \omega$. Then $\langle p_i \mid i \in \omega\rangle \in M$ and $p_i \Vdash_{Q_1} r_i = \underset{\sim}{\delta}(i)$. That is, as $p_i \Vdash r_i \in Q_1/\underset{\sim}{G}_1$, p_i "knows" that r_i is the i-th member of the derived sequence. Consequently, p_i determines $\underset{\sim}{h} \restriction i$ (in Q_1 forcing) as r_i determines $\underset{\sim}{f} \restriction i$ (in Q_2 forcing). Namely

$$\mathrm{int}(\langle p_i \mid i \in \omega\rangle, \underset{\sim}{h}) = \mathrm{int}(\bar{r}, \underset{\sim}{f}).$$

Hence, $\mathrm{int}(\langle p_i \mid i \in \omega\rangle, \underset{\sim}{h}) <_0 g$, and so by the surprising theorem there exist $s \geq p_0$ in $Q_1 \cap M$ so that $s \Vdash_{Q_1} \underset{\sim}{h} <_0 g$. That is,

$$s \Vdash_{Q_1} \mathrm{int}(\underset{\sim}{\delta}_{Q_1}(\bar{r}, \underset{\sim}{f}), \underset{\sim}{f}) <_0 g$$

Since $s \geq p_0$, s forces that r_0 is the first member of the derived sequence, and hence that all members of the derived sequence extend p in Q_2. $\dashv$

In the following we shall apply the previous lemma in a slightly more complex situation in which the discussion is not in V, but rather in V^{Q_0}, where $Q_0 \lhd Q_1 \lhd Q_2$. We will use $\underset{\sim}{G}_0$ and $\underset{\sim}{G}_1$ as canonical names for the Q_0 and Q_1 generic filters.

3.4 Lemma. *Suppose that $Q_0 \lhd Q_1 \lhd Q_2$ are posets with commuting projections $\pi_{i,j} : Q_i \to Q_j$, for $0 \leq j < i \leq 2$. Assume that Q_1 (and hence Q_0) is proper and ${}^\omega\omega$-bounding. Suppose that:*

1. *$\underset{\sim}{f} \in V^{Q_2}$ is a name of a real.*
2. *$M \prec H_\kappa$ is countable with $Q_0, Q_1, Q_2, \underset{\sim}{f} \in M$. Let $q_0 \in Q_0$ be an (M, Q_0)-generic condition. Assume that $g \in {}^\omega\omega$ bounds all the reals of M.*
3. *$\underset{\sim}{p} \in V^{Q_0}$ is given such that $q_0 \Vdash_{Q_0} \underset{\sim}{p} \in (Q_2/\underset{\sim}{G}_0) \cap M$.*
4. *q_0 forces that there is in $M[\underset{\sim}{G}_0]$ a Q_2-increasing sequence of conditions in $Q_2/\underset{\sim}{G}_0$ that interprets $\underset{\sim}{f}$, respects g, and is above $\underset{\sim}{p}$ in the Q_2 ordering.*

Then there is some $q_1 \in Q_1$, an (M, Q_1)-generic condition, so that

1. *$\pi_{1,0}(q_1) = q_0$.*
2. *$q_1 \Vdash_{Q_1} \pi_{2,1}(\underset{\sim}{p}) \in \underset{\sim}{G}_1$*

3. *q_1 forces that there is in $M[\underset{\sim}{G}_1]$ a Q_2-increasing sequence of conditions in $Q_2/\underset{\sim}{G}_1$ that interprets $\underset{\sim}{f}$ and respects g, and is above $\underset{\sim}{p}$ in the Q_2 ordering.*

Proof. Observe that what the lemma does is to push the situation described in item 4 from $\underset{\sim}{G}_0$ to $\underset{\sim}{G}_1$. That is, the basic object that interests us is a sequence of conditions in Q_2, increasing in the ordering of Q_2 and interpreting $\underset{\sim}{f}$ (in $\Vdash_{Q_2}$) as a function that respects g. The assumed sequence (in item 4) is compatible with $\underset{\sim}{G}_0$, and the resulting sequence is compatible with $\underset{\sim}{G}_1$.

Let G_0 be some (V, Q_0) generic filter containing q_0. Work in $M[G_0] \prec H_\kappa[G_0]$ and apply the previous lemma as follows. Observe first that g dominates all the reals in $M[G_0]$, since q_0 is (M, Q_0)-generic. Observe also that $\pi_{2,1} : Q_2/G_0 \to Q_1/G_0$ is a projection (see Lemma 1.3). Then $\underset{\sim}{f}/G_0 \in V[G_0]^{Q_2/G_0}$ is a name of a real. Following assumption 4 of the lemma, let $\bar{r} \in M[G_0]$ be an increasing sequence of conditions in Q_2/G_0 that interprets $\underset{\sim}{f}/G_0$ (since it interprets $\underset{\sim}{f}$) and respects g (and is above $\underset{\sim}{p}[G_0]$ in $\leq_{Q_2}$). We apply the previous lemma to Q_2/G_0, Q_1/G_0, and $\underset{\sim}{f}/G_0$ in $V[G_0]$. Thus by that lemma some condition $s \in (Q_1/G_0) \cap M[G_0]$ exists which extends $\pi_{2,1}(\underset{\sim}{p}[G_0])$ and forces in Q_1/G_0 that the derived sequence $\underset{\sim}{\delta}_{Q_1/G_0}(\bar{r}, \underset{\sim}{f}/G_0)$ respects g in its interpretation of $\underset{\sim}{f}/G_0$, and is above $\underset{\sim}{p}[G_0]$ in Q_2/G_0.

Now let $\underset{\sim}{s} \in V^{Q_0}$ be a name of s, forced to have all the properties of s described above. (Observe that $\underset{\sim}{s}$ is not in M since its definition involves g, but it is forced to become a condition in M.) By the Properness Extension Lemma there is $q_1 \in Q_1$ that is (M, Q_1)-generic such that $\pi_{1,0}(q_1) = q_0$ and $q_1 \Vdash_{Q_1} \underset{\sim}{s} \in \underset{\sim}{G}_1$. So $q_1 \Vdash_{Q_1} \pi_{2,1}(\underset{\sim}{p}) \in \underset{\sim}{G}_1$. We claim that q_1 is as required.

Let G be any (V, Q_1)-generic filter containing q_1. Then $G_0 = \pi_{1,0}``G$ is (V, Q_0) generic and $p = \underset{\sim}{p}[G_0]$, $s = \underset{\sim}{s}[G_0]$ can be formed. Since $q_1 \in G$ and $q_1 \Vdash \underset{\sim}{s} \in \underset{\sim}{G}_1$, $s \in G$. Now G is also $(V[G_0], Q_1/G_0)$-generic, and we write $G_1 = G$ to emphasize that $G_1 \subset Q_1/G_0$. Since $s \in G_1$, whatever s forces holds in $V[G_1]$. Namely, there is in $M[G_1]$ a Q_2/G_0 increasing sequence in $(Q_2/G_0)/G_1 = Q_2/G_1$ that respects g in its interpretation of $\underset{\sim}{f}/G_0$ and is above p in the ordering of Q_2/G_0. If p_n is the n-th member of this sequence, then for some finite function e we have $p_n \Vdash_{Q_2/G_0} (\underset{\sim}{f}/G_0) \restriction n = e$. So there is some $g \in G_0$ such that $g + p_n \Vdash_{Q_2} \underset{\sim}{f} \restriction n = e$. Using Lemma 1.1, we can ammend the derived sequence (that is, replace conditions p with conditions of the form $g + p$ where $g \in G_0$) and obtain a sequence of conditions in Q_2/G_1 that is increasing in Q_2 and respects g in its interpretation of $\underset{\sim}{f}$ (in Q_2 forcing) and is above p in Q_2. $\dashv$

3.5 Theorem. *Let $\langle P_i \mid i \leq \delta \rangle$ be a countable support iteration of proper ${}^{\omega}\omega$-bounding posets. Then P_δ is (proper and) ${}^{\omega}\omega$-bounding.*

Proof. We know that P_δ is proper, and the ${}^{\omega}\omega$-bounding property of P_δ is proved by induction on δ. The successor case is obvious and so we assume that δ is a limit ordinal. Let $\underset{\sim}{f}$ be a P_δ-name of a real; we must find a condition (extending a given condition $p_0 \in P_\delta$) that forces $\underset{\sim}{f} <_0 g$ for some ground-model $g \in {}^{\omega}\omega$. Pick $M \prec H_\kappa$, countable, with $P_\delta, \underset{\sim}{f}, p_0 \in M$. Let $\gamma_n \in \delta \cap M$ for $n \in \omega$ be increasing and cofinal in $M \cap \delta$. For simplicity, start with $\gamma_0 = 0$.

Find in M an increasing sequence $\bar{r} = \langle r_i \mid i \in \omega \rangle$, of conditions in P_δ that interprets $\underset{\sim}{f}$ as $f^* \in {}^{\omega}\omega$, starting with the given condition p_0. Find $g \in {}^{\omega}\omega$ that bounds the reals of M, and such that $f^* <_0 g$. To prove the theorem, we will find $q \in P_\delta$ (extending p_0) that forces $\underset{\sim}{f} <_0 g$.

We intend to define by induction on $n \in \omega$ conditions $q_n \in P_{\gamma_n}$, and names $\underset{\sim}{p}_n \in V^{P_{\gamma_n}}$, that satisfy the following four properties (the first two are as in the Properness Extension Lemma (2.8) but the dense sets are not needed).

1. $q_0 \in P_0$ is the trivial condition. $q_n \in P_{\gamma_n}$ is (M, P_{γ_n})-generic.

$$q_{n+1} \restriction \gamma_n = q_n.$$

2. $\underset{\sim}{p}_0 = p_0$ is given in P_δ. $\underset{\sim}{p}_n$ is a P_{γ_n}-name such that

$$q_n \Vdash_{P_{\gamma_n}} \quad \underset{\sim}{p}_n \text{ is a condition in } P_\delta \cap M \text{ such that:}$$

$$\text{(a) } \underset{\sim}{p}_n \restriction \gamma_n \in \underset{\sim}{G}_{\gamma_n},$$

$$\text{(b) } \underset{\sim}{p}_{n-1} \leq_\delta \underset{\sim}{p}_n,$$

3.

$$q_n \Vdash_{\gamma_n} \underset{\sim}{p}_n \textit{ determines } \underset{\sim}{f} \restriction n \text{ in } P_\delta\textit{-forcing to be totally bounded by } g \restriction n. \tag{I.10}$$

4.

$$q_n \Vdash_{\gamma_n} \quad \text{some } r \in M[\underset{\sim}{G}_{\gamma_n}] \text{ is a } P_\delta \text{ -increasing sequence of conditions in } P_\delta/\underset{\sim}{G}_{\gamma_n} \text{ that interprets } \underset{\sim}{f},\text{ respects } g,\text{ and is above } \underset{\sim}{p}_n.$$

In words, 4 means that if G is a (V, P_{γ_n})-generic filter containing q_n, then there is in $M[G]$ a P_δ increasing sequence, r, of conditions in P_δ/G that interpret $\underset{\sim}{f}$ (in P_δ forcing) and respect g, and are all above $\underset{\sim}{p}_n[G]$ in P_δ.

If we succeed in this, and define $q = \bigcup_n q_n$, then $q \Vdash_\delta \underset{\sim}{p}_n \in \underset{\sim}{G}_\delta$, as in (I.7). So (I.10) implies that $q \Vdash \underset{\sim}{f} <_0 g$.

To start the induction observe that $\bar{r}$ respects g so that condition 4 holds for $n = 0$. Suppose that $q_n \in P_{\gamma_n}$, and $\underset{\sim}{p}_n \in V^{P_{\gamma_n}}$ are defined. We first define $\underset{\sim}{p}_{n+1}$ and then q_{n+1}.

Let G be a (V, P_{γ_n})-generic filter containing q_n. Then (by item 4) there is in $M[G]$ a P_δ-increasing sequence r of conditions in P_δ/G that interpret $\underset{\sim}{f}$, respect g, and are above $\underset{\sim}{p}_n[G]$. Let p_{n+1} be $r(n+1)$, that is the $(n+1)$-th member of r, which determines $\underset{\sim}{f} \restriction n+1 = v$ for some $v : n+1 \to \omega$ with $v <_0 g \restriction n+1$. Then $\underset{\sim}{p}_{n+1}$ is defined to be a $V^{P_{\gamma_n}}$ name of p_{n+1}.

Now for $Q_0 = P_{\gamma_n}$, $Q_1 = P_{\gamma_{n+1}}$, $Q_2 = P_\delta$, Lemma 3.4 can be applied to q_n (as $q_0 \in Q_0$) and $\underset{\sim}{p}_{n+1}$ (as $\underset{\sim}{p} \in V^{Q_0}$). So there exists some $q_{n+1} \in P_{\gamma_{n+1}}$ that is $(M, P_{\gamma_{n+1}})$-generic with $q_n = \pi(q_{n+1})$ and such that the required inductive assumptions hold.

$\dashv$

3.1. Application: Non-isomorphism of ultrapowers

The significance of Theorem 3.6 proved in this section is clarified by comparing the following two theorems of Keisler and of Shelah concerning the notion of elementary equivalence (see the book by Comfort and Negrepontis [2]).

> **Theorem** (Keisler [9]). *If $2^\lambda = \lambda^+$, and $\mathcal{A}$, $\mathcal{B}$ are structures of size $\le \lambda^+$ (in a language of size λ), then $\mathcal{A} \equiv \mathcal{B}$ implies that $\mathcal{A}$ and $\mathcal{B}$ have isomorphic ultrapowers,*
>
> $$\mathcal{A}^\lambda/p \cong \mathcal{B}^\lambda/p$$
>
> *obtained by some ultrafilter p on λ.*

Keisler also showed that it is not possible to obtain this result if the language has size λ^+.

The following was proved by Shelah [12]:

> If $\mathcal{A} \equiv \mathcal{B}$ are both of size $\le k$, then $\mathcal{A}^\alpha/p \cong \mathcal{B}^\alpha/p$ for some ultrafilter p on $2^k = \alpha$.

In particular, for countable elementarily equivalent structures, Shelah's theorem provides an ultrafilter on a set of size $2^{\aleph_0}$ that makes their ultrapowers isomorphic, and Keisler's theorem obtains an ultrafilter on ω, provided that $2^{\aleph_0} = \aleph_1$. The theorem proved in this section shows that CH is indeed necessary for obtaining the ultrafilter to be on ω (see Shelah [14]).

3.6 Theorem. *Assuming CH, there are two countable elementarily equivalent structures $\mathcal{A} \equiv \mathcal{B}$ and there is a generic extension in which $2^{\aleph_0} = \aleph_2$ and for every ultrafilters p, q on ω*

$$\mathcal{A}^\omega/p \ \not\cong\ \mathcal{B}^\omega/q. \tag{I.11}$$

To prove this theorem, we will consider two propositions, P_1 and P_2, show that they imply the existence of $\mathcal{A} \equiv \mathcal{B}$ as in the theorem, and then prove their consistency.

Let DP (Diverging and Positive) denotes the set of functions $h \in {}^\omega\omega$ diverging to infinity, with $h(n) > 0$ for every n. If $\langle A_n \mid n \in \omega\rangle$ is a sequence of finite (non-empty) sets, then $\Pi_{n<\omega} A_n$ is the set of all functions f defined on ω with $f(n) \in A_n$ for every n.

(P_1) If $\langle A_n \mid n \in \omega\rangle$ is a sequence of finite sets, and $\{f_\alpha \mid \alpha \in \omega_1\} \subseteq \Pi_{n<\omega} A_n$, then, for every $h \in DP$, there is a choice of subsets $H_n \subseteq A_n$, $n \in \omega$, such that

1. $|H_n| \leq h(n)$ for all n,
2. $\forall \alpha < \omega_1\ \exists n_0\ \forall n \geq n_0\ (f_\alpha(n) \in H_n)$.

(P_2) $({}^\omega\omega, <^*)$ has a cofinal sequence of length ω_1.

It is left as an exercise to prove that Martin's Axiom + $2^{\aleph_0} > \aleph_1$ implies P_1 but negates P_2, and CH implies P_2 but negates P_1.

Our aim is to prove the following theorem.

3.7 Theorem. *$P_1 \wedge P_2$ implies the existence of two elementarily equivalent countable structures that have no isomorphic ultrapowers with ultrafilters on ω.*

We first investigate a consequence of P_2 concerning the structure of ultrapowers of a certain type of graph. Let Δ be the bipartite graph obtained by taking U and V to be two copies of ω, with edges such that every $n \in V$ is connected exactly to those k in U such that $k \leq n$. Let p be any nonprincipal ultrafilter over ω, and form the ultrapower $P = \Delta^\omega/p$ Consider the cofinal sequence $\langle f_\alpha \mid \alpha \in \omega_1\rangle$ from P_2, and form for every f_α an element a_α of U^P obtained by viewing $f_\alpha(n) \in U$ (and taking the equivalence class of f_α). The fact that the sequence of functions is cofinal in ${}^\omega\omega$ implies the following property of P (expressed with U and V as predicates):

There is a sequence $a_i \in U$, $i \in \omega_1$ such that there is no $b \in V$ edge connected with every a_i. (I.12)

Indeed, any $[b] \in V^P$ is the equivalence class of some function $n \mapsto b(n)$, and there is some f_α that bounds b. Hence b is not connected to f_α in the ultrapower.

In fact, we can redo this argument even in the following slightly more general situation. Suppose that Δ is a bipartite graph built on two copies U and V of ω, just as before, but now we know that the following holds.

For every finite set $X \subset V$ there is some $u \in U$ such that no $x \in X$ is connected with u. (I.13)

Then define a sequence $u_n \in U$ by induction on n such that u_n is not connected to any one of the first n nodes of V. Here too (I.12) holds in any ultrapower. That is in Δ^ω/p there is a set of ω_1 members of U such that there is no $b \in V$ that is connected to all of them. In fact, this result on ultrapowers can be generalized to ultraproducts of countable bipartite graphs that satisfy property (I.13) above. These ultraproducts must satisfy (I.12).

Define $\Gamma_{k,\ell}$ to be the finite bipartite graph with two disjoint sets of vertices U and V, where $|U| = k$, and the vertices in V are obtained by associating with every $x \subseteq U$ of size $\leq \ell$ a vertex a_x in V that is edge connected exactly with the vertices of x.

