Infinite games on finite sets

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Abstract

We study a family of infinite games with imperfect information introduced by B. Model for two players that alternately remove and add points to a finite set. We investigate the existence of imperfect information strategies for the remover for different ambient cardinalities. We also study a variant of a game of D. Gale introduced by Scheepers and Weiss.

1 Introduction

In [3] Boris Model introduced the following game on $X = \Re$ (the set of real numbers). The Model game $\mathrm{MG}(X, m, \ell)$ is defined with two additional parameters $m, \ell \in \omega$ with $0 < \ell < m$. There are two players which we call Adder and Remover. A play of this game consists of ω moves played alternately by the two players. A finite set $X_0 \subset X$ of cardinality m is presented first to the players. Remover then chooses a set $R_0 \subset X_0$ of cardinality ℓ, and Adder completes $X_0 \setminus R_0$ by choosing ℓ points from X and adding them to $X_0 \setminus R_0$. Thus producing $X_1 \subset X$ of cardinality m again. In general, the two players define a sequence $\langle X_i \mid i \in \omega \rangle$ with $X_i \in [X]^m$ (where $[X]^m$ denotes the collection of all subsets of X of cardinality m). In its k-th move (for $k \geq 0$), Remover chooses $R_k \subset X_k$ of size ℓ, and Adder then defines $X_{k+1} \in [X]^m$ so that $X_k \setminus R_k \subset X_{k+1}$. The resulting ω-sequence

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of sets $P = \langle X_k, R_k \mid k \in \omega \rangle$ is called a play, and sometimes we refer to a finite initial sequence $\langle X_k, R_k \mid k < k_0 \rangle$ as a partial (initial) play.

Let $P = \langle X_k, R_k \mid k \in \omega \rangle$ be a play. The set of fixed points of P is defined by

$$\mathit{fixed}(P) = \bigcup_{n \in \omega} \bigcap_{k \geq n} X_k \setminus R_k.$$

In simple words, $x \in \mathit{fixed}(P)$ iff for some $n \in \omega$ x is in X_n and is never removed in any of the subsequent moves of Remover.

The aim of Remover in this game is to make $\mathit{fixed}(P)$ as small as possible, and the aim of Adder is to make it as large as possible.

It is clear that the game can be defined and played on any set X not just the reals. Model was mostly interested in $X = \Re$, but we found that investigating the possibilities of $X = \aleph_n$ reveals some interesting questions and puts the results about $\Re$ in context.

The game is trivial if perfect information is assumed for Remover. In this case Remover can ensure that $\mathit{fixed}(P) = \emptyset$ for any play P. The game is also of lesser interest if the moves of Remover depend only on the position X_n (without knowledge of n). The point of Model in introducing these games is to study games with partial information in which one of the players (Remover in this game) knows the stage number but does not recall the previous moves. That is, in deciding R_n, Remover knows X_n and n, but not the sequence of X_i's or the points removed so far. So a *strategy* for Remover in the game $\mathrm{MG}(X, m, \ell)$ is a "choice" function $\sigma : [X]^m \times \omega \to [X]^\ell$ such that $\sigma(A, k) \subset A$ for every $A \in [X]^m$ and $k \in \omega$. A play of the game according to strategy σ is a play in which Remover always responds with $R_k = \sigma(X_k, k)$. When the strategy is known, we refer to the sequence $\langle X_k \mid k \in \omega \rangle$ as a play. That is, the sets removed, R_k, are omitted since they can be reconstructed as $R_k = \sigma(X_k, k)$.

Model investigated the number $c = c(\Re, m, \ell)$ which is the minimal number c such that for some strategy σ for Remover, in any play P of $\mathrm{MG}(\Re, m, \ell)$, $|\mathit{fixed}(P)| \leq c$.

The following interesting results of Model [4] motivate our present work ([4] contains additional results).

1. If X is uncountable and $m \in \omega$, then for any strategy σ for Remover in the game $\mathrm{MG}(X, m, 1)$ there is a play P with $\mathit{fixed}(P) \geq 1$. So the remover has no strategy to ensure emptiness of fixed points if he is

allowed to remove single points in each move. Model asked if a similar result holds when the number of points removed is $\ell > 1$.

2. If the continuum hypothesis (CH) is assumed, then, for every $m \in \omega$, Remover has a strategy in the game $\mathrm{MG}(\Re, m, 1)$ that ensures $|\mathit{fixed}(P)| \leq 1$ in every play of the game. Model asked about the role of CH in this result.

We have the corresponding results.