We are particularly interested in graphs of the form $\Gamma_{n^2+1,n}$ because they have the following property: for any $X \subseteq V$ with cardinality $\leq n$ there is some $a \in U$ that is not connected to any $x \in X$.

Let Γ be the disjoint union of the graphs $G_n = \Gamma_{n^2+1,n}$ for $2 \leq n < \omega$. The language of the structure Γ includes not only the edge relation but also the predicates U and V, and a partial order $<_\Gamma$ that puts the vertices of G_n below those of G_m for $n < m$. The nodes in G_n are incomparable in $<_\Gamma$.

The connected components of the graph Γ are the copies of the $\Gamma_{n^2+1,n}$. (Because for $n \geq 2$, $\Gamma_{k,n}$ is connected. In fact, any two nodes that are in the same G_n are connected by a sequence of at most four edges.) So the connected components of Γ are exactly the maximal antichains of $<_\Gamma$ and this fact can be expressed by a single sentence.

Let Γ_{NS} be some countable nonstandard elementary extension of Γ. Then $\Gamma_{NS} \equiv \Gamma$, but Γ_{NS} contains also infinite connected components. The connected components of Γ_{NS} are, again, its $<_\Gamma$ antichains. Assuming P_1 and P_2, we are going to prove that Γ_{NS} and Γ have no isomorphic ultrapowers on ω. Observe that any non-standard (infinite) component of Γ_{NS} has property (I.13). Hence the following statement is true in any nonprincipal ultrapower $(\Gamma_{NS})^\omega/q$. The set of components C that satisfy the following property is cofinal in the ordering $<_\Gamma$.

There is a sequence $a_i \in U \cap C$, $i \in \omega_1$ such that there is no $b \in V$ edge connected with every a_i. (I.14)

Indeed, the set of components that are in fact ultraproducts of nonstandard components of Γ_{NS} is such a cofinal set in the ordering $<_\Gamma$ of all components.

On the other hand no ultrapower of Γ over ω can satisfy this property, because the following holds in any ultrapower Γ^ω/p. There is a complement of initial set of components C in the $<_\Gamma$ ordering for which:

If $a_i \in C \cap U$ for $i \in \omega_1$, then there is some $b \in V$ connected to all the a_i's. (I.15)

It is easy to establish (I.15) for Γ^ω/p once the following observation is made. Suppose that $h \in DP$, p is a nonprincipal ultrafilter on ω, and $G = (\Pi_n G_{h(n)})/p$. Then (P_1) implies that G is a bipartite graph with the following property.

If $a_i \in G \cap U$, for $i \in \omega_1$, then there is $b \in G \cap V$ connected to all the a_i's. (I.16)

This follows by applying P_1 to $A_n = h(n)^2 + 1$, $\{a_\alpha \mid \alpha \in \omega_1\}$, and h.

Now we turn to the consistency result itself.

3.8 Theorem. *Let $\langle A_n \mid n \in \omega\rangle$ be a sequence of finite sets. Let $h \in DP$ (diverging to ∞ with $h(n) \geq 1$). There is a proper, ${}^\omega\omega$-bounding forcing poset P, of size continuum, such that in any generic extension $V[G]$ via P the following holds.*

> *There is in $V[G]$ a sequence $\langle H_n \mid n \in \omega\rangle$, with $H_n \subseteq A_n$, $|H_n| \leq h(n)$, that eventually bounds every ground-model $f \in \Pi_n A_n$. I.e., if $f \in V \cap \Pi_{n<\omega} A_n$, then for some k for all $n \geq k$ $f(n) \in H_n$.*

To obtain the desired model where P_1 and P_2 hold, assume CH $+2^{\aleph_1} = \aleph_2$ and iterate with countable support ω_2 many posets as in the theorem. By CH, the resulting poset satisfies the $\aleph_2$-c.c (Theorem 2.10). This ensures P_1, since any parameters for P_1 appears in an initial segment of the iteration, so that a suitable bookkeeping device takes care of any possible sequences. P_2 is a consequence of the fact that the resulting poset is ${}^\omega\omega$-bounding and hence the ground reals give a bounding sequence of length ω_1 (by the preservation theorem). We turn to the proof of the theorem.

For any finite set A let $\mathcal{P}(A)$ be the power set of A, and $\mathcal{P}_m(A)$ be the collection of subsets of A of cardinality $\leq m$. We say that $E \subseteq \mathcal{P}(A)$ is a k-cover (k a natural number) if for every $x \subseteq A$ of size $\leq k$, for some $e \in E$, $x \subseteq e$.

Referring to the given DP function h, define

$$S = \bigcup_{\ell<\omega} \prod_{0\leq i<\ell} \mathcal{P}_{h(i)}(A_i).$$

That is, $\eta \in S$ iff η is a finite sequence such that for $i \in \mathrm{dom}(\eta)$, $\eta(i)$ is a subset of A_i of size $\leq h(i)$. S forms a tree under extension (inclusion). We will force with infinite subtrees of S that are good in the following sense.

Let $T \subseteq S$ be a subtree (that is, a collection of sequences in S closed under initial segments). We make the following definitions.

1. If $\eta \in T$ then the number $\mathrm{dom}(\eta)$ is also called the height of the node η. Let $T \restriction m$ be the collection of all nodes in T of height $< m$.

2. We say that $\eta \in T$ is the stem of T if η is comparable to all nodes of T (under inclusion), and η is maximal with this property.

3. A node $\eta \in T$ is said to be k-covering in T iff its (immediate) successors in T form a k-cover of the appropriate A_i ($i = \mathrm{dom}(\eta)$). That is, $\{\mu(i) \mid \mu \in T \text{ extends } \eta\}$ is a k-cover of A_i.

Let P be the poset (under inclusion) of all infinite subtrees $T \subseteq S$ that have a stem $\sigma(T)$, and each node $\eta \in T$ is at least 1-covering (except the nodes below the stem which are not 1-covering), and such that, for every k, except for finitely many nodes, all nodes of T are k-covering.

In any forcing extension via P, the generic sequence of stems provides a sequence $H_n \subseteq A_n$ with $|H_n| \leq h(n)$. A density argument shows that every ground-model $f \in \Pi_n A_n$ is eventually bounded by the H_n's. In this argument, use the obvious remark that if $E \subseteq P_m(A)$ is a k-cover, and $a \in A$, then the collection of $e \in E$ such that $a \in e$ form a $k-1$-cover of A. Both properness and the ${}^\omega\omega$-bounding property follow once we prove that P satisfies Axiom A^* of Baumgartner (see 2.3). For this we define relations $\leq_k$, for $0 \leq k < \omega$, on the trees in P.

$\leq_0$ is just the poset ordering (inverse inclusion).

Define $T_1 \leq_1 T_2$ iff T_2 is a pure extension of T_1: that is, T_2 extends T_1, and they have the same stem.

Define $T_1 \leq_k T_2$ for $k > 1$ iff $T_1 \leq_1 T_2$, $T_1 \restriction k = T_2 \restriction k$ and for every $i \leq k$, for any $\eta \in T_2$, if η is i-covering in T_1 then η remains i-covering in T_2. The following is a direct consequence of the definitions.

3.9 Lemma. *$\leq_k$ is transitive, and $k < \ell$ implies that $\leq_\ell \subseteq \leq_k$.*

3.10 Lemma. *If $T_1 \leq_1 T_2 \leq_2, \ldots, T_n \leq_n T_{n+1}, \ldots$, then a fusion $T \in P$ can be defined such that $T_i \leq_i T$ for all i.*

Proof. Indeed, $T = \bigcup_{1 \leq i < \omega} T_i \restriction i$ works. ⊣

Given a name $\underset{\sim}{\tau}$ of an ordinal we must show that every $T \in P$ has a $\leq_k$ extension that decides $\underset{\sim}{\tau}$ up to finitely many possibilities. (This can be seen to be an equivalent formulation of item 2 of property A^*.)

Say that a tree T is m-covering if every node in T (not below the stem) is m-covering. For any $T \in P$ and $\eta \in T$, let $T(\eta)$ be the subtree of T obtained by letting η to be the stem. The following lemma suffices to prove Axiom A^*.

3.11 Lemma. *Let $\underset{\sim}{\tau}$ be a P-name of an ordinal. If $m \geq 2k$ and T is m-covering, then T has a pure extension T' that is k-covering and such that for some finite set of ordinals, B, $T' \Vdash \underset{\sim}{\tau} \in B$.*

Proof. A node η of T is *good* iff $T(\eta)$ has a pure, k-covering extension that decides $\underset{\sim}{\tau}$ up to finitely many possibilities. η is *bad* if it is not good. So the Lemma says that the stem of T is good.

Let X be a set of successors of some η in T; we say that X is a *majority set* if X is k-covering. More formally, X is a majority set if for $i = |\eta|$ the collection $\{\mu(i) \mid \mu \in X\}$ is a k-cover of A_i.

Observe that since any $\eta \in T$ is m-covering and $m \geq 2k$, if the set of successors of η is given as a union $X_1 \cup X_2$, then X_1 or X_2 is a majority. (For otherwise there are sets $x_1, x_2 \subseteq A_i$ of size k each such that x_1 is not covered by the nodes of X_1 and x_2 is not covered by the nodes of X_2. But then $x_1 \cup x_2$ is of size $\leq m$ and is not covered by any successor of η!) Hence if $\eta \in T$ is bad, then the bad successors of η form a majority.

For any trees T_1, T_2 we say that T_2 is a majority extension of T_1 if $T_2 \geq T_1$ is obtained by taking only majority of successors in T_1. Equivalently, T_2 is a pure extension of T_1 which is k-covering.

Now if the lemma does not hold and the stem is bad, then there is a majority extension T' of T consisting entirely of bad nodes. This is impossible: pick any $T'' \geq T'$ that decides $\underset{\sim}{\tau}$, and find in T'' a node η such that $T(\eta)$ is k-covering. Then η must be good (already in T). $\dashv$

4. Preservation of unboundedness

This section is adapted from [13] (reworked in Chapter VI of [15]). A forcing poset P is said to be *weakly* ${}^\omega\omega$-bounding if the old reals are not bounded in the extension. That is, the following holds in every extension $V[G]$ via P: for any $f \in {}^\omega\omega \cap V[G]$ there exists $g \in V$, such that $\{n \mid f(n) \leq g(n)\}$ is infinite. For example, the Cohen-real forcing is weakly ${}^\omega\omega$-bounding. (Given $f \in V^P$, let $\{c_n \mid n \in \omega\}$ enumerate all Cohen conditions, and define $g(n)$ so that some extension of c_n forces that $g(n) = \underset{\sim}{f}(n)$.)

4.1 Theorem. *The weak ${}^\omega\omega$-bounding property is preserved by the limit of a countable support iteration of proper posets if each initial part of the iteration is weakly ${}^\omega\omega$-bounding.*

Thus, if δ is limit, if $\langle P_i \mid i \leq \delta\rangle$ is a countable support iteration of proper posets, and every P_i, for $i < \delta$, is weakly ${}^\omega\omega$-bounding, then P_δ is weakly ${}^\omega\omega$-bounding too.

Observe the difference between the formulation of this theorem and that of Theorem 3.5: here we speak about initial parts of the iteration–not about the iterands. In the next subsection we will explain why the iteration of weakly ${}^\omega\omega$-bounding posets is not necessarily weakly ${}^\omega\omega$-bounding, and we will define the notion of almost bounding and show that the iteration of almost bounding posets is weakly ω-bounding.

Theorem 4.1 is proved by induction on δ. Let $\underset{\sim}{f}$ be a name for a real in V^{P_δ}, and $p_0 \in P_\delta$ an arbitrary condition. Pick $M \prec H_\kappa$ countable, with P_δ , p_0 , $\underset{\sim}{f} \in M$ as usual. Fix an increasing sequence $\gamma_i \in \delta \cap M$ converging to $\sup(\delta \cap M)$. Let $g \in {}^\omega\omega$ $<^*$-dominate all the reals of M. We will find in P_δ an extension q of p_0 that forces

$$\{n \in \omega \mid \underset{\sim}{f}(n) \leq g(n)\} \quad \text{is infinite.}$$

As before, we define by induction conditions $q_n \in P_{\gamma_n}$ that are (M, P_{γ_n})-generic, and names $\underset{\sim}{p}_n \in V^{P_{\gamma_n}}$ such that:

1. $q_{n+1} \restriction \gamma_n = q_n$
2. $q_n \Vdash_{\gamma_n}$ "$\underset{\sim}{p}_n$ is in $P_\delta \cap M$, it extends $\underset{\sim}{p}_{n-1}$, and $\underset{\sim}{p}_n \restriction \gamma_n \in \underset{\sim}{G}_n$ (the generic filter over P_{γ_n})".
3. $q_n \Vdash_{\gamma_n}$ "$\underset{\sim}{p}_n \Vdash_\delta$ for some $k \geq n$, $\underset{\sim}{f}(k) \leq g(k)$ ".

When done, $q = \bigcup_n q_n$ is in P_δ, and for every n

$$q \Vdash_\delta \underset{\sim}{p}_n \in G_\delta$$

(we have seen that in proving the Properness Extension Lemma, 2.8). Hence q "knows" what every interpretation of $\underset{\sim}{p}_n$ knows, i.e., $q \Vdash_\delta$ *for some* $k \geq n$, $\underset{\sim}{f}(k) \leq g(k)$. This holds for every n. Hence q is as required.

We now turn to the inductive definition. To begin with, $\underset{\sim}{p}_0$ is in fact a condition—the given p_0— and $q_0 \in P_{\gamma_0}$ is an (M, P_{γ_0})-generic condition extending $p_0 \restriction \gamma_0$.

Suppose that q_n and $\underset{\sim}{p}_n$ are defined, we shall obtain $\underset{\sim}{p}_{n+1}$ and then q_{n+1}. Imagine a generic extension $V[G_n]$, where $q_n \in G_n \subseteq P_{\gamma_n}$. Then $\underset{\sim}{p}_n$ is realized as some condition $p_n \in P_\delta \cap M$, such that $p_n \restriction \gamma_n \in G_n$.

In $M[G_n]$, define an increasing sequence, $\langle r_i \mid i \in \omega\rangle$, beginning with $r_0 = p_n$, of conditions in P_δ that decide the values of $\underset{\sim}{f}$, and such that $r_i \restriction \gamma_n \in G_n$ (use Lemma 1.2). Let f^* be the real thus interpreted; so for every $k < \omega$, r_k forces (in P_δ forcing) that $\underset{\sim}{f} \restriction k = f^* \restriction k$. Obviously $f^* \in M[G_n]$.

Since P_{γ_n} is weakly ${}^\omega\omega$-bounding, for some $h \in {}^\omega\omega \cap M$, $h(i)$ is above $f^*(i)$ for infinitely many i's. But $h <^* g$, and hence for some $i_0 \geq n+1$, $f^*(i_0) < g(i_0)$. For $j = i_0 + 1$, r_j fixes the value of $\underset{\sim}{f}(i_0)$ to be $f^*(i_0)$. Now define $\underset{\sim}{p}_{n+1}$ to be a P_{γ_n}-name of r_j. Finally, q_{n+1} is defined by the Properness Extension Lemma to be a condition in $P_{\gamma_{n+1}}$ such that $q_{n+1} \restriction \gamma_n = q_n$ and

$$q_{n+1} \Vdash_{\gamma_{n+1}} \underset{\sim}{p}_{n+1} \restriction \gamma_{n+1} \in \underset{\sim}{G}_{n+1}.$$

The proof is finished, but it is worth remarking that the definition of $\underset{\sim}{p}_{n+1}$ depends on g, the function that dominates M, hence $\underset{\sim}{p}_{n+1}$ cannot be defined in M. But of course it is always realized as some condition in M.

4.1. The almost bounding property

The successor case, which causes no problem for the ${}^{\omega}\omega$-bounding property, is not obvious at all for the weakly bounding property. In fact, it is possible to have Q_1 weakly ${}^{\omega}\omega$-bounding, $Q_2 \in V$ weakly ${}^{\omega}\omega$-bounding in V^{Q_1}, yet $Q_1 \times Q_2$ adds a dominating real. For example, add $\aleph_1$ many Cohen reals (this is Q_1), and then do the Hechler forcing with conditions from V. (Hechler [6] posets adds a generic function in ${}^{\omega}\omega$ by giving finite information on the generic function, and a function in ${}^{\omega}\omega$ which the generic must from now on dominate. See also Jech [7]) Now, though Q_2 adds a dominating real to V, it is an exercise to see that Q_2 is weakly ${}^{\omega}\omega$-bounding in V^{Q_1}, because the Q_2-name of any real is already in $V^{Q_1 \restriction \alpha}$ for some countable α

In order to tackle the successor stage, we introduce a notion that is of intermediate power in between weakly ${}^{\omega}\omega$-bounding and ${}^{\omega}\omega$-bounding — the *almost* ${}^{\omega}\omega$-bounding.

4.2 Definition. A poset Q is called almost ${}^{\omega}\omega$-bounding iff for every Q-name, $\underset{\sim}{f} \in {}^{\omega}\omega$, and condition $q \in Q$, there exists $g \in {}^{\omega}\omega$ such that

($*$) For every infinite $A \subseteq \omega$, there is $q' \geq q$ such that:

$$q' \Vdash \text{ for infinitely many } n \in A, \underset{\sim}{f}(n) \leq g(n). \qquad \text{(I.17)}$$

Notice the order of quantification: $\exists g \in {}^{\omega}\omega \forall A \subseteq \omega$. If it is reversed, then (for proper posets) this property becomes the weak bounding property.