1. If X is uncountable and $\ell < m \in \omega$, then for any strategy σ for Remover in the game $\mathrm{MG}(X, m, \ell)$ there is a play P with $\mathit{fixed}(P) \geq 1$. So the remover has no strategy to ensure emptiness of fixed points even if he is allowed to remove ℓ points.

2. The CH result of Model is in fact about ω_1. Generalizing this to arbitrary $n \in \omega$, we prove for $X = \aleph_n$ that there is a strategy for Remover in the game $\mathrm{MG}(\aleph_n, m, 1)$ to ensure that the number of fixed points in any play never exceeds n (this results uses no assumptions beyond ZFC).

3. We prove that $c(\omega_2, 3, 1) = 1$ is equivalent to the negation of the continuum hypothesis.

Model's games resemble in spirit to a game invented independently by D. Gale which is also a game with limited memory. These games were introduced in [2] where K. Ciesielski and R. Laver answer several questions of Gale. Further investigation of variant games were published in a series of three papers by Scheepers and Scheepers and Weiss. The third paper ([7]) makes a conjecture which we answer positively in section 5.

We conclude with questions for further investigation in section 6.

The authors would like to thank Boris Model for patiently explaining to them his results and open questions.

2 How to remove points from $\aleph_n$

Theorem 2.1 *For every $m \in \omega$ and $1 \leq \ell < m$, for every $n \in \omega$, $c(\omega_n, m, \ell) \leq n$. That is, there is a strategy for Remover which ensures that there is no play in $\mathrm{MG}(\omega_n, m, \ell)$ with $n + 1$ fixed points.*

The proof is by induction on n. For $n = 0$, the games are played on $\omega_0 = \omega$, and Remover's strategy for $\mathrm{MG}(\omega, m, \ell)$ is trivial. Enumerate ω in a sequence $\langle a_n \mid n \in \omega \rangle$ so that each $x \in \omega$ appears infinitely often. For its s-th move, Remover defines $R_s = \sigma_\omega(A, s)$ as some subset $X \subseteq A$ of cardinality ℓ so that $a_s \in X$ if $a_s \in A$. Clearly, in any play $P = \langle A_i, R_i \mid i \in \omega \rangle$ played with σ_ω, *fixed*$(P) = \emptyset$.

The following property will be needed later on to crank the induction.

> For every $x \in \omega$ there exists k so that, if $A_0, \ldots, A_{k-1}$ is any sequence of length k, with $A_i \in [\omega]^m$ such that
>
> 1. $x \in A_0$, and
> 2. $A_i \setminus \sigma_\omega(A_i, i) \subseteq A_{i+1}$ for every $0 \leq i \leq k-2$, (1)
>
> then
> $$x \in \bigcup_{0 \leq i < k} \sigma_\omega(A_i, i).$$

In simpler terms, this says that for any x there is k so that segments of length k of plays cannot contain x continuously. In order to prove that any play has an empty set of fixed points, we must consider arbitrary $a \in \omega$ and the part $\langle A_i, R_i \mid i \geq a \rangle$ of the play. The statement that we need for this is a slight strengthening of (1) namely that for every x and a there exists k such that if A_i, for $a \leq i < a + k$, form a partial play then $x \in \bigcup_{a \leq i < a+k} \sigma_\omega(A_i, i)$. This follows from the assumption that every $x \in \omega$ is enumerated infinitely often.

This strategy σ_ω for the games $\mathrm{MG}(\omega_0, m, \ell)$ is uniform: it does not depend on the value of m (but ℓ is needed to determine the size of the set returned). For $n > 0$ we will need strategies $\sigma^{m,\ell}_{\omega_n}$ that take the values of m and ℓ into account. For uniformity of expression, however, it is convenient to define $\sigma^{m,\ell}_{\omega}$ as σ_ω restricted to sets of size m (and returning sets of size ℓ).

The following overview is designed to motivate the proof for ω_n where $n \geq 1$. The strategy of Remover is to work in cycles which get longer and longer. Presented with $X_s \in [\omega_n]^m$ for its s-th move, Remover looks at the maximal element α of X_s and considers the set $X_s \cap \alpha$ of $m-1$ elements. Since α has cardinality $\leq \omega_{n-1}$ there exists a strategy for $\mathrm{MG}(\alpha, m-1, \ell)$ obtained by the induction hypothesis, and transferred from ω_{n-1} onto α. This strategy tells Remover which subset of $X_s \cap \alpha$ to remove. Remover continues for a while to use the maximal element of the set presented, and to employ the transferred strategy from the lower cardinal. (Remark however that the maximal elements of the sets X_s presented need not be the same all the way.) Then Remover enters the second phase of this cycle, and considers the second largest member of X presented to him. It is this second largest member α that is used as a pivot to determine which transferred strategy to employ over $X_s \cap \alpha$. (It is a strategy for $\mathrm{MG}(\alpha, m-2, \ell)$.) Remover plays this second phase for a while, and then moves to the third phase etc. Remover continues to lower its pivot point and plays transferred strategies on partial plays, until he reaches the $(m-\ell)$-th phase, and then he just removes the first ℓ ordinals in X_s all at once. Remover then finishes its cycle, and restarts another cycle choosing again the maximal element as pivot. Now however each phase is played longer.