4.3 Lemma. *If P is weakly ${}^{\omega}\omega$-bounding, and $Q \in V^P$ is almost ${}^{\omega}\omega$-bounding (in V^P), then $P * Q$ is weakly ${}^{\omega}\omega$-bounding.*

Proof. Let $\underset{\sim}{f}$ be a $P * Q$-name, and $(p, q) \in P * Q$ a condition that forces $\underset{\sim}{f} \in {}^{\omega}\omega$. We will find a generic extension via $P * Q$ with a filter containing (p, q) in which $\underset{\sim}{f}$ is weakly bounded by some function in V. First take a (V, P)-generic G with $p \in G$. Working in $V[G]$, $\underset{\sim}{f}$ "becomes" a name in Q-forcing of a real, and we continue to denote this name by $\underset{\sim}{f}$. Q "is" now an almost ${}^{\omega}\omega$-bounding forcing poset, and $q \in Q$ a condition. By definition, there is in $V[G]$ a function $g \in {}^{\omega}\omega$ such that ($*$) (in Definition 4.2) holds. Since P is weakly ${}^{\omega}\omega$-bounding, g is weakly bounded, say by $h \in V$, and

$$A = \{n \mid g(n) \leq h(n)\}$$

is infinite, in $V[G]$. So there is an extension q' of q in Q for which (I.17) of ($*$) holds. If the second generic extension is done with q' in the generic filter, then for an infinite subset $A_0 \subseteq A$, $f(n) \leq g(n)$ holds for $n \in A_0$. Thus $\underset{\sim}{f}$ is weakly dominated by $h \in V$. $\dashv$

By combining 4.1 and 4.3 we get the following theorem.

4.4 Theorem. *The iteration of almost bounding, proper posets is weakly ${}^{\omega}\omega$-bounding.*

4.2. Application to cardinal invariants of the continuum

This section deals with two cardinal invariants $\mathfrak{b}$ and $\mathfrak{s}$. For additional information on these cardinals the reader may consult Blass's chapter (Combinatorial Cardinal Characteristics of the Continuum). Following [13] we will establish here the consistency of

$$\textit{bounding number} < \textit{splitting number}.$$

The bounding number, $\mathfrak{b}$, is the smallest cardinality of an unbounded subset of ${}^{\omega}\omega$ (in the eventual dominance ordering $<^*$). In what follows, $[\omega]^{\omega}$ denotes the set of infinite subsets of ω, and $\subseteq^*$ between members of $[\omega]^{\omega}$ denotes eventual inclusion, i.e. $A \subseteq^* B$ *iff* $A \setminus B$ is finite.

The splitting number, $\mathfrak{s}$, is the smallest cardinality of a "splitting" set $S \subseteq [\omega]^{\omega}$, where S is splitting iff for every infinite $A \subseteq \omega$, some $B \in S$ splits A, that is, both $A \cap B$ and $A \setminus B$ are infinite. In other words, we say that $A \subseteq \omega$ makes an ultrafilter on $S \subseteq [\omega]^{\omega}$ if for every $B \in S$ either $A \subseteq^* B$ or $A \subseteq^* \omega \setminus B$. Thus S is splitting iff no A makes an ultrafilter on S.

4.5 Theorem. *Assume CH. There is a generic extension in which $2^{\aleph_0} = \aleph_2$, cardinals are not collapsed, and $\mathfrak{b} < \mathfrak{s}$.*

The general structure of the consistency proof for this theorem is to iterate ω_2 almost-bounding proper forcings that "kill" the old reals as a splitting family. Finally, by Theorem 4.4, the reals of the ground model are still undominated, and hence $\mathfrak{b} = \aleph_1$, but $s = \aleph_2$ because no set of size $\aleph_1$ can be splitting. This is so because every set of reals of size $\aleph_1$ is included in some stage $\gamma < \omega_2$ of the iteration and hence was "killed" at the following stage (by introducing some A that makes an ultrafilter on $\mathcal{P}(\omega) \cap V_{\gamma}$). Thus we only need the following.

4.6 Theorem. *There is a proper, almost ${}^{\omega}\omega$-bounding poset Q of size continuum such that in V^Q*

> *There is an infinite set $A \subseteq \omega$ such that for every $B \subseteq \omega$ from V, $A \subseteq^* B$ or $A \subseteq^* \omega \setminus B$.*

Proof. The first forcing notion that comes to mind is the Mathias forcing [11]. It consists of pairs (u, E) where u is a finite and E an infinite subset of ω. Extension is defined so that $(u_1, E_1) \leq (u_2, E_2)$ iff $E_2 \subseteq E_1$, u_2 is an end-extension of u_1, and $u_2 \setminus u_1 \subseteq E_1$. If G is a generic filter, then $U = \bigcup\{u \mid (u, E) \in G \text{ for some } E\}$ makes an ultrafilter on $\mathcal{P}(\omega) \cap V$ (it is not split by any subset of ω in the ground model). However, this forcing introduces a dominating real— the enumeration of the generic subset — and hence we must search for another solution.

The conditions in Q will be pairs (u, T) such that $u \subseteq \omega$ is finite, and $T = \langle t_i \mid i \in \omega \rangle$ is a sequence of "logarithmic measures". Each t_i consists of a finite subset of ω, s_i which is denoted $\mathrm{int}(t_i)$, and a finite measure, specified below, defined on all subsets of $\mathrm{int}(t_i)$ and taking natural number values. We have that $\max(u) < \min(s_i) \leq \max(s_i) < \min(s_{i+1})$. Define $\mathrm{int}(T) = \bigcup_i \mathrm{int}(t_i)$; this is an infinite set of numbers in ω above u, and the order on Q will be such that if $(u_1, T_1) \leq (u_2, T_2)$ holds, then $(u_1, \mathrm{int}(T_1)) \leq (u_2, \mathrm{int}(T_2))$ as Mathias conditions. The reason that the reals in V do not split the generic real $U = \bigcup\{u \mid \exists T(u, T) \in G\}$ is the same as for the Mathias forcing: it will be shown that if $(u_1, T_1) \in Q$ then whenever $\mathrm{int}(T_1) = x \cup y$, there is an extension (u_2, T_2) of (u_1, T_1) in Q such that $\mathrm{int}(T_2) \subseteq x$ or $\mathrm{int}(T_2) \subseteq y$. To define Q we need first the notion of "logarithmic measure".

A *logarithmic measure* on $S \subseteq \omega$ (S is usually finite) is a function $h : \mathcal{P}_\omega(S) \to \omega$ (where $\mathcal{P}_\omega(S)$ is the collection of all *finite* subsets of S) and such that if $A \cup B \subseteq S$ is finite and $h(A \cup B) \geq \ell + 1$ then $h(A) \geq \ell$ or $h(B) \geq \ell$. It follows that if $h(A_0 \cup \cdots \cup A_{n-1}) > \ell$ then $h(A_j) \geq \ell - j$ for some $0 \leq j < n$.

When h is a logarithmic measure on S and S is finite, then $h(S)$ is called the level of h, and is denoted $\mathrm{level}(h)$.

Actually, our measures will all be induced by a collection of positive sets as follows: Given a collection $P \subseteq \mathcal{P}_\omega(S)$ that is closed upwards ($a \in P$ and $a \subseteq b$ imply $b \in P$), a logarithmic measure h *induced by* P is inductively defined as follows on $\mathcal{P}_\omega(S)$:

1. $h(e) \geq 0$ for every $e \in \mathcal{P}_\omega(S)$.

2. $h(e) > 0$ iff $e \in P$.

3. For $\ell \geq 1$, $h(e) \geq \ell + 1$ iff $|e| > 1$ and whenever $e = e_1 \cup e_2$ then $h(e_1) \geq \ell$ or $h(e_2) \geq \ell$.

Then $h(e) = \ell$ iff ℓ is the maximal natural number such that $h(e) \geq \ell$ (there has to be such ℓ and $h(e) < \infty$).

We have, for measures defined by positive sets, that if $h(e) = k$ and $e \subseteq a$ then $h(a) \geq k$.

For example, if the positive sets are those containing at least two points, then $h(X)$ is the least i for which $|X| \leq 2^i$. We will use the following observation.

4.7 Lemma. *Let $P \subseteq \mathcal{P}_\omega(\omega)$ be an upwards closed collection (of finite nonempty sets), the following condition implies that the measure h induced by P on $\mathcal{P}_\omega(\omega)$ has arbitrarily high values:*

> *For every decomposition $\omega = A_1 \cup \ldots \cup A_n$ into finitely many sets, for some $i, \mathcal{P}_\omega(A_i) \cap P \neq \emptyset$.*

Assuming this condition, for every natural number k and decomposition $\omega = \bigcup_{i<n} A_i$ into $n < \omega$ sets, for some $i < n$, $h(e) \geq k$ for some $e \subseteq A_i$.

Proof. Suppose that P satisfies the condition of the lemma, and we shall prove by induction on $k < \omega$ the required conclusion.

For $k = 1$, this is just the assumed condition. Assume the claim for $k = \ell$, and let's prove it for $\ell + 1$. Let $\omega = \bigcup_{i<n} A_i$ be a decomposition such that (contrary to our lemma) for every $j < \omega$, for all $i < n$, $h(A_i \cap j) \not\geq \ell + 1$. Thus for every $i < n$ there are e_1 and e_2 such that $A_i \cap j = e_1 \cup e_2$ and both $h(e_1) \not\geq \ell$ and $h(e_2) \not\geq \ell$. König's lemma can be used to find a decomposition $A_i = E^i_1 \cup E^i_2$ such that there is no x with $h(x) \geq \ell$ included in E^i_1 or in E^i_2. Hence a decomposition of ω into $2n$ sets contradicts the inductive assumption for ℓ. $\dashv$

To prove Theorem 4.6, we define the poset Q which was informally described above. Q consists of all pairs $p = (u, T)$ where

1. $u \subseteq \omega$ is finite (called the stem of p) and
2. $T = \langle t_i \mid i \in \omega \rangle$ is a sequence of measures, $t_i = (s_i, h_i)$ where h_i is a logarithmic measure on s_i, which is a finite subset of ω ($s_i = \mathrm{int}(t_i)$) such that
   (a) $\max(u) < \min(s_0)$
   (b) $\max(s_i) < \min(s_{i+1})$
   (c) The level of the measures diverges to infinity, and, moreover, $\mathrm{level}(h_i) < \mathrm{level}(h_{i+1})$. (We defined $\mathrm{level}(h_i) = h_i(s_i)$.)

Recall that $\mathrm{int}(T) = \bigcup\{s_i \mid i \in \omega\}$. For convenience, we write $p = (u, T)$ even when $\max(u)$ is not below $\min(s_0)$. This p refers then to the condition obtained by throwing away sufficiently many t_i's, that is, $p = (u, \langle t_i \mid i \geq k \rangle)$ where k is first such that $\max(u) < \min(s_k)$.

The extension relation for Q is defined by:

$$(u_1, T_1) \leq (u_2, T_2),$$

where $T_\ell = \langle t^\ell_i \mid i \in \omega \rangle$, $t^\ell_i = (s^\ell_i, h^\ell_i)$ for $\ell = 1, 2$, if and only if

1. u_2 is an end extension of u_1, and $u_2 \setminus u_1 \subseteq \mathrm{int}(T_1)$.
2. $\mathrm{int}(T_2) \subseteq \mathrm{int}(T_1)$. Moreover, there is a sequence of finite subsets of ω, $\langle B_i \mid i \in \omega \rangle$ with $\max(B_i) < \min(B_{i+1})$ and $\max(u_2) < \min(s^1_j)$ for $j = \min(B_0)$, such that $s^2_i \subseteq \bigcup\{s^1_j \mid j \in B_i\}$.

3. For every i: if $e \subseteq s_i^2$ is h_i^2-positive (i.e., $h_i^2(e) > 0$), then for some j, $e \cap s_j^1$ is h_j^1-positive.

The reader may check that this defines an order on Q. An extension that does not change the stem is called a pure extension. Observe that if (w, R) extends (v, T) then (w, R) extends (w, T) as well. That is any extension can be formed by first extending the stem and then a pure extension. We shall prove that Q is proper and almost ${}^{\omega}\omega$-bounding.

For properness, Axiom A will be shown to hold. The $\leq_n$ relations needed are defined as follows. $<_0$ is the extension relation on Q just defined. For $n > 0$

$$(u_1, T_1) \leq_n (u_2, T_2)$$

iff

$$(u_2, T_2) \text{ is a pure extension of } (u_1, T_1) \text{ and for } 0 \leq i < n-1,\ h_i^1 = h_i^2,$$

that is, the stem and the first $n-1$ measures (and sets) are the same in both conditions. In particular $(u_1, T_1) \leq_1 (u_2, T_2)$ iff (u_2, T_2) is a pure extension of (u_1, T_1), that is an extension such that $u_1 = u_2$.

We can check the fusion property. Suppose that $p_0 \leq_1 p_1 \leq_2 \cdots p_{i-1} \leq_i p_i \ldots$ is a fusion sequence, where $p_\ell = (u_\ell, \langle t_i^\ell \mid i \in \omega \rangle)$. Then set $p = (u, T)$ by $u = u_0$, and $T = \langle t_i \mid i \in \omega \rangle$ defined as $t_i = t_i^{i+1}$ (p takes the common stem u, and the measure t_i that is common to all the conditions with indices above i) then $p \in Q$ and $p_i \leq_i p$ for all i.

To show Axiom A, we have to prove that for any $n \in \omega$, $p = (u, T)$, and dense open set D, there are a countable $D_0 \subseteq D$ and an extension $p \leq_n p_0$, such that D_0 is predense above p_0.

We say that a condition (u, T) (with $T = \langle t_i \mid i \in \omega \rangle$) is *preprocessed* for D and i iff for every $v \subseteq i$ that is an end extension of u, if $(v, \langle t_j \mid j \geq i \rangle)$ has a pure extension in D, then $(v, \langle t_j \mid j \geq i \rangle)$ is already in D. The following can be easily proved:

1. If (u, T) is preprocessed for D and i, then any extension is also preprocessed for D and i.

2. Any given condition has a $\leq_{i+1}$ extension that is preprocessed for D and i.

3. Hence by taking the fusion of a sequence, one may obtain an extension of any given condition that is preprocessed for every i.

Now if $p_0 = (u, T)$ is preprocessed for every i and D_0 is the set of all conditions in D of the form $(v, \langle t_j \mid j \geq i \rangle)$, then D_0 is predense above p_0. Thus Axiom A holds.

The almost ${}^{\omega}\omega$-bounding property of Q is of course the main point.

4.8 Lemma (Main Lemma). *Let $\underset{\sim}{f}$ be a Q-name for a function in ${}^{\omega}\omega$, and $q \in Q$ a condition. There is a pure extension $p \geq q$, $p = (u, T)$, $T = \langle t_i \mid i \in \omega\rangle$, with the following property:*

> *For every i and $s \subseteq \mathrm{int}(t_i)$ that is t_i-positive, if $v \subseteq i$ then for some $w \subseteq s$, $((v \cup w), T)$ determines the value of $\underset{\sim}{f}(i)$.*

We first show how this lemma implies the almost bounding property (Definition 4.2). Given $\underset{\sim}{f}$ and q, let $p \in Q$ be as in the lemma. For every i, define

$$g(i) \quad = \quad \max\left\{k \mid \begin{array}{l} \text{for some } v \subseteq i \text{ and } w \subseteq \mathrm{int}(t_i) \\ (v \cup w, T) \text{ forces that } \underset{\sim}{f}(i) = k \end{array}\right\}$$

Now let $A \subseteq \omega$ be any infinite set. Put $p' = (u\,,\, \langle t_i \mid i \in A\rangle)$. Then p' extends p and

$$p' \Vdash \text{ for infinitely many } i \in A, \underset{\sim}{f}(i) \leq g(i).$$

To see this, let p'' be any extension of p' and k an arbitrarily high integer. Say $p'' = (v, R)$ where $R = \langle r_i \mid i \in \omega\rangle$. Find $i > k$, $i \in A$, such that $v \subseteq i$, and $\mathrm{int}(R) \cap \mathrm{int}(t_i)$ is t_i-positive (there is such an i by the definition of extension in Q). Using the property of the Main Lemma for $s = \mathrm{int}(R) \cap \mathrm{int}(t_i)$, let $w \subseteq s$ be such that $(v \cup w, T)$ decides the value of $\underset{\sim}{f}(i)$. Then $(v \cup w, R)$ extends p'' and makes the same decision because it is also an extension of $(v \cup w\,, T)$.

Now we turn to the proof of the Main Lemma. The required condition p is obtained as a fusion of a sequence defined inductively in ω steps. At the ith step we have a condition $p_i = (u, \langle t_j \mid j \in \omega\rangle)$ and we define p_{i+1} so that $p_i \leq_{i+1} p_{i+1}$ That is, u and $t_0, \ldots, t_{i-1}$ are not touched in the extension. We start with $T = \langle t_j \mid j \geq i\rangle$ and apply the following lemma 2^i times, considering each $v \subseteq i$ in turn.

4.9 Lemma. *Let $(\emptyset, T)$ be a condition, $\underset{\sim}{f}$ a name for a function in ${}^{\omega}\omega$, and i any natural number. Fix $v \subseteq i$. There is a pure extension $(\emptyset, R)$ of $(\emptyset, T)$ with $R = \langle r_\ell \mid \ell \in \omega\rangle$ such that for every ℓ and r_ℓ-positive $s \subseteq \mathrm{int}(r_\ell)$, for some $w \subseteq s$, $(v \cup w, \langle r_m \mid m > \ell\rangle)$ determines the value of $\underset{\sim}{f}(i)$. (Observe that any further pure extension of $(\emptyset, R)$ retains this property.)*

Proof. We may assume that $(\emptyset, T)$ (and thence any extension) is preprocessed for $\underset{\sim}{f}(i)$: If an extension (w, R) determines the value of $\underset{\sim}{f}(i)$, then already (w, T) determines that value.

Define a measure h on $\mathrm{int}(T)$ induced by the following positive sets. A finite set $x \subseteq \mathrm{int}(T)$ is "positive" iff the following two conditions hold.

1. for some l, $x \cap \mathrm{int}(t_l)$ is t_l-positive. (t_l are the measures composing T.)

2. For some $w \subseteq x$, $(v \cup w, T)$ determines the value of $\underset{\sim}{f}(i)$.

Property 1 ensures that if $(\emptyset, R)$ is obtained by taking a sequence of subsets of $\mathrm{int}(T)$ with increasing h-measures, then $(\emptyset, R)$ is an extension of $(\emptyset, T)$. Property 2 ensures that this extension has the required properties.

It remains to check that the basic property required to obtained arbitrarily high values of h holds (Lemma 4.7). So let $\mathrm{int}(T) = A_0 \cup \cdots \cup A_{n-1}$ be a partition, and we will find some A_ℓ that contains a positive set. Because the measures t_i are logarithmic and increasing to infinity, for some $\ell < n$ there exists an infinite index set $I \subseteq \omega$ such that the t_i-measures of $A_\ell \cap \mathrm{int}(t_i)$ for $i \in I$ are diverging to infinity. (Otherwise, for every $\ell < n$ there is a finite bound on the t_i measures of $A_\ell \cap \mathrm{int}(t_i)$ for $i \in \omega$, and hence there is a bound k on the measures of $A_\ell \cap \mathrm{int}(t_i)$ where $\ell < n$ and $i \in \omega$. But this is impossible when the measure of t_i is greater than $k + n$.) We may thus find an extension of (v, T), of the form (v, R) such that $\mathrm{int}(R) \subseteq A_\ell$. Now pick any extension $(v \cup w\,,\, R')$ of (v, R) that decides the value of $\underset{\sim}{f}(i)$. Then already $(v \cup w, T)$ decides that value. This shows the existence of a positive subset of A_ℓ (namely a finite union, x, of $\mathrm{int}(r_m)$'s such that $w \subseteq x$. ⊣

⊣

5. No new reals

This last section deals with proper posets that add no new reals, that is, introduce no new subsets of ω in any generic extension. Such posets add no countable sequences of ordinals either, but the shorter expression is the custom. It follows from the work of Jensen and Johnsbråten [8] that the countable support iteration of forcing posets that add no new reals may well add a new real. This shows the need for more complex schemes for iterating posets that add no new reals, and the notion of Dee-completeness is simpler than any other scheme introduced by Shelah for that purpose. The preservation proof that we present here uses the notion of α-properness, and we therefore begin with this notion (following chapter V of Shelah [15]). Our aim is to explain Dee-completeness by means of a simple example (in 5.2), and then to give a rather detailed proof of the Dee-completeness iteration theorem 5.17.

A simple example is given in Section to accompany the definition and discussion of Dee-completeness.

5.1. α-properness

Let $\alpha > 0$ be a countable ordinal, and $\bar{M} = \langle M_i \mid i < \alpha \rangle$ be a sequence of countable, elementary substructures of H_λ (where λ is some fixed regular cardinal). We say that $\bar{M}$ is an α-tower if and only if

1. For every $\delta < \alpha$ limit, $M_\delta = \bigcup_{i<\delta} M_i$.
2. For every $j < \alpha$, $\langle M_i \mid i \leq j \rangle \in M_{j+1}$.

Since λ is regular (or, at least, $\mathrm{cf}(\lambda) > \aleph_0$) if $M \subset H_\lambda$ is countable then $M \in H_\lambda$. Thus, for $j < \alpha$, $\langle M_i \mid i \leq j \rangle \in H_\lambda$ so that (2) makes sense.

5.1 Definition. Let $\alpha > 0$ be a countable ordinal. A forcing poset P is α-proper iff for (every) sufficiently large λ, for every α-tower $\bar{M} = \langle M_i \mid i < \alpha \rangle$ of countable, elementary substructures of H_λ such that $P \in M_0$ the following holds: Every $p \in P \cap M_0$ has an extension $q \geq p$ that is (M_i, P)-generic for every $i < \alpha$. We say that q is $(\bar{M}, P)$-generic in this case.

Clearly properness is 1-properness. We say that P is $< \omega_1$ proper if it is α-proper for every countable ordinal α.

Any *c.c.c* poset is $< \omega_1$ proper. Any countably closed poset is $< \omega_1$ proper. In proving this, one sees why each successor structure in the tower needs to contain the sequence of structures up to that point.

Another example is given by Axiom A posets (Definition 2.3). Let P be an Axiom A poset and prove by induction on $\alpha < \omega_1$ that P is α-proper. For $\alpha = \omega$ we argue as follows. Let $\bar{M} = \langle M_i \mid i < \omega \rangle$ be a tower of countable, elementary substructures of H_λ with $P \in M_0$, and let $p_0 \in P \cap M_0$ be a given condition. Construct by induction conditions $p_i \in P$ such that:

$$p_i \leq_i p_{i+1}, \text{ and } p_{i+1} \in M_{i+1} \text{ is } (M_i, P)\text{-generic.}$$

Let q be the fusion condition, satisfying $p_i \leq_i q$ for all i. Then $p_i \leq q$ and q is thence (M_i, P)-generic.

It is not difficult to check that if P is α-proper than it is $\alpha + 1$-proper. So properness implies n-properness for every $n < \omega$. It does not imply ω-properness. If $\alpha = \beta_1 + \beta_2$ is a sum of two smaller ordinals, then any poset that is both β_1 and β_2 proper is also α proper. So, for α-properness, the values that really count are indecomposable countable ordinals.

Equivalent Definition

As for properness, it is useful to know that if P and Q are posets, P is α-proper and Q is α-proper in V^P, then Q is α-proper already in V. A suitable notion of closed unbounded sets is introduced which is the basis for an equivalent definition of α-properness, from which that useful fact follows. Recall that $\mathcal{P}_{\aleph_1}(A)$ is the collection of all countable subsets of A.

5.2 Definition. Let A be an uncountable set and α a countable ordinal.

1. $P^\alpha_{\aleph_1}(A)$ is the set of all increasing and continuous sequences $\langle a_i \mid i < \alpha \rangle$ where $a_i \in \mathcal{P}_{\aleph_1}(A)$ for all $i < \alpha$. (The sequence is increasing if $a_i \subseteq a_j$ for $i < j$, and it is continuous if for limit $\delta < \alpha$, $a_\delta = \bigcup_{i<\delta} a_i$.)

2. Let $F : (\bigcup_{\beta<\alpha} P^{\beta}_{\aleph_1}(A)) \times [A]^{<\aleph_0} \to \mathcal{P}_{\aleph_1}(A)$ be given. We say that F is an α-function. A sequence $\langle a_i \mid i < \alpha\rangle \in P^{\alpha}_{\aleph_1}(A)$ is said to be *closed* under F if for every $\beta < \alpha$ that is a successor ordinal or zero, for every $x \in [a_\beta]^{<\aleph_0}$, $F(\langle a_i \mid i < \beta\rangle, x) \subseteq a_\beta$. So a_0 is closed under the function taking $x \in [a_0]^{<\aleph_0}$ to $F(\emptyset, x)$; a_1 is closed under the function taking $x \in [a_1]^{<\aleph_0}$ to $F(\langle a_0\rangle, x)$ etc.

3. Let $G(F) \subseteq P^{\alpha}_{\aleph_1}(A)$ be the collection of all α-sequences that are closed under F. Then $\{G(F) \mid F \text{ is an } \alpha - \text{function}\}$ generates a countably closed filter on $P^{\alpha}_{\aleph_1}(A)$, which is denoted $\mathcal{D}^{\alpha}_{\aleph_1}(A)$.

4. We say that $S \subseteq P^{\alpha}_{\aleph_1}(A)$ is *stationary* if its complement is not in $\mathcal{D}^{\alpha}_{\aleph_1}(A)$.

Useful examples of $\mathcal{D}^{\alpha}_{\aleph_1}(A)$ sets are the following:

1. The collection of all α-towers $\langle M_i \mid i < \alpha\rangle$ of countable elementary substructures of H_λ. Here A is the set H_λ, and M_i refers to the universe of that structure.

2. For a closed unbounded set $C \subseteq \mathcal{P}_{\aleph_1}(A)$, collect all sequences $\langle a_i \mid i < \alpha\rangle \in P^{\alpha}_{\aleph_1}(A)$ such that $a_i \in C$ for all i.

In a sense, $\mathcal{D}^{\alpha}_{\aleph_1}(A)$ is normal. If $g : \mathcal{P}_{\aleph_1}(A) \to A$ is a choice function (namely $g(x) \in x$ whenever x is non-empty) and $S \subseteq P^{\alpha}_{\aleph_1}(A)$ is stationary, then for some fixed $v \in A$, $\{\langle a_i \mid i < \alpha\rangle \in S \mid g(a_0) = v\}$ is stationary.

The following is standard.

5.3 Lemma. *Suppose that $A_0 \subset A_1$ are uncountable and $C_1 \in \mathcal{D}^{\alpha}_{\aleph_1}(A_1)$. Define $C_0 = \{\langle a_i \cap A_0 \mid i < \alpha\rangle \mid \langle a_i \mid i < \alpha\rangle \in C_1\}$. Then $C_0 \in \mathcal{D}^{\alpha}_{\aleph_1}(A_0)$.*

The proof of the following theorem resembles that of the Properness Equivalent Theorem 2.13.

5.4 Theorem. *For any poset P and countable ordinal α the following are equivalent.*

1. *P is α-proper (as in Definition 5.1).*

2. *For some $\lambda > 2^{|P|}$, for every α-tower $\bar{M}$ of countable elementary substructures of H_λ, any condition in M_0 has an extension that is $(\bar{M}, P)$-generic.*

3. *For every uncountable λ, P preserves stationary subsets of $P^{\alpha}_{\aleph_1}(\lambda)$.*

4. *For $\lambda_0 = 2^{|P|}$, P preserves stationary subsets of $P^{\alpha}_{\aleph_1}(\lambda_0)$.*

5. *The α-test set for P, as defined below, is in $\mathcal{D}^{\alpha}_{\aleph_1}(A)$.*

   Form $A = P \cup \mathcal{P}(P)$. Then $\langle a_i \mid i < \alpha \rangle \in P^{\alpha}_{\aleph_1}(A)$ is in the α-test set for P iff for every $p_0 \in a_0 \cap P$ there is $p \in P$ that is a_i-generic for every $i < \alpha$. (That is, for every $D \in a_i \cap \mathcal{P}(P)$, if D is dense in P, then D is pre-dense above p.)

Preservation of α-properness

We shall prove the following

5.5 Theorem. *Let $\alpha < \omega_1$ be a countable ordinal, and $\langle P_i \mid i \leq \gamma \rangle$ be a countable support iteration of α-proper posets, then the limit P_γ is α-proper.*

The theorem is obtained as a particular case of the α-Extension Lemma which is proved by induction on α. As the case $\alpha = \omega$ involves almost all the essential ideas of the general case, the reader may concentrate on ω-properness.

5.6 Lemma. *(The α-Extension Lemma). Let α be any countable ordinal, and $\langle P_i \mid i \leq \gamma \rangle$ be a countable support iteration of α-proper posets. Let λ be a sufficiently large cardinal. Let $\bar{M} = \langle M_\xi \mid \xi \leq \alpha \rangle$ be an $\alpha+1$-tower of countable elementary substructures of H_λ, with $\gamma, P_\gamma, \alpha \in M_0$. For every $\gamma_0 \in \gamma \cap M_0$ and $q_0 \in P_{\gamma_0}$ that is $(\bar{M}, P_{\gamma_0})$-generic the following holds.*

If $\underset{\sim}{p}_0 \in V^{P_{\gamma_0}}$ is such that

$$q_0 \Vdash_{\gamma_0} \underset{\sim}{p}_0 \in P_\gamma \cap M_0 \wedge \underset{\sim}{p}_0 \restriction \gamma_0 \in \underset{\sim}{G}_0$$

(where $\underset{\sim}{G}_0$ is the canonical name for the generic filter over P_{γ_0}),

then there is an $(\bar{M}, P_\gamma)$−generic condition q such that

$$q \restriction \gamma_0 = q_0 \text{ and } q \Vdash_\gamma \underset{\sim}{p}_0 \in \underset{\sim}{G}$$

(where $\underset{\sim}{G}$ is the canonical name of the generic filter over P_γ, and the name $\underset{\sim}{p}_0 \in V^{P_{\gamma_0}}$ is now viewed as member of V^{P_γ}).

Proof. The proof is by induction on $\alpha < \omega_1$ and for any fixed α by induction on γ. We begin with $\alpha = \alpha' + 1$ a successor ordinal. We are given a tower $\bar{M} = \langle M_\xi \mid \xi \leq \alpha' + 1 \rangle$, a condition q_0, and a name $\underset{\sim}{p}_0$ as in the lemma. We intend to define a name $\underset{\sim}{r} \in V^{P_{\gamma_0}}$ such that q_0 forces (in P_{γ_0}) the following sentences.

1. $\underset{\sim}{r} \in M_\alpha \cap P_\gamma$ is $\langle M_\xi \mid \xi \leq \alpha' \rangle$-generic.

2. $\underset{\sim}{r}\restriction\gamma_0 \in \underset{\sim}{G}_0$, and $\underset{\sim}{p}_0 <_{P_\gamma} \underset{\sim}{r}$.

Then, using the Properness Extension Lemma 2.8, we can define $q \in P_\gamma$ that is (M_α, P_γ)-generic and such that $q\restriction\gamma_0 = q_0$ and $q \Vdash \underset{\sim}{r} \in \underset{\sim}{G}$. It follows that q is as required.

To define $\underset{\sim}{r}$, let G_0 be a (V, P_{γ_0})-generic filter with $q_0 \in G_0$, and we shall describe $\underset{\sim}{r}[G_0]$. Let $p_0 \in P_\gamma \cap M_0$ be the interpretation of $\underset{\sim}{p}_0$. Then $p_0\restriction\gamma_0 \in G_0$ and we can find $q_0' \in G_0$ that extends both q_0 and $p_0\restriction\gamma_0$. Now we can apply the inductive assumption on α' to the tower $\langle M_\xi \mid \xi \leq \alpha'\rangle$, to q_0' and to p_0, and we find $q' \in P_\gamma$ such that $q'\restriction\gamma_0 = q_0'$, q' is $(\langle M_\xi \mid \xi \leq \alpha'\rangle, P_\gamma)$ generic, and $p_0 <_{P_\gamma} q'$. Since $M_\alpha[G_0] \prec H_\lambda[G_0]$, we can find $q^* \in M_\alpha[G_0]$ with similar properties as q'. Namely, $q^* \in P_\gamma$ (and so $q^* \in M_\alpha$, as $M_\alpha[G_0] \cap V = M_\alpha$), $q^*\restriction\gamma_0 \in G_0$, $p_0 <_{P_\gamma} q^*$, and q^* is $(\langle M_\xi \mid \xi \leq \alpha'\rangle, P_\gamma)$ generic. Let $\underset{\sim}{r}$ be a name of q^* forced by q_0 to have these properties. Then $\underset{\sim}{r}$ is as required.

Assume now that α is a limit ordinal, and let α_n, for $n \in \omega$, be an increasing sequence with limit α. In case γ is a successor ordinal, we shall use the following two-step iteration lemma whose proof is similar to the corresponding case of proper forcing (and is hence not given).

5.7 Lemma. *Suppose that P_0 is α-proper, and $P_1 \in V^{P_0}$ is α-proper in V^{P_0}. Then $R = P_0 * P_1$ is α-proper, and the following holds. Suppose that $\bar{M} \prec H_\lambda$ is an α-tower and $R \in M_0$. Let $\underset{\sim}{r} \in V^{P_0}$ be a name, and $p_0 \in P_0$ be an $(\bar{M}, P_0)$-generic condition such that*

$$p_0 \Vdash_{P_0} \underset{\sim}{r} \in M_0 \cap R \text{ and } \pi(\underset{\sim}{r}) \in \underset{\sim}{G}_0$$

where $\pi : R \to P_0$ is the projection (taking $(a, b) \in R$ to a), and $\underset{\sim}{G}_0$ is the canonical name for the P_0-generic filter. Then there is a name $\underset{\sim}{p}_1 \in V^{P_0}$ such that

1. *$(p_0, \underset{\sim}{p}_1)$ is $(\bar{M}, R)$-generic, and*
2. *$(p_0, \underset{\sim}{p}_1) \Vdash_R \underset{\sim}{r} \in \underset{\sim}{G}$*

where $\underset{\sim}{G}$ is the canonical name of the R-generic filter.

In case $\gamma = \gamma' + 1$ is a successor ordinal, the α-Extension Lemma is proved in two steps: first we use the inductive assumption to define q in $P_{\gamma'}$, and then we use the lemma above to further extend q in $P_{\gamma'+1}$.

We continue the proof of the α-Extension Lemma, and assume now that γ is a limit ordinal. We fix a sequence $\langle \gamma_i \mid i \in \omega\rangle$ increasing and cofinal in γ and such that $\gamma_n \in M_{\alpha_n}$ with γ_0 the given ordinal. (If $\mathrm{cf}(\gamma) = \omega$, let $\langle \gamma_i \mid i \in \omega\rangle$ be an increasing, cofinal in γ sequence in $M_0 \cap \gamma$, with γ_0 as given. If $\mathrm{cf}(\gamma) > \omega$, define $\gamma_n = \sup(\gamma \cap M_{\alpha_{n-1}})$ for $n \geq 1$.) We intend to define by induction on $n < \omega$ conditions $q_n \in P_{\gamma_n}$ and names $\underset{\sim}{p}_n \in V^{P_{\gamma_n}}$ such that:

1. $q_0 \in P_{\gamma_0}$ is the given condition. And for $n > 0$, $q_n \in P_{\gamma_n}$ is $(\langle M_\xi \mid \alpha_n < \xi \leq \alpha \rangle, P_{\gamma_n})$-generic and $q_{n+1} \upharpoonright \gamma_n = q_n$. (In fact, q_n is generic for the complete tower, but this follows from item 2 below.)

2. $\underset{\sim}{p}_0$ is given. $\underset{\sim}{p}_n$ is a P_{γ_n}-name such that

$$q_n \Vdash_{\gamma_n} \quad \underset{\sim}{p}_n \text{ is a condition in } P_\gamma \cap M_{\alpha_n+1} \text{ such that:}$$

   (a) $\underset{\sim}{p}_n \upharpoonright \gamma_n \in \underset{\sim}{G}_{\gamma_n}$,

   (b) $\underset{\sim}{p}_{n-1} \leq_\gamma \underset{\sim}{p}_n$,

   (c) $\underset{\sim}{p}_n$ is an $(\langle M_\xi \mid \xi \leq \alpha_n \rangle, P_\gamma)$-generic condition (when $n > 0$).

When this sequence is defined, $q = \bigcup_n q_n$ is a condition in P_γ and

$$q \Vdash_\gamma \underset{\sim}{p}_n \in \underset{\sim}{G}_\gamma$$

as we have seen in the proof of the Properness Extension Lemma. But as q forces that $\underset{\sim}{p}_n$ is $(\langle M_\xi \mid \xi \leq \alpha_n \rangle, P_\gamma)$-generic, q itself is $(\langle M_\xi \mid \xi \leq \alpha_n \rangle, P_\gamma)$-generic, for every $n \in \omega$, and the proof of the lemma is concluded since q is then $(\langle M_\xi \mid \xi < \alpha \rangle, P_\gamma)$ and hence $(\bar{M}, P_\gamma)$-generic (as α is a limit ordinal).

We turn now to the inductive construction. Suppose that q_n and $\underset{\sim}{p}_n$ are defined. As before, we shall first define $\underset{\sim}{p}_{n+1}$ and then q_{n+1}.

We define $\underset{\sim}{p}_{n+1}$ as a P_{γ_n}-name by the following requirements. If G is any (V, P_{γ_n})-generic filter containing q_n, form

$$M_{\alpha_{n+1}+1}[G] \prec H_\lambda[G], \tag{I.18}$$

and let $p \in P_\gamma$ be the interpretation of $\underset{\sim}{p}_n$. Then $p \in P_\gamma \cap M_{\alpha_n+1}$ and $p \upharpoonright \gamma_n \in G$. As $q_n, p \upharpoonright \gamma_n \in P_{\gamma_n}$ are in G, there is $q'_n \in G$ that extends both q_n and $p \upharpoonright \gamma_n$. By the α_{n+1}-Properness Extension Lemma applied to q'_n and p there is $q^*_n \in P_\gamma$ such that $q^*_n \upharpoonright \gamma_n = q'_n$, $p \leq_\gamma q^*_n$, and q^*_n is $(\langle M_\xi \mid \alpha_n + 1 \leq \xi \leq \alpha_{n+1} \rangle, P_\gamma)$-generic. It follows from (I.18) that (similarly to q^*_n) there is in $M_{\alpha_{n+1}+1}$ a condition $q^* \in P_\gamma$ such that $q^* \upharpoonright \gamma_n \in G$, $p \leq_\gamma q^*$, and q^* is $(\langle M_\xi \mid \alpha_n + 1 \leq \xi \leq \alpha_{n+1} \rangle, P_\gamma)$-generic. Then we define the interpretation of $\underset{\sim}{p}_{n+1}$ in $V[G]$ to be q^*.

Clearly q_n forces that $\underset{\sim}{p}_{n+1}$ is in $P_\gamma \cap M_{\alpha_{n+1}+1}$ and

1. $\underset{\sim}{p}_{n+1} \upharpoonright \gamma_n \in \underset{\sim}{G}_{\gamma_n}$,

2. $\underset{\sim}{p}_n \leq \underset{\sim}{p}_{n+1}$ in P_γ,

3. $\underset{\sim}{p}_{n+1}$ is $(\langle M_\xi \mid \alpha_n + 1 \leq \xi \leq \alpha_{n+1} \rangle, P_\gamma)$-generic (and so by item 2 it is $(\langle M_\xi \mid \xi \leq \alpha_{n+1} \rangle, P_\gamma)$-generic.

Now $\underset{\sim}{p}_{n+1} \restriction \gamma_{n+1}$ is forced by q_n to be in $M_{\alpha_{n+1}+1}$ and the inductive assumption for γ_{n+1} can be applied to yield a condition $q_{n+1} \in P_{\gamma_{n+1}}$ that is $(\langle M_\xi \mid \alpha_{n+1} < \xi \leq \alpha \rangle, P_{\gamma_{n+1}})$ generic and such that

$$q_{n+1} \Vdash_{\gamma_{n+1}} \underset{\sim}{p}_{n+1} \restriction \gamma_{n+1} \in \underset{\sim}{G}_{\gamma_{n+1}}.$$

Hence q_{n+1} is also $(\langle M_\xi \mid \xi \leq \alpha_{n+1} \rangle, P_{\gamma_{n+1}})$-generic. This ends the proof of the α-Extension Lemma, and hence of the α-properness preservation theorem. $\dashv$

5.2. A coloring problem

The definitions needed for Dee-completeness are quite complex and they can be better understood with an example. Hence, before presenting the general definition of Dee-completeness (in 5.3) we bring a particular case. The simplest that I know is a problem of Hajnal and Máté concerning the chromatic number of graphs in a certain family of graphs on ω_1 described below. (There is also a nostalgic reason for bringing this example: Theorem 5.8 is my first result in set theory). The *chromatic number* of any (non-directed) graph $g = (V, E)$ is the least cardinal κ such that there is a function $f : V \to \kappa$ from the set of vertices V into κ such that for every $\alpha \neq \beta$ in V, if $\alpha E \beta$ then $f(\alpha) \neq f(\beta)$.

Hajnal and Máté investigated in [5] the following family of graphs $g = (V, E)$ with set of vertices $V = \omega_1$, and in which for every limit $\delta \in \omega_1$ the set $\chi^g_\delta = \{\alpha \mid \alpha \in \delta \text{ and } \alpha \; E \; \delta\}$ forms an ω-sequence cofinal in δ (and for non limit $\beta \in \omega_1$ there is no $\alpha < \beta$ such that $\alpha E \beta$. We shall call such graphs Hajnal–Máté graphs. They had shown that if the diamond principle $\diamondsuit$ holds, then there is an Hajnal–Máté graph of chromatic number $\aleph_1$, and if MA+$2^{\aleph_0} > \aleph_1$ holds, then every Hajnal–Máté graph has countable chromatic number. They had suggested that Jensen's method [3] for proving the consistency of CH + "there are no Souslin trees" may lead to a consistency proof for CH + "the chromatic number of every Hajnal–Máté graph is $\aleph_0$". This turned out to be true.

5.8 Theorem. *Assume $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$. There is a generic extension that adds no new reals, collapses no cardinals, and such that every Hajnal–Máté graph in the extension has a countable chromatic number and $2^{\aleph_1} = \aleph_2$.*

Let $g = (V, E)$ be an Hajnal–Máté graph. Define P_g as the poset for making the chromatic number of g countable. That is, $h \in P_g$ iff for some countable ordinal γ, $h : \gamma + 1 \to \omega$ is such that whenever $\alpha < \beta \le \gamma$ and $\alpha \mathrel{E} \beta$ then $h(\alpha) \ne h(\beta)$. So the domain, $\mathrm{dom}(h)$, of a condition in P_g is always a countable successor ordinal.

The ordering on P_g is extension.

Clearly if $h \in P_g$ and $\gamma' < \gamma = \mathrm{dom}(h)$ is a successor ordinal then $h \restriction \gamma' \in P_g$. It is easy to check that any condition in P_g has extensions with arbitrarily high domain in ω_1.

If $h \in P_g$ and x is a finite function from $\mathrm{dom}(x) \subset \omega_1$ to ω, we say that x is compatible with h if $h \cup x$ is a function that assigns distinct values (in ω) to connected vertices.

5.9 Lemma. *If $h \in P_g$ and $\gamma + 1 = \mathrm{dom}(h)$, then for every countable μ above γ there is some condition $h' > h$ with $\mu + 1 = \mathrm{dom}(h')$. Moreover, for every finite x that is compatible with h there is an extension h' of $h \cup x$ in P_g.*

Thus if G is generic over P_g then $\bigcup G$ is defined on ω_1 and the chromatic number of g in the extension is countable.

We plan to prove that P_g is proper, α-proper for every countable α, and show how to iterate P_g forcings without adding new reals. Then $2^{\aleph_1} = \aleph_2$ implies that an iteration of length ω_2 suffices to ensure that all Hajnal–Máté graphs produced are taken care of, and the theorem can be concluded.

5.10 Lemma. *P_g is proper. Moreover, if $M \prec H_\lambda$ is countable, $P_g \in M$, and $h_0 \in P_g \cap M$, then for any finite x compatible with h_0 there is an (M, P_g)-generic condition h compatible with x. (x is not necessarily in M.)*

Proof. Given $h_0 \in P_g \cap M$ with $\mathrm{dom}(h_0) = \gamma_0 + 1$, list all dense sets $\langle D_i \mid i \in \omega \rangle$ that are in M. Let $\delta = M \cap \omega_1$ and let χ^g_δ be the ω-sequence of ordinals that are connected in g to δ. Our aim is to define an increasing sequence of conditions $h_n \in M$ so that $h_{n+1} \in D_n$ and then to define $h = \bigcup_{n \in \omega} h_n$ and to extend h on δ as well, in order to obtain an (M, P_g)-generic condition. The problem, of course, is that h may map χ^g_δ onto ω and leave no place for the value of δ, or that it assigns a value that is incompatible with x. The solution is based on the fact that δ "is" ω_1 for M.

We make an observation. Given a condition $f \in P_g$, and v, a finite function compatible with f, let $H_f(v)$ be the first (in some well-ordering) condition in P_g that extends $f \cup v$.

5.11 Claim. *Let $D \subseteq P_g$ be dense. For every $f \in P_g$ there is a closed unbounded set $C_f \subseteq \omega_1$ such that for every $\gamma \in C$, for every finite function v defined on a subset of γ and compatible with f, if $h = H_f(v)$, then h has an extension $h' \in D$ such that* $\mathrm{dom}(h') < \gamma$.

This claim is not difficult to prove.

Consider the given finite function x compatible with h_0. We may assume that $\delta \in \mathrm{dom}(x)$ (or else extend x). Let $x_0 = x \cap M$ and $x_1 = x \setminus M$ be the lower and upper parts of x. By further extending h_0 in M we may assume that $x_0 \subset h_0$. We can also assume that if $\alpha < \delta$ and α is connected in the graph to some point in the domain of x_1 above δ then $\alpha \in \mathrm{dom}(h_0)$ (as there are only finitely many such αs). So, in fact, in defining h_n's we must only be careful to avoid $k = x(\delta)$ on χ^g_δ. This k is "reserved" as the value for δ. Assume that $h_n \in M$ is defined and $k \notin h_n``\chi^g_\delta$. For $f = h_n$ and $D = D_n$ there is an unbounded set $C_f \subseteq \omega_1$ as in the claim above. As C_f is definable, $C_f \in M$. Let γ be the first member of C_f above $\mathrm{dom}(h_n)$. Let $u = \chi^g_\delta \cap \gamma \setminus \mathrm{dom}(h_n)$. Then u is finite since $\gamma < \delta$. Let v be a function defined on u, compatible with h_n, and avoiding the value k. Then $h = H_f(v)$ has an extension $h' = h_{n+1} \in D_n$ that lives in γ, so that $\{\langle \delta, k\rangle\}$ is still compatible with h_{n+1}. Finally $h = \bigcup_{n\in\omega} h_n \cup \{\langle\delta, k\rangle\}$ is (M, P_g)-generic. This ends the properness proof for P_g. $\dashv$

The generic condition h thus obtained is a "completely generic" condition, which means that it actually defines an (M, P_g)-generic filter. This shows that P_g adds no new reals.

5.12 Definition. If $P \in M$ is a poset, then $q \in P$ is *completely* $(M, P)-$*generic* iff $\{p \in P \cap M \mid p \leq q\}$ is an (M, P)-generic filter.

We say that G is bounded by q and also that q induces G.

We say that a poset P is "completely proper" iff P is proper and the properness definition applies to P with "completely generic condition" replacing "generic condition". Clearly completely proper posets do not add new reals. In fact, P is completely proper iff P is proper and adds no new reals. (If the latter condition holds and M is as in the definition of properness, find q that is (M, P)-generic and then further extend it to some condition that determines $\underset{\sim}{G} \cap M$.)

Thus we know that P_g adds no new reals, and now we prove by induction on $\alpha < \omega_1$ that

5.13 Lemma. *P_g is α-proper. In fact, if $\langle M_\xi \mid \xi \leq \alpha\rangle$ is any α tower of countable elementary submodels of H_λ with $P_g \in M_0$ and $h_0 \in M_0$ is any condition, then for every finite x compatible with h_0 there is an extension $h \in P_g$ that is completely (M_ξ, P_g)-generic for every $\xi \leq \alpha$ and is compatible with x.*

Proof. The proof of the lemma is by induction on $\omega \leq \alpha < \omega_1$. Using the properness lemma 5.10 we assume that α is a limit ordinal, and pick an increasing and cofinal sequence α_n. We define an increasing sequence h_n of conditions compatible with x such that $h_{n+1} \in M_{\alpha_{n+1}+1}$ is $\langle M_\xi \mid \xi \leq \alpha_{n+1}$. Then $\bigcup_{n\in\omega} h_n \cup \{\langle\delta, x(\delta)\rangle\}$ is as required. $\dashv$

Now we make a crucial observation which is the key to the proof that the iteration of P_g posets adds no new reals. Let $M \prec H_\lambda$ be a countable elementary substructure with $P_g \in M$, and let $p_0 \in P_g \cap M$ be given. Recall that $G^* \subset P_g \cap M$ is an (M, P_g)-generic filter if it is a filter over $P_g \cap M$ that intersects every dense set in M. Define

$$\mathrm{Gen}_{p_0}(M, P_g)$$

to be the set of all (M, P_g)-generic filters containing p_0 (we write $\mathrm{Gen}(M, P_g)$ for the collection of all M generic filters). We say that $G^* \in \mathrm{Gen}(M, P_g)$ is *extendible* to a condition in P_g iff for some $q \in P_g$, $G^* = \{p \in P_g \cap M \mid p < q\}$. We say in this case that q bounds G^*. Clearly G^* is extendible to a condition in P_g iff its range of values on χ^g_δ (where $\delta = \omega_1 \cap M$) is not all of ω. For any ω-sequence x cofinal in δ, we shall say that G^* is *appropriate* for x iff $\bigcup G^* \restriction x$ omits at least one value. So G^* is extendible to a condition in P_g iff G^* is appropriate for χ^g_δ.

For an ω-sequence x cofinal in $\delta = \omega_1 \cap M$ define

$$A^{p_0}_x = \{G^* \in \mathrm{Gen}_{p_0}(M, P_g) \mid G^* \text{ is appropriate for } x\}.$$

Thus $A^{p_0}_{\chi^g_\delta}$ is the collection of all $G^* \in \mathrm{Gen}_{p_0}(M, P_g)$ extendible to a condition in P_g.

We claim that (for a fixed p_0) the collection

$$\{A^{p_0}_x \mid x \subset \delta \ \text{ is a cofinal } \omega \text{ sequence}\}$$

has the countable intersection property. That is, if $X = \{x_i \mid i \in \omega\}$ is some countable collection of ω-sequences cofinal in $\delta = \omega_1 \cap M$, then there is some G^* in $\bigcap_{i \in \omega} A^{p_0}_{x_i}$. To prove this fact, find some ω-sequence x converging to δ and such that $\mathrm{range}(x_i) \setminus \mathrm{range}(x)$ is finite for every i. It is easy to define such x that almost contains each x_i by induction. If G^* is some (M, P_g)-generic filter containing p_0 that omits infinitely many values on x (and now we can easily define such a filter), then $G^* \in A^{p_0}_{x_i}$ for every i.

We describe in general terms what is involved in proving that the iteration adds no new reals. In proving that the iteration of P_g-like forcings does not add any new real we will be asked to produce a completely generic condition for some $M \prec H_\lambda$ without knowing the value of χ^g_δ. This ω-sequence of ordinals connected to δ will only be given as a name. So instead of χ_δ we will be offered a countable collection $\{x_i \mid i \in \omega\}$ of ω-sequences with the assurance that χ^g_δ is forced to be among them. We will still be able to find G^* by taking into consideration all the x_i's as was shown above. The essence of this argument is embodied in the following lemma.

5.14 Lemma. *Suppose that $M_0 \prec M_1 \prec H_\lambda$ are countable elementary substructures with $M_0 \in M_1$. Suppose that $P \in M_0$ is a poset that adds no new reals, and that $g \in V^P \cap M_0$ is a name for some Hajnal–Máté graph. Let $G_0 \in M_1$ be some (M_0, P)-generic filter. Then there exists an $(M_0, P * P_g)$-generic filter G_1 extending G_0, and there exists a name $\underset{\sim}{r} \in V^P$ such that the following holds.*

> *If $q \in P$ is a (plain) (M_1, P)-generic condition that bounds G_0 (and is thence completely generic over M_0), then the condition $(q, \underset{\sim}{r}) \in P * P_g$ bounds G_1.*

In fact, both the filter G_1 and the name $\underset{\sim}{r}$ are definable from parameters in M_1 and a countable enumeration of M_1. Thus, if $M_1 \in M_2 \prec H_\lambda$ then $G_1, \underset{\sim}{r} \in M_2$.

Proof. Let $\mu : M_0 \to N_0$ be the Mostowski collapsing function of M_0 onto a transitive structure N_0. Then $N_0 \in M_1$. Let $\mathcal{G}_0 = \mu``G_0 \subset \mu(P)$ be the image of G_0 under the collapsing map. Forming $N_0[\mathcal{G}_0]$ as a generic extension, $\mu(g)[\mathcal{G}_0]$ is a Hajnal–Máté graph there denoted h. This graph is clearly on $\delta = \omega_1 \cap M_0$. As $\mathcal{G}_0 \in M_1$, $N_0[\mathcal{G}_0] \in M_1$.

Let $X = \{x_i \mid i \in \omega\}$ be an enumeration (the least in some global well order) of all ω-sequences x in M_1 that are cofinal in δ. We know how to find an $(N_0[\mathcal{G}_0], P_h)$-generic filter H that is appropriate for every x_i. Now form $\mathcal{G}_1 = \mathcal{G}_0 * H$. Then $\mathcal{G}_1$ is an N_0 generic filter extending $\mathcal{G}_0$. The required filter G_1 is the μ pre-image of $\mathcal{G}_1$.

The name $\underset{\sim}{r} \in V^P$ is defined by the following requirement as a condition in P_g with domain $\delta + 1$. It is easier to describe the interpretation of $\underset{\sim}{r}$ in (V, P) generic extensions $V[G]$. If $\bigcup H$ (which is a function on δ that lies in V) can be extended (by assigning a value to δ) to a condition in $P_{g[G]}$, then let $\underset{\sim}{r}[G]$ be that condition.

Assume now that $q \in P$ is as in the lemma, a bound of $G_0 = \mu^{-1}``\mathcal{G}_0$ that is also an (M_1, P)-generic condition. Since P adds no new countable sequences, there exists in M_1 a dense set of conditions in P that determine the value of χ^g_δ in V. Thus $q \Vdash \chi^g_\delta \in M_1$, and hence $q \Vdash \exists i \in \omega\ (\chi^g_\delta = x_i)$. Hence $q \Vdash H$ is bounded by $\underset{\sim}{r}$. ⊣

Observe that $\underset{\sim}{r}$ is a name of a function defined on $\delta + 1$. Although $\underset{\sim}{r} \restriction \delta$ is, in a sense, $\bigcup H$, $\underset{\sim}{r}(\delta)$ is just a name and any specific value for $\underset{\sim}{r}(\delta)$ may conflict with some x_i. Only after determining (generically) χ^g_δ can we assign a compatible value to $\underset{\sim}{r}(\delta)$.

In applications we need a slightly stronger version of this lemma, in which a condition $(p_0, p_1) \in P * P_g \cap M_0$ is also given, and $p_0 < q$ is assumed. Then $\underset{\sim}{r}$ is also required to satisfy $q \Vdash_P \underset{\sim}{r} > p_1$, so that the given condition (p_0, q_0) is in the (M_0, P)-generic filter determined by $(q, \underset{\sim}{r})$.

The final model of CH + "*every Hajnal–Máté graph has countable chromatic number*" is obtained through an iteration with countable support of posets of the form P_g, done over a ground model in which CH holds. Since each P_g has size $\aleph_1$ the iteration satisfies the $\aleph_2$-c.c., and a suitable bookkeeping device ensures that every possible Hajnal–Máté graph is dealt with.

As a preparation for the final iteration we prove here that P_ω (the iteration of the first $\aleph_0$ posets) adds no new reals. So let P_n be defined by

$P_{n+1} \equiv P_n * P_{g_n}$ where P_{g_n} is a name for some Hajnal–Máté graph. Specifically, the members of P_n are functions defined on n and P_ω is the countable support limit.

Let $M_0 \prec H_\lambda$ be some countable elementary substructure with $P_\omega \in M_0$, and $p_0 \in P_\omega \cap M_0$ a given condition. It suffices to find an extension of p_0 that is completely generic over M_0. Let $\langle D_n \mid n \in \omega \rangle$ be an enumeration of all dense subsets of P_ω that are in M_0.

Starting with M_0 build a tower $\langle M_n \mid n \in \omega \rangle$ of countable elementary substructures of H_λ. We plan to define a sequence of conditions $q_n \in P_n$ and $p_n \in P_\omega \cap M_0$ such that

1. $q_n \in P_n$ is completely generic over (M_0, P_n), and it is (M_k, P_n)-generic for every $k \geq n$;
2. $p_n \restriction n \leq q_n$, and $q_n = q_{n+1} \restriction n$.
3. $p_n \leq p_{n+1}$ in P_ω, and $p_{n+1} \in D_n \cap M_0$.

When the construction is done, define $q = \bigcup_n q_n$. Then $q \geq p_k \restriction n$ for every n and hence $q \geq p_k$ for every k. This shows that $q \in P_\omega$ is completely generic over M_0, because the p_n's visit every dense set in M_0.

Suppose that p_n and q_n are defined. The assumption that q_n is completely generic means that

$$G_0 = \{p \in P_n \cap M_0 \mid p < q_n\}$$

is (M_0, P_n)-generic. Use Lemma 1.2 to find $p_{n+1} \geq p_n$ with $p_{n+1} \in D_n \cap M_0$ and such that $p_{n+1} \restriction n \leq q_n$.

Notice that $G_0 \in M_n$ because the set D of conditions $q \in P_n$ that are completely generic over M_0 is pre-dense above q_n. Since $D \in M_n$, q_n is compatible with a member q of $D \cap D_n$ and, in the definition of G_0, q can replace q_n.

The previous lemma is applicable to $M_0 \in M_n \in M_{n+1}$ and to $P = P_n$. So there is a name $\underset{\sim}{r} \in M_{n+1}$ such that $(q_n, \underset{\sim}{r}) \in P_{n+1}$ is completely generic over M_0 and $p_{n+1} \restriction n+1 \leq (q_n, \underset{\sim}{r})$. As P_{g_n} is (forced to be) ω-proper, there is a name $r' \in V^{P_n}$ such that

$$q_n \Vdash \underset{\sim}{r} < r' \ \&\ r' \text{ is } (\langle M_i[\underset{\sim}{G}_{P_n}] \mid i \geq n+1 \rangle, P_{g_n})\text{-generic}$$

Define $q_{n+1} = q_n^\frown \langle r' \rangle$. Then $q_{n+1} \in P_{n+1}$ is tower generic for $\langle M_k \mid k \geq n+1 \rangle$, $q_n = q_{n+1} \restriction n$, and $q_{n+1} \geq p_{n+1} \restriction n+1$.

5.3. Dee-completeness

The aim of Dee-completeness is to provide a framework for obtaining models of CH. It allows countable support iteration of a large family of posets that add no new countable sets. Our definitions here are slightly different from those originally given by Shelah [15], Chapter V, but we have kept the original names believing that our interpretation of the basic ideas is accurate.

A *completeness system* is a three argument function $\mathbb{D}(N, P, p_0)$ defined when

- N is a countable transitive model of ZFC$^-$ (ZFC minus the power-set axiom).
- $P \in N$ is a forcing poset in N, and
- $p_0 \in P$.

Then $\mathbb{D}(N, P, p_0)$ is a non-empty collection of non-empty subsets of $\mathrm{Gen}_{p_0}(N, P)$. That is, if $A \in \mathbb{D}(N, P, p_0)$ then every $G \in A$ is a filter over P containing p_0 and intersecting every dense subset of P in N.

For example, if $g \in N$ is some Hajnal–Mátè graph and $P = P_g$ is in N the poset for coloring g with countably many colors, then we define $\mathbb{D}(N, P, p_0) = \{A_X \mid X \subset N\}$ where $A_X \subseteq \mathrm{Gen}_{p_0}(N, P)$ is defined as follows. In case X is an ω-sequence cofinal in ω_1^N, then $G \in A_X$ iff $G \in \mathrm{Gen}_{p_0}(N, P)$ is such that $\bigcup G \restriction X$ omits infinitely many colors. If X is not as above, then $A_X = \mathrm{Gen}_{p_0}(N, P)$. It is reasonable to have g as a parameter, although in our case it is reconstructible from P.

We apply $\mathbb{D}$ to non-transitive structures as well: if $M \prec H_\lambda$ is a countable elementary submodel, $P \in M$ a poset, and $p_0 \in P \cap M$, then we let $\pi : M \to N$ be the transitive collapsing isomorphism and for each $X \in \mathbb{D}(N, \pi(P), \pi(p_0))$ we define $\pi^{-1}(X) = \{\pi^{-1}``G \mid G \in X\}$. This yields $\mathbb{D}(M, P, p_0) = \{\pi^{-1}(X) \mid X \in \mathbb{D}(N, \pi(P), \pi(p_0))\}$. In simple terms, $\mathbb{D}(M, P, p_0)$ is defined by viewing the argument (M, P, p_0) as a representation of its isomorphism type.

We say that a poset P is *Dee-complete* (or just complete, for brevity) with respect to a completeness system $\mathbb{D}$ if for every sufficiently large λ, for every countable $M \prec H_\lambda$ with $P \in M$ and for every $p_0 \in P \cap M$, there is $X \in \mathbb{D}(M, P, p_0)$ such that every $G \in X$ is bounded in P. (Thus P is proper.)

Repeating this definition, now with transitive structures, we obtain that P is Dee-complete with respect to $\mathbb{D}$ if the following holds for every countable $M \prec H_\lambda$ with $P \in M$:

for every $p_0 \in P \cap M$, if $\pi : M \to N$ is the transitive collapse, there is $X \in \mathbb{D}(N, \pi(P), \pi(p_0))$ such that (I.19)

$$\text{for every } \mathcal{G} \in X,\ \pi^{-1}``\mathcal{G} \text{ is bounded in } P.$$

We say that a completeness system is *countably complete* iff whenever $A_i \in \mathbb{D}(N, P, p)$ for $i \in \omega$ then $\bigcap_{i \in \omega} A_i \neq \emptyset$. We have seen that for every

Hajnal–Máté graph g the system defined above is countably complete. Thus every P_g is Dee-complete with respect to a countably complete system.

It is sometimes convenient to add a parameter to $\mathbb{D}$. We shall say that $\mathbb{D}$ is a completeness system with a parameter if $\mathbb{D}$ is a four argument function: $\mathbb{D}(N, P, p_0, c)$ is defined when N, P, p_0 are as in the definition given above, and $c \in N$ is the parameter. As before, $\mathbb{D}(N, P, p_0, c)$ is a non-empty collection of non-empty subsets of $\mathrm{Gen}_{p_0}(N, P)$. We say that a poset P is Dee-complete with respect to a completeness system $\mathbb{D}$ with parameter iff for every sufficiently large λ there is $c \in H_\lambda$ such that:

> for every countable $M \prec H_\lambda$ with $P, c \in M$, for every $p_0 \in P \cap M$, if $\pi : M \to N$ is the transitive collapse, then there is $X \in \mathbb{D}(N, \pi(P), \pi(p_0), \pi(c))$ such that, for every $\mathcal{G} \in X$, $\pi^{-1}``\mathcal{G}$ is bounded in P.

(I.20)

In fact, the parameters are dispensable by the following lemma. Recall that by H_λ we mean the structure $(H_\lambda, \in, <)$ where $<$ is a fixed well-ordering of H_λ.

5.15 Lemma. *Let P be a poset that is Dee-complete with respect to some countably complete completeness system $\mathbb{D}$ with parameter. Then P is also Dee-complete with respect to some three-argument countably complete completeness system $\mathbb{D}'$.*

Proof. Let λ be sufficiently large so that H_λ with parameter $c \in H_\lambda$ shows the completeness of P with respect to $\mathbb{D}$ (as in (I.20)). Then, for $\lambda' > \lambda^{<\lambda} + 2^{2^{\mathfrak{c}}}$, $H_\lambda \in H_{\lambda'}$ and $\mathbb{D} \in H_{\lambda'}$ (since $\mathbb{D}$ is a function from $H_{\aleph_1}$ to $\mathcal{P}\mathcal{P}(H_{\aleph_1})$). So $H_{\lambda'}$ satisfies the statement $\psi(P, \lambda, c, \mathbb{D})$ saying that λ is a cardinal with $P, c \in H_\lambda$ and $\mathbb{D}$ is a four-argument system such that (I.20) holds.

Using the assumed well-ordering of $H_{\lambda'}$, let λ_0, c_0, and $\mathbb{D}_0$ be minimal such objects for which $\psi(P, \lambda_0, c_0, \mathbb{D}_0)$ holds. If $M \prec H_{\lambda'}$ is any countable elementary substructure with $P \in M$, then $\lambda_0, c_0, \mathbb{D}_0 \in M$ since they are definable in $H_{\lambda'}$, and moreover, these objects are minimal in M to satisfy $\psi(P, \lambda_0, c_0, \mathbb{D}_0)$. Observe that $H^M_{\lambda_0} = M \cap H_{\lambda_0} \prec H_{\lambda_0}$, and also that if $D \in M$ is a subset of P then $D \in H^M_{\lambda_0}$.

Let $\pi : M \to N$ be the collapse onto a transitive structure. Then $\pi_0 = \pi \restriction H_{\lambda_0}$ is the collapse of a countable elementary substructure of H_{λ_0}, namely $H^M_{\lambda_0}$ onto $H^N_{\pi(\lambda_0)} = N_0$.

So $\mathbb{D}_0(N_0, \pi(P), \pi(p_0), \pi(c_0))$ has the required good properties, and in particular each $G \in X \in \mathbb{D}_0(N_0, \pi(P), \pi(p_0), \pi(c_0))$ is generic not only over N_0 but also over N. This leads to the following definition of $\mathbb{D}'$ as required by the lemma.

If N is any countable transitive structure, $R \in N$ a poset and $r \in R$, define $\mathbb{D}'(N, R, r)$ as follows. Look for λ'_0, c'_0, that are minimal to satisfy

$\exists D\psi(R, \lambda_0', c_0', D)$ in N, and apply $\mathbb{D}_0(H^N_{\lambda_0'}, R, r, c_0')$. If there is no such λ_0', then $\mathbb{D}'(N, R, r)$ is arbitrarily defined. ⊣

The definition of Dee-completeness has the form "for every λ sufficiently large etc." It is not difficult to see that if there is a completeness system that works for one λ, there is one that works for every higher λ as well. We are going to argue now that it is always possible in this case to take $\lambda = (2^{\|P\|})^+$ (Shelah, personal communication).

5.16 Lemma. *If poset P is complete with respect to a countably complete completeness system $\mathbb{D}$ then there is a countably complete (four-argument) completeness system $\mathbb{D}'$ so that already $\lambda_0 = (2^{\|P\|})^+$ suffices to demonstrate the completeness of P. That is, for some $c \in H_{\lambda_0}$, (I.20) holds.*

Proof. The definition of $\mathbb{D}'$ is simple. For any countable transitive structure N, poset $R \in N$, condition $r \in R$, and parameter $p \in N$, define

$$\mathbb{D}'(N, R, r, p) = \mathbb{D}(p, R, r)$$

if this makes sense, that is, if p is transitive, $R, r \in p$ and $\mathbb{D}(p, R, r)$ is indeed a collection of sets of N-generic filters as required. In case this definition doesn't make sense, let $\mathbb{D}'(N, R, r, p)$ be defined as an arbitrary collection of subsets of $\mathrm{Gen}_r(N, R)$ with the countable intersection property. We must define a good parameter $c \in H_{\lambda_0}$ that will work.

Let λ be sufficiently large so that for every countable $M \prec H_\lambda$ with $P \in H_\lambda$ (I.19) holds. Let $\kappa = \|P\|$ be the cardinality of P, and assume for simplicity that κ is the universe of P. We may assume that $\lambda > \lambda_0 = (2^\kappa)^+$ (or else there is nothing to show). Let K be an elementary substructure of H_λ of cardinality 2^κ (containing all subsets of κ). Clearly, every elementary substructure of K is also an elementary substructure of H_λ, so that if we let $\pi : K \to \bar{K}$ be the transitive collapse of K then, for every $M \prec \bar{K}$, the transitive collapse of M is the transitive collapse of an elementary substructure of K, namely the pre-image of M.

We claim that $c = \bar{K} \in H_{\lambda_0}$ works. Let $M \prec H_{\lambda_0}$ be countable with $P, \bar{K} \in M$, and $p_0 \in P \cap M$ be given. Then $M \cap \bar{K} \prec \bar{K}$. Let $\pi : M \to N$ be the collapsing function onto a transitive structure. $\pi(\bar{K})$ is the transitive collapse of $M \cap \bar{K}$, and $\mathbb{D}'(N, \pi(P), \pi(p_0), \pi(\bar{K})) = \mathbb{D}(\pi(\bar{K}), \pi(P), \pi(p_0))$. The point is that if $G \in X \in \mathbb{D}(\pi(\bar{K}), \pi(P), \pi(p_0))$, then G is not only $(\pi(P), \pi(\bar{K}))$-generic filter but also N-generic, since any subset of $\pi(\kappa)$ in N is already in $\pi(\bar{K})$ (as any subset of κ in M is already in $M \cap \bar{K}$). ⊣

Our aim is to prove the following

5.17 Theorem (Dee Completeness Iteration). *The iteration, of any length γ, with countable support of $(< \omega_1)$-proper posets that are Dee-complete with respect to countably complete systems* that lie in the ground model V *does not add any new reals.*

Note the inductive character of this theorem. For $Q_i \in V^{P_i}$ to be Dee-complete with respect to a system that lies in V, one needs that P_i adds no new countable sets–so that every countable transitive set in V^{P_i} is in V.

To prove the theorem we shall first define for every countable $M \prec H_\lambda$ (with $P_\gamma \in M$) an (M, P_γ)-generic filter G_γ. Then we will prove that G_γ is bounded in P_γ. That is, we will find a condition in P_γ that is completely generic for M. The definition of G_γ is by induction, and we shall actually have to define for every $\gamma_0 < \gamma$ and G_{γ_0} that is (M, P_{γ_0})-generic a filter G_γ that extends G_{γ_0}. There will be two main cases in this definition: γ successor and γ limit, and likewise there will be two cases in the proof that G_γ is bounded. We start with what is needed for the successor case.

Two step iteration

Let P be a poset and $\underset{\sim}{Q} \in V^P$ a name forced (by 0_P) to be a poset. Let λ be sufficiently large and $M_0 \prec H_\lambda$ be a countable elementary submodel such that $P, \underset{\sim}{Q} \in M_0$. We want to find a criterion for when a condition $(q_0, q_1) \in P * \underset{\sim}{Q}$ is completely $(M_0, P * \underset{\sim}{Q})$-generic. A first guess is: q_0 is completely generic for (M_0, P) and q_0 forces that q_1 is completely generic for $(M_0[\underset{\sim}{G}], \underset{\sim}{Q})$. But a moment's reflection reveals that this is not sufficient for (q_0, q_1) to determine, in V, an $(M_0, P * \underset{\sim}{Q})$ generic filter. So we need a finer criterion.

Let $\pi : M_0 \to N_0$ be the transitive collapsing map. Suppose that $q_0 \in P$ is completely generic over (M_0, P) and let $G_P \subseteq P \cap M_0$ be the (M_0, P) generic filter induced by q_0. Then $\mathcal{G}_0 = \pi``G_P$ is an $(N_0, \pi(P))$-generic filter and we can form the (transitive) extension $N_0^* = N_0[\mathcal{G}_0]$. In N_0, $\pi(\underset{\sim}{Q})$ is a name, and its interpretation $Q_0^* = \pi(\underset{\sim}{Q})[\mathcal{G}_0]$ is a poset in N_0^*.

Let $\underset{\sim}{G} \in V^P$ be the canonical name of the generic filter over P. If F is any (V, P) generic filter containing q_0, then $M_0[F] \prec H_\lambda[F]$ can be formed and the collapsing map π on M_0 can be extended to collapse $M_0[F]$ onto N_0^*. Let $\underset{\sim}{\pi}$ be the name of this extended collapse. Then $q_0 \Vdash_P \underset{\sim}{\pi} : M_0[\underset{\sim}{G}] \to N_0^*$. We phrase now the desired criterion but omit the routine proof.

5.18 Lemma. *Using the above notations,* (q_0, q_1) *is completely generic over* $(M_0, P * \underset{\sim}{Q})$ *iff*

1. q_0 *is* (M_0, P) *completely generic and*

2. *for some* $\mathcal{G}_1 \subset Q_0^*$ *that is* (N_0^*, Q_0^*)*-generic,* $q_0 \Vdash \underset{\sim}{\pi}^{-1}``\mathcal{G}_1$ *is bounded by* q_1.

In this case, the filter induced by (q_0, q_1) *over* $M_0 \cap P * \underset{\sim}{Q}$ *is* $\pi^{-1}``\,\mathcal{G}_0 * \mathcal{G}_1$.

Given a countable $M_0 \prec H_\lambda$ such that the two step iteration $P * \underset{\sim}{Q}$ is in M_0, our aim (under some assumptions stated in the following definition) is to extend each (M_0, P)-generic filter G_0 to an $(M_0, P * \underset{\sim}{Q})$-generic filter. This definition depends not only on M_0, but also on another countable elementary submodel $M_1 \prec H_\lambda$ such that $M_0 \in M_1$. In addition, we assume some $p_0 \in P * \underset{\sim}{Q}$ which we want to include in the extended filter. All of this leads to a five place function $\mathbb{E}(M_0, M_1, P * \underset{\sim}{Q}, G_0, p_0)$ that we define now.

5.19 Definition. Let P be a poset that adds no new countable sets of ordinals, and suppose that $\underset{\sim}{Q}, \underset{\sim}{\mathbb{D}} \in V^P$ are such that

$$\Vdash_P \quad \begin{array}{l} \underset{\sim}{\mathbb{D}} \in V \text{ is a countably complete system} \\ \text{and } \underset{\sim}{Q} \text{ is Dee-complete with respect to } \underset{\sim}{\mathbb{D}}. \end{array}$$

Let λ be sufficiently large, and $M_0 \prec M_1 \prec H_\lambda$ be countable elementary submodels with $M_0 \in M_1$ and $P, \underset{\sim}{Q}, \underset{\sim}{\mathbb{D}} \in M_0$. Let $G_0 \subset M_0 \cap P$ be (M_0, P)-generic and suppose that $G_0 \in M_1$. Let $p_0 \in (P * \underset{\sim}{Q}) \cap M_0$ be given, $p_0 = (a, \underset{\sim}{b})$ with $a \in G_0$. Then we define

$$G = \mathbb{E}(M_0, M_1, P * \underset{\sim}{Q}, G_0, p_0) \tag{I.21}$$

an $(M_0, P * \underset{\sim}{Q})$-generic filter containing p_0, by the following procedure.

Let $\pi : M_0 \to N_0$ be the transitive collapse, and $\mathcal{G}_0 = \pi``G_0$. Form $N_0^* = N_0[\mathcal{G}_0]$. Observe that $N_0^* \in M_1$. Let $Q_0^* = \pi(\underset{\sim}{Q})[\mathcal{G}_0]$, and $\mathbb{D}_0 = \pi(\underset{\sim}{\mathbb{D}})[\mathcal{G}_0]$. Then $\mathbb{D}_0 \in N_0$ because it is forced to be in the ground model. So $\mathbb{D}_0 = \pi(\mathbb{D})$ where $\mathbb{D} \in M_0$ is a countably complete completeness system. Thus $\mathbb{D}(N_0^*, Q_0^*, b^*)$ is defined in M_1 where $b^* = \pi(\underset{\sim}{b})[\mathcal{G}_0]$ is a condition in Q_0^*. Since $M_1 \cap \mathbb{D}(N_0^*, Q_0^*, b^*)$ is countable, $\exists \mathcal{G}_1 \in \bigcap(M_1 \cap \mathbb{D}(N_0^*, Q_0^*, b^*))$. $\mathcal{G}_1$ is (N_0^*, Q_0^*)-generic and $b^* \in \mathcal{G}_1$.

Form $\mathcal{G}_0 * \mathcal{G}_1 = \mathcal{G}$, an $(N_0, \pi(P * \underset{\sim}{Q}))$-generic filter. Then $\pi(p_0) \in \mathcal{G}$. Finally define

$$\mathbb{E}(M_0, M_1, P * \underset{\sim}{Q}, G_0, p_0) = \pi^{-1}``\mathcal{G}.$$

End of Definition 5.19.

In fact, we want to define a formula ψ so that

$$H_\lambda \models \psi(G, M_0, M_1, P * \underset{\sim}{Q}, G_0, p_0)$$

iff (I.21) holds. That is, we want to define $\mathbb{E}$ in H_λ. We can't take the above definition verbally because it relies on the assumption that M_0 and M_1 are elementary substructures of H_λ, something which is not expressible in H_λ itself. So we redo that definition for any countable subsets M_0 and M_1 of H_λ (or models of ZF$^-$). Whenever Definition 5.19 above relies on some fact that happens not to hold, we let G have an arbitrary value. For example, if N_0^* is not in M_1 or if $M_1 \cap \mathbb{D}(N_0^*, Q_0^*, b^*)$ is empty, then we let G be some arbitrary fixed M_0 generic filter.

The following is a main lemma which shows the crux of the argument (compare with Lemma 5.14). It analyzes the iteration of two posets when the second is Dee-complete.

5.20 Lemma (The Gambit Lemma). *Let P be a poset and suppose that $\underset{\sim}{Q}, \underset{\sim}{\mathbb{D}} \in V^P$ are such that*

$$\Vdash_P \quad \underset{\sim}{\mathbb{D}} \in V \text{ is a countably complete system} \\ \text{and } \underset{\sim}{Q} \text{ is Dee-complete with respect to } \underset{\sim}{\mathbb{D}}.$$

Let λ be sufficiently large, and $M_0 \prec M_1 \prec H_\lambda$ be countable elementary submodel with $M_0 \in M_1$ and $P, \underset{\sim}{Q}, \underset{\sim}{\mathbb{D}} \in M_0$. Suppose that $q_0 \in P$ is (M_1, P)-generic as well as (M_0, P) completely generic, and let $G_0 \subset M_0 \cap P$ be the M_0 filter over $M_0 \cap P$ induced by q_0. Let $p_0 \in P * \underset{\sim}{Q}$, $p_0 \in M_0$ be given so that $p_0 = (a, \underset{\sim}{b})$ and $a \in G_0$. Then there is $q_1 \in V^P$ so that (q_0, q_1) is completely generic over $(M_0, P * \underset{\sim}{Q})$ and $p_0 < (q_0, q_1)$. In fact (q_0, q_1) bounds $G = \mathbb{E}(M_0, M_1, P * \underset{\sim}{Q}, G_0, p_0)$.

Proof. Notice that $G_0 \in M_1$ by the following argument. Let R be the collection of all conditions $r \in P$ that are completely generic over (M_0, P). Then $R \in M_1$ and $q_0 \in R$. Since q_0 is (M_1, P)-generic it follows that it is compatible with some $r \in R \cap M_1$. But any two compatible conditions in R induce the same filter, and hence G_0 is the filter induced by r.

As in Definition 5.19, let $\pi : M_0 \to N_0$ be the transitive collapse, and $\mathcal{G}_0 = \pi``G_0$. We recall the definition of $\mathbb{E}(M_0, M_1, P * \underset{\sim}{Q}, G_0, p_0)$. Form $N_0^* = N_0[\mathcal{G}_0]$. Let $Q_0^* = \pi(\underset{\sim}{Q})[\mathcal{G}_0]$, and $\mathbb{D}_0 = \pi(\underset{\sim}{\mathbb{D}})[\mathcal{G}_0]$. $\mathbb{D}_0 \in N_0$ and $\mathbb{D}_0 = \pi(\mathbb{D})$ where $\mathbb{D} \in M_0$ is a countably complete completeness system. Thus $\mathbb{D}(N_0^*, Q_0^*, b^*)$ is defined in M_1 where $b^* = \pi(\underset{\sim}{b})[\mathcal{G}_0]$ is a condition in Q_0^*. Since $M_1 \cap \mathbb{D}(N_0^*, Q_0^*, b^*)$ is countable and non-empty, we were able to pick $\mathcal{G}_1 \in \bigcap(M_1 \cap \mathbb{D}(N_0^*, Q_0^*, b^*))$, (N_0^*, Q_0^*)-generic with $b^* \in \mathcal{G}_1$. We defined $\mathcal{G} = \mathcal{G}_0 * \mathcal{G}_1$, and defined $G = \mathbb{E}(M_0, M_1, P * \underset{\sim}{Q}, G_0, p_0)$ as $\pi^{-1}``\mathcal{G}$.

Let $\underset{\sim}{G} \in V^P$ be the canonical name of the generic filter over P. Then q_0 forces that π can be extended to a collapse $\underset{\sim}{\pi}$ which is onto N_0^*: that is,

$$q_0 \Vdash_P \ \underset{\sim}{\pi} : M_0[\underset{\sim}{G}] \to N_0^*.$$

The conclusion of our lemma follows if we show that

$$q_0 \Vdash_P \underset{\sim}{\pi}^{-1}``\, \mathcal{G}_1 \text{ is bounded in } \underset{\sim}{Q}. \tag{I.22}$$

In this case, if we define $q_1 \in V^P$ so that $q_0 \Vdash_P \ q_1$ bounds $\underset{\sim}{\pi}^{-1}``\mathcal{G}_1$, then the previous lemma (5.18) implies that the $(M_0, P * \underset{\sim}{Q})$-generic filter induced by $(q_0, q_1)$ is $\pi^{-1}``\mathcal{G}_0 * \mathcal{G}_1$.

So let F be (V, P)-generic with $q_0 \in F$. $\underset{\sim}{\pi}[F]$ collapses $M_0[F]$ onto N_0^*, and there is a set $X \in \mathbb{D}(N_0^*, Q_0^*, b^*)$ so that if $\mathcal{H} \in X$ is any filter then $\underset{\sim}{\pi}^{-1}$"$\mathcal{H}$ is bounded in $\underset{\sim}{Q}[F]$. As $M_1[F] \prec H_\lambda[F]$, we can have $X \in M_1[F]$. But since $\mathbb{D}$ is in the ground model, $X \in M_1$. Thus $\mathcal{G}_1 \in X$, where $\mathcal{G}_1$ is the filter defined above. This proves (I.22).

$\dashv$

Proof of Theorem 5.17

Let P_γ be a countable support iteration of length γ, obtained by iterating $Q_i \in V^{P_i}$ as in the theorem. That is, each Q_i is Dee-complete in V^{P_i} for some countably complete system taken from V. Let λ be a sufficiently large cardinal. To prove the theorem we first describe a machinery for obtaining generic filters over countable submodels of H_λ. We define a function $\mathbb{E}$ that takes five arguments $\mathbb{E}(M_0, \bar{M}, P_\gamma, G_0, p_0)$, of the following types.

1. $M_0 \prec H_\lambda$ is countable, $P_\gamma \in M_0$ (so $\gamma \in M_0$), and $p_0 \in P_\gamma \cap M_0$.
2. For some $\gamma_0 \in M_0 \cap \gamma$, G_0 is an (M_0, P_{γ_0})-generic filter, and such that $p_0 \restriction \gamma_0 \in G_0$. We assume that $G_0 \in M_1$.
3. The order type of $M_0 \cap [\gamma_0, \gamma)$ is α.
4. $\bar{M} = \langle M_\xi \mid 1 \leq \xi \leq \alpha \rangle$ is a tower of countable elementary submodels of H_λ, and $M_0 \in M_1$. It will be clear later why we separate M_0 from the rest of the tower.

The value returned, $G_\gamma = \mathbb{E}(M_0, \bar{M}, P_\gamma, G_0, p_0)$ is an (M_0, P_γ)-generic filter that extends G_0 and contains p_0. Formally, in saying that G_γ extends G_0 we mean that the restriction projection takes G_γ onto G_0. The definition of $\mathbb{E}(M_0, \bar{M}, P_\gamma, G_0, p_0)$ is by induction on $\alpha < \omega_1$.

Assume that $\alpha = \alpha' + 1$ is a successor ordinal. Then $\gamma = \gamma' + 1$ is also a successor. Assume first that $\gamma_0 = \gamma'$. Then $\alpha = 1$ and we have only two structures: M_0 and M_1. Since P_γ is isomorphic to $P_{\gamma_0} * Q_{\gamma_0}$, we can define G_γ by Definition 5.19. So

$$G_\gamma = \mathbb{E}(M_0, M_1, P_{\gamma_0} * Q_{\gamma_0}, G_0, p_0).$$

Assume next that $\gamma_0 < \gamma'$. Then $G_{\gamma'} = \mathbb{E}(M_0, \langle M_\xi \mid 1 \leq \xi \leq \alpha' \rangle, P_{\gamma'}, G_0, p_0 \restriction \gamma')$ is defined and is an $(M_0, P_{\gamma'})$ generic filter that extends G_0 and contains $p_0 \restriction \gamma'$. Moreover, we assume that $G_{\gamma'} \in M_\alpha$, for otherwise the inductive definition stops. (When we finish this definition it will be evident that it continues for every $\alpha < \omega_1$ since $M_\alpha \prec H_\lambda$ and the parameters are all in M_α).

This brings us to the previous case and we define

$$G_\gamma = \mathbb{E}(M_0, M_\alpha, P_{\gamma'} * Q_{\gamma'}, G_{\gamma'}, p_0). \tag{I.23}$$

Now suppose that α is a limit ordinal, and let $\langle \alpha_n \mid n \in \omega \rangle$ be an increasing and cofinal sequence with $\alpha_0 = 0$. Let $\gamma_n \in M_0$ be the corresponding increasing and cofinal in γ sequence (so that α_n is the order-type of $M_0 \cap [\gamma_0, \gamma_n)$). Let $\langle D_n \mid n \in \omega \rangle$ be an enumeration of all dense subsets of P_γ that are in M_0.

We define $G_\gamma = \mathbb{E}(M_0, \bar{M}, P_\gamma, G_0, p_0)$ as follows. We define by induction on $n \in \omega$ a condition $p_n \in P_\gamma \cap M_0$ and an (M_0, P_{γ_n})-generic filter $G_n \in M_{\alpha_n+1}$ such that:

1. G_0 and p_0 are given. $p_n \restriction \gamma_n \in G_n$.
2. $p_n \leq p_{n+1}$ and $p_{n+1} \in D_n$.

Suppose that G_n and p_n are defined. First, we can easily find $p_{n+1} \in D_n \cap M_0$ such that $p_{n+1} \restriction \gamma_n \in G_n$. Now define

$$G_{n+1} = \mathbb{E}(M_0, \langle M_\xi \mid \alpha_n + 1 \leq \xi \leq \alpha_{n+1} \rangle, P_{\gamma_{n+1}}, G_n, p_{n+1} \restriction \gamma_{n+1}). \quad \text{(I.24)}$$

Finally, let G_γ be the generic filter generated by $\{p_n \mid n \in \omega\}$.
This ends the definition of $\mathbb{E}(M_0, \bar{M}, P_\gamma, G_0, p_0)$.

Theorem 5.17 is a direct consequence of the following lemma.

5.21 Lemma. *(Dee-Properness Extension Lemma) Let $\langle P_i \mid i \leq \gamma \rangle$ be a countable support iteration of forcing posets (γ is any ordinal) where each iterand Q_i satisfies the following in V^{P_i}:*

1. *Q_i is α-proper for every countable α.*
2. *Q_i is Dee-complete with respect to some countably complete completeness system in the ground model V.*

Suppose that $M_0 \prec H_\lambda$ is countable, $P_\gamma \in M_0$ and $p_0 \in P_\gamma \cap M_0$. For any $\gamma_0 \in \gamma \cap M_0$, if α is the order-type of $M_0 \cap [\gamma_0, \gamma)$ and $\bar{M} = \langle M_k \mid k \leq \alpha \rangle$ is a tower of countable elementary substructures (starting with the given M_0) then the following holds. For every $q_0 \in P_{\gamma_0}$ that is completely (M_0, P_{γ_0})-generic as well as $(\bar{M}, P_{\gamma_0})$-generic, if $p_0 \restriction \gamma_0 < q_0$ then there is some $q \in P_\gamma$ such that $q_0 = q \restriction \gamma_0$, $p_0 < q$ and q is completely (M_0, P_γ)-generic. In fact, the filter induced by q is $\mathbb{E}(M_0, \langle M_\xi \mid 1 \leq \xi \leq \alpha \rangle, P_\gamma, G_0, p_0)$ where $G_0 \subset P_{\gamma_0} \cap M_0$ is the filter induced by q_0.

Proof. Let $G_0 \subseteq P_{\gamma_0} \cap M_0$ be the M_0 generic filter induced by q_0. Observe that $G_0 \in M_1$ follows from the assumption that q_0 is (also) M_1 generic. We shall prove by induction on α (the order-type of $M_0 \cap [\gamma_0, \gamma)$) that q can be found which bounds $G_\gamma = \mathbb{E}(M_0, \langle M_\xi \mid 1 \leq \xi \leq \alpha \rangle, P_\gamma, G_0, p_0)$.

Suppose first that $\alpha = \alpha' + 1$ and consequently $\gamma = \gamma' + 1$ are successor ordinals. Define, in M_α, $X \subset P_{\gamma_0}$ a maximal antichain of conditions r such that

1. r bounds G_0,
2. r is $\langle M_\xi \mid 1 \leq \xi \leq \alpha' \rangle$ generic.

Then $X \in M_\alpha$ is pre-dense above q_0. By our inductive assumption every $r_0 \in X$ has a prolongation $r_1 \in P_{\gamma'}$ that bounds $G_{\gamma'} = \mathbb{E}(M_0, \langle M_\xi \mid 1 \leq \xi \leq \alpha' \rangle, P_{\gamma'}, G_0, p_0 \restriction \gamma')$. Since all the parameters are in M_α, we get that $G_{\gamma'} \in M_\alpha$. Since $M_\alpha \prec H_\lambda$, we can choose $r_1 \in M_\alpha$ whenever $r_0 \in X \cap M_\alpha$. This defines a name $\underset{\sim}{r}_1 \in V^{P_{\gamma_0}}$, forced by q_0 to be in $M_\alpha \cap P_{\gamma'}$. Namely, if G is any (V, P_{γ_0})-generic filter containing q_0, then $X \cap G$ contains a unique condition r_0, and we let $\underset{\sim}{r}_1[G] = r_1$. By the Properness Extension Lemma we can find $q_1 \in P_{\gamma'}$, $q_1 \restriction \gamma_0 = q_0$, q_1 is $(M_\alpha, P_{\gamma'})$ generic, and $q_1 \Vdash \underset{\sim}{r}_1$ *is in the generic filter*. It follows that q_1 bounds $G_{\gamma'}$. We must define $q_2 \in P_\gamma$ such that $q_2 \restriction \gamma' = q_1$ and q_2 bounds G_γ. In order to define $q_2(\gamma')$ use Lemma 5.20 and (I.23).

Now assume that α is a limit ordinal. We follow the definition of G_γ (see (I.24)). Recall that we had an ω-sequence $\langle \alpha_n \mid n \in \omega \rangle$ cofinal in α, and we defined γ_n cofinal in γ as the resulting sequence. We defined by induction $p_n \in P_\gamma \cap M_0$ and filters $G_n \subseteq P_{\gamma_n}$, $G_n \in M_{\alpha_n+1}$, and defined G_γ as the filter generated by the p_n's. We shall define now $q_n \in P_{\gamma_n}$ by induction on $n \in \omega$ so that the following hold.

1. q_n bounds G_n,
2. $p_n \restriction \gamma_n < q_n$,
3. $q_n = q_{n+1} \restriction \gamma_n$,
4. q_n is $\langle M_\xi \mid \alpha_n + 1 \leq \xi \leq \alpha \rangle$ generic over P_{γ_n}.

Thus, as q_n gains in length, it looses its status as an M_ξ generic condition for $0 < \xi \leq \alpha_n$. So, finally, $q = \bigcup_{n \in \omega} q_n$ is not M_ξ generic for any $\xi > 0$. But these M_ξ's are not needed anymore as q gained its complete genericity over M_0.

Suppose that q_n is defined. Let X in $M_{\alpha_{n+1}+1}$ be a maximal antichain in P_{γ_n} of conditions r that induce G_n and are $\langle M_\xi \mid \alpha_n + 1 \leq \xi \leq \alpha_{n+1} \rangle$ generic over P_{γ_n}. Observe that X is pre-dense above q_n. For each $r_0 \in X$ find $r_1 \in P_{\gamma_{n+1}}$ such that r_1 bounds G_{n+1}, $p_{n+1} \restriction \gamma_{n+1} < r_1$, and $r_1 \restriction \gamma_n = r_0$ (use the inductive assumption). If $r_0 \in X \cap M_{\alpha_{n+1}+1}$ then r_1 is taken from $M_{\alpha_{n+1}+1}$. Now view $\{r_1 \mid r_0 \in X\}$ as a name $\underset{\sim}{r}$ for a condition forced by q_n to lie in $M_{\alpha_{n+1}+1}$. By the α-Extension Lemma define q_{n+1} that satisfies 2–4 above and such that $q_{n+1} \Vdash \underset{\sim}{r} \in \underset{\sim}{G}$. Then q_{n+1} bounds G_{n+1} and is as required. $\dashv$

Simple completeness systems

An important chapter in the theory of forcing is the study of forcing axioms. These are axioms in the spirit of Martin's Axiom, and the best known among those related to proper forcing is probably the Proper Forcing Axiom (PFA) which uses a supercompact cardinal for its consistency. PFA was defined by Baumgartner, and [1] contains many applications and a consistency proof. (The reader can also read a consistency proof in Cummings' article in this volume.) Chapters VII, VIII, XVII of [15] discuss many of these proper forcing axioms, and we shall restrict our attention here to just one axiom: a variant of Axiom II of Chapter VII which is used to obtain consistency results with CH. This will motivate the notion of *simple* completeness systems and will lead to the p.i.c.

Assume that κ is a supercompact cardinal in the ground model V. Define a countable support iteration P_γ, for $\gamma \leq \kappa$ of $< \omega_1$-proper posets, of cardinality $< \kappa$ each, that are Dee-complete for countably complete completeness systems from V. The actual choice of the iterand is done by some "Laver diamond" function.

Let $P = P_\kappa$ be the resulting countable support iteration. It follows by arguments that are very similar to those used for the PFA consistency result that if G is (V, P) generic, then $V[G]$ satisfies the following axiom, formulated with the ground model V as a predicate:

> GCH holds. V is a ZFC subuniverse containing all reals, and so that the following holds. Let P be any poset such that
>
> 1. P is $< \omega_1$-proper.
> 2. P is Dee-complete with respect to some countably complete completeness system from V
>
> Then for any sequence $D_\alpha \subseteq P$ of dense subsets of P, for $\alpha < \omega_1$, there is some filter $G \subset P$ such that $G \cap D_\alpha \neq \emptyset$ for every $\alpha < \omega_1$

It can be maintained that an axiom should not relate to a subuniverse V in its formulation, and that a more local axiom is required. For this the notion of simplicity is introduced, and we bring here an axiom which is a variant of the original formulation of Shelah [15].

Let $HC = (H_{\aleph_1}, \in)$ be the structure consisting of the universe of all hereditarily countable sets together with the membership relation $\in$. We say that a completeness system $\mathbb{D}$ is *simple over HC* if there is a first-order formula $\psi(y_0, \ldots, y_4)$ in the $\in$ language such that for every transitive and countable model N of ZFC$^-$, poset $P \in N$, and $p \in P$

$$\mathbb{D}(N, P, p) = \{A_X \mid X \in H_{\aleph_1}\}$$

where

$$A_X = \{G \in \mathrm{Gen}_p(N, P) \mid HC \models \psi(G, X, N, P, p)\}. \tag{I.25}$$

For example, $\psi(G, X, N, P, p)$ could say the following.

> Assume that:
>
> 1. In N, g is a Hajnal–Máté graph, $P = P_g$, and $p \in P$.
> 2. X is an unbounded ω-sequence in ω_1^N.
>
> Then $G \subset P$ is N generic, $p \in G$, and the restriction of $\bigcup G$ to X omits infinitely many colors. If the above assumptions do not hold, then G is any (N, P) generic filter containing p.

Now the axiom that can be used to obtain results that are consistent with CH is the following (PFA for countably complete simple completeness systems).

> The continuum hypothesis holds. If P is any $< \omega_1$ proper poset that is Dee-complete for some countably complete completeness system that is simple over HC, and if $\{D_i \mid i \in \omega_1\}$ is a collection of dense subsets of P, then there is a filter $G \subset P$ that meets all of the D_i's. (I.26)

As mentioned, the consistency of this axiom relies on a supercompact cardinal in the ground model, yet in many of the specific applications of this axiom, the supercompact cardinal is not needed. For example, the Souslin hypothesis (which says there are no Souslin trees) is a consequence of the axiom, and in fact the axiom implies that every Aronszajn tree is special. For every Aronszajn tree T, there is a $(< \omega_1)$ proper poset which is Dee-complete with respect to some countably complete *simple* over HC system, and which specializes T (see Chapter V of Shelah [15]). Hence the axiom quoted above implies this strong form of the Souslin hypothesis. Yet, to get the consistency with CH of "every Aronszajn tree is special" no large cardinal is needed–this is the result of Jensen [3]. We may, without any large cardinal assumption, iterate such specializing posets (by the Dee Completeness Iteration Theorem) and obtain the same consistency result. If we do so, we encounter a small difficulty which is discussed in the following subsection, namely that the specializing posets have size $2^{\aleph_1}$ each and so it is unclear, at this stage, that the iteration satisfies the $\aleph_2$-c.c. Although we shall not describe the specializing posets, it turns out that they satisfy the $\aleph_2$-p.i.c. (a strong form of the chain condition described below) and hence the $\aleph_2$-c.c of the iteration follows. Another use of the $\aleph_2$-p.i.c. (which is the one that will be exemplified) is to obtain extensions in which $2^{\aleph_1} > \aleph_2$.

5.4. The Properness Isomorphism Condition

Using the iteration scheme of the previous section we know how to obtain models of ZFC+ CH + $2^{\aleph_1} = \aleph_2$ + *Every Hajnal–Máté graph has countable* *chromatic number*. In this section we modify a little the construction in order to obtain such models with $2^{\aleph_1}$ arbitrarily large. To obtain this, Shelah uses the following simple idea. Starting with $2^{\aleph_1}$ already large, form a countable support *product* of all posets of the form P_g that are in the ground model. This takes care of all Hajnal–Máté graphs in V. Now iterate such large products ω_2 times, and obtain a model of CH + $2^{\aleph_1}$ large + *Every Hajnal–Máté graph has countable chromatic number.* The main technical problem in this approach is to prove that the iteration satisfies the $\aleph_2$-c.c. After we prove that this is the case, we will argue that every Hajnal–Máté graph in the extension already appears in some intermediate stage and hence acquired a countable chromatic number at the following stage. To prove the $\aleph_2$-c.c. we will use the condition named $\aleph_2$-p.i.c. (for Properness Isomorphism Condition), introduced in Chapter VIII of [15] exactly for such applications in mind.

We employ the following terminology. Suppose $M_0, M_1 \prec H_\lambda$ are countable, isomorphic, elementary submodels, and $\aleph_2 \leq \kappa < \lambda$ is a regular cardinal such that $\kappa \in M_0 \cap M_1$; typically $\kappa = \aleph_2$. We say that M_0 and M_1 are in *standard situation* (with respect to κ) if:

1. The sets $A = M_0 \cap M_1 \cap \kappa$, $B = (M_0 \setminus M_1) \cap \kappa$, and $C = (M_1 \setminus M_0) \cap \kappa$ are arranged $A < B < C$ (where $X < Y$ means that $\forall x \in X \forall y \in Y (x < y)$).

2. The isomorphism, denoted $h : M_0 \to M_1$, is the identity function on $M_0 \cap M_1$ (so in particular on A).

5.22 Definition. Let M_0 and M_1 be in standard situation with $h : M_0 \to M_1$ the isomorphism. Suppose that $P \in M_0 \cap M_1$ is a poset. We say that a condition $q \in P$ is *simultaneously* (M_0, P) and (M_1, P) generic iff

1. q is both (M_0, P) and (M_1, P) generic,

2. $$q \Vdash_P (\forall r \in M_0 \cap P)\ r \in \underset{\sim}{G} \text{ iff } h(r) \in \underset{\sim}{G} \tag{I.27}$$

   where $\underset{\sim}{G}$ is the name of the P generic filter. Equivalently, (I.27) can be stated as: for every $q' \geq q$ and $r \in M_0 \cap P$, if $r < q'$ then $h(r) < q'$. Yet another equivalent formulation is that p forces that h can be extended to an isomorphism of $M_0[\underset{\sim}{G}]$ onto $M_1[\underset{\sim}{G}]$.

Remark that the requirement that q is (M_1, P)-generic is dispensable since it follows from (2) if q is (M_0, P)-generic.

5.23 Definition. Let κ be an uncountable regular cardinal. A poset P satisfies the κ-p.i.c. if the following holds for every sufficiently large cardinal λ and any two isomorphic countable elementary submodels $M_0, M_1 \prec H_\lambda$ with $P \in M_0 \cap M_1$ that are in standard situation. For every $p \in M_0 \cap P$

there exists $q > p$ in P that is simultaneously (M_0, P) and (M_1, P) generic. (Hence, in particular, $q > h(p)$.)

This definition is phrased so that the κ-p.i.c. of P implies its properness (take $M_0 = M_1$), but for clarity we shall use the expression "P is a proper κ-p.i.c. poset".

For example, any proper poset of size $\aleph_1$ is $\aleph_2$-p.i.c. because $M_0 \cap P = M_1 \cap P$. So our discussion here generalizes section 2.2. In fact, if P is proper and $|P| < \kappa$ then P satisfies the κ-p.i.c. (see ([15] Chapter VIII), but $\mu^{\aleph_0} < \kappa$ for $\mu < \kappa$ is needed for the lemma that derives the κ chain condition (see below). The Cohen forcing poset for adding $\aleph_2$ reals (with finite conditions), while c.c.c., is not $\aleph_2$-p.i.c. In contrast, the poset for adding subsets of ω_1 with countable conditions is $\aleph_2$-p.i.c.

5.24 Lemma. *If κ is a regular cardinal such that $\mu^{\aleph_0} < \kappa$ for every $\mu < \kappa$, then any κ-p.i.c. poset satisfies the usual κ-c.c.*

Proof. In fact, every collection $\{p_i \mid i < \kappa\} \subset P$ contains a subcollection of size κ of pairwise compatible conditions. For each $i < \kappa$ pick some countable $M_i \prec H_\lambda$ with $p_i \in M_i$. Since $\mu^{\aleph_0} < \kappa$ for every $\mu < \kappa$, a standard Δ-system argument yields a set $I \subset \kappa$ of cardinality κ such that $\{M_i \cap \kappa \mid i \in I\}$ form a Δ-system. So for $i, j \in I$ with $i < j$, M_i and M_j are in standard situation. We may also assume that (M_i, p_i) and (M_j, p_j) are isomorphic (so $h : M_i \to M_j$ is an isomorphism such that $h(p_i) = p_j$). Now the κ-p.i.c. implies that some $q \in P$ extends both p_i and p_j. ⊣

We shall prove now the main theorem using a short argument followed by a series of lemmas which are brought without proof or with a short proof since they resemble those of section 2.

5.25 Theorem. *Suppose that κ is a regular cardinal and $\mu^{\aleph_0} < \kappa$ for every $\mu < \kappa$.*

1. *If P_κ is a countable support iteration of length κ of proper κ-p.i.c. posets, then P_κ satisfies the κ-c.c.*
2. *If P_γ is a countable support iteration of length $\gamma < \kappa$ of proper κ-p.i.c. posets, then P_γ satisfies the κ-p.i.c. (For this we don't need the assumption on $\mu^{\aleph_0}$.)*

Proof. To prove the first part, consider a collection $\{p_i \mid i < \kappa\} \subset P_\kappa$. Use Fodor's theorem to fix a bound $i_0 < \kappa$ on sup $i \cap \mathrm{dom}(p_i)$ on a stationary set of indices i's with uncountable cofinality, so that $\mathrm{dom}(p_i)$ form a Δ-system. This shows that it suffices to prove that P_{i_0}, the iteration of the first $i_0 < \kappa$ posets, is κ-c.c. In fact it is κ-p.i.c. as the second part of the theorem shows. The proof of this part is in the following sequence of lemmas. ⊣

5.26 Lemma. *Let P be a poset and $Q \in V^P$ a name of a poset. Form $R = P * Q$. Then for any countable $M_0, M_1 \prec H_\lambda$ in standard situation and such that $R \in M_0 \cap M_1$ we have the following characterization: $(p, q) \in R$ is simultaneously (M_0, R) and (M_1, R) generic iff*

1. *$p \in P$ is simultaneously (M_0, P) and (M_1, P) generic and*
2. *$p \Vdash_P q$ is simultaneously $(M_0[\underset{\sim}{G}_P], Q)$ and $(M_1[\underset{\sim}{G}_P], Q)$ generic.*

For the proof, use Lemma 2.5 and the equivalent statement following (I.27).

5.27 Lemma. *Suppose that P is a κ-p.i.c. poset and $Q \in V^P$ is (forced to be) κ-p.i.c. there. Then $R = P * Q$ is also κ-p.i.c. and the following stronger form of Lemma 2.5 holds.*
Suppose that

1. *$M_0, M_1 \prec H_\lambda$ with $R \in M_0 \cap M_1$ are countable elementary submodels in standard situation.*
2. *$p_0 \in P$ is a simultaneously (M_0, P) and (M_1, P) generic.*
3. *$\underset{\sim}{r} \in V^P$ is a name such that*
$$p_0 \Vdash_P \underset{\sim}{r} \in M_0 \cap R \text{ and } \pi(\underset{\sim}{r}) \in \underset{\sim}{G}_0$$
where $\pi : P * Q \to P$ is the projection, and $\underset{\sim}{G}_0$ is the canonical name for the P-generic filter.

Then there is some $q_0 \in V^P$ such that (p_0, q_0) is simultaneously (M_0, R) and (M_1, R) generic and
$$(p_0, q_0) \Vdash_R \underset{\sim}{r} \in \underset{\sim}{G}.$$
So also
$$(p_0, q_0) \Vdash_R h(\underset{\sim}{r}) \in \underset{\sim}{G}.$$

The following lemma ends the proof of the second item of Theorem 5.25.

5.28 Lemma. *(Extension of p.i.c.) Let P_γ be a countable support iteration of length $\gamma < \kappa$ of proper posets that are κ-p.i.c. Let λ be sufficiently large and $M_0, M_1 \prec H_\lambda$ be countable with $P_\gamma \in M_0 \cap M_1$ and that are in standard situation. For any $\gamma_0 \in \gamma \cap M_0$ and $q_0 \in P_{\gamma_0}$ that is simultaneously (M_0, P_{γ_0}) and (M_1, P_{γ_0}) generic the following holds. If $\underset{\sim}{p}_0 \in V^{P_{\gamma_0}}$ is such that*
$$q_0 \Vdash_{\gamma_0} \underset{\sim}{p}_0 \in P_\gamma \cap M_0 \text{ and } \underset{\sim}{p}_0 \restriction \gamma_0 \in \underset{\sim}{G}_0$$
(where $\underset{\sim}{G}_0$ is the name of the P_{γ_0} generic filter) then there is a condition $q \in P_\gamma$ such that $q \restriction \gamma_0 = q_0$, q is simultaneously (M_0, P_γ) and (M_1, P_γ) generic, and $q \Vdash_\gamma \underset{\sim}{p}_0 \in \underset{\sim}{G}$. (Thus also $q \Vdash_\gamma h(\underset{\sim}{p}_0) \in \underset{\sim}{G}$).

The proof follows the same steps of the Properness Extension Lemma 2.8.

The following discussion can help to clarify some of the definitions and statements described above. Let *HM* be the collection of all Hajnal– Máté graphs. Recall that for every $g \in HM$, P_g is the poset for making the chromatic number of g countable. Let $P = \Pi^{\aleph_0}_{g \in HM} P_g$ be the countable support product of all ground model P_g's. That is, $f \in P$ iff f is a function defined on *HM* with $f(g) \in P_g$ and such that $f(g) = \emptyset$ for all but countably many g's. The ordering is coordinate wise extension.

The reader can go over the corresponding steps for P_g and prove that P is

1. proper,
2. α-proper for every $\alpha < \omega_1$,
3. Dee-complete for a countably complete completeness system $\mathbb{D}$.
4. an $\aleph_2$-p.i.c. poset.

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