There is an obvious problem with this description. Consider the first phase, for example, when the pivot is the largest element of X. This pivot ordinal may change in the next move as a result of the ordinals added by Adder. It may increase or stay the same, but it will never decrease while Remover is in the same phase. Anyhow the transferred strategies can change, and the game of Remover becomes incoherent. What saves the day is a certain stability property: if there is a fixed ordinal in a cycle then there must be an honest partial play of some phase with a fixed strategy.

We continue now with some definitions that will be used in the proof. Assume that strategies $\sigma_\alpha^{m,\ell}$ for Remover have been defined on every $\alpha \le \omega_{n-1}$ for all games $\mathrm{MG}(\alpha, m, \ell)$ where $1 \le \ell \le m$. (In case $\ell = m$, Remover just removes all points at once.) For every ordinal $\alpha \in \omega_n \setminus \omega_{n-1}$ fix a one-to-one correspondence between α and ω_{n-1}, and transfer the assumed strategy for Remover in the game $\mathrm{MG}(\omega_{n-1}, m, \ell)$ to the game $\mathrm{MG}(\alpha, m, \ell)$. This transferred strategy on α is denoted $\sigma_\alpha^{m,\ell}$. So the strategies $\sigma_\alpha^{m,\ell}$ are defined for every $\alpha < \omega_n$, and we want to define $\sigma = \sigma_{\omega_n}^{m,\ell}$ on ω_n as well.

Fix integers $1 \le \ell < m < \omega$. A *cycle number* is an integer $q \ge 1$. A *phase number* is an integer p such that $\ell \le p < m$ (so there are $m - \ell$ phases). A *step number*, or *move number*, (for cycle q) is an integer $0 \le r < q$. Remover works in cycles, $1, 2, \ldots, q, \ldots$, and for each cycle q it goes through phases $m-1$ down to ℓ. At each phase p it executes q steps, $0, \ldots, q-1$. For every cycle q, phase p, and step r we calculate the stage number associated with the r-th step in the p-th phase of the q-th cycle:

$$\mathit{stage}(q, p, r) = (m-\ell)q(q-1)/2 + (m-p-1)q + r.$$

We explain this formula. Each cycle contains $m - \ell$ phases. Each phase in cycle q contains q steps, and hence cycle q contains $q(m-\ell)$ steps. The first cycle, for $q = 1$, contains $(m - \ell)$ steps (one per phase), the second cycle contains $2(m-\ell)$ steps etc. Hence $(m-\ell)q(q-1)/2$ is the number of steps in the first $q-1$ cycles. The number $(m-p-1)q$ is the number of steps related to phases $(m-1), (m-2), \ldots, p+1$, and r is the number of steps associated with phase p. The important thing about *stage* is that it is one-to-one and onto ω and that $q_1 \leq q_2$, $p_2 \leq p_1$, $r_1 \leq r_2$ imply $\mathit{stage}(q_1, p_1, r_1) \leq \mathit{stage}(q_2, p_2, r_2)$.

The interval $[\mathit{stage}(q, m-1, 0), \ldots, \mathit{stage}(q, \ell, q-1)]$ is called the q-th cycle. The interval $[\mathit{stage}(q, p, 0), \ldots, \mathit{stage}(q, p, q-1)]$ of q numbers is called the p-th phase of cycle q. It members are called p-th phase numbers or indices.

The following lemma is obvious.

Lemma 2.2 *For every $a \in \omega$ there is $k \in \omega$ such that the interval $[a, a+k]$ contains a cycle. So for every a and q_0 there is k so that the interval $[a, a+k]$ contains the q-th cycle for some $q \geq q_0$.*

For every $A \in [\omega_n]^m$ we let $A(0) < A(1) \cdots A(m-1)$ be the increasing enumeration of the members of A. So $|A \cap A(k)| = k$. If $x \in A$, then $\lambda(x, A) = |x \cap A|$ is the place of x in A. So $A(p) = x$ iff $\lambda(x, A) = p$.

For every stage $s \in \omega$ there are unique cycle q, phase p, and step number r so that $s = \mathit{stage}(q, p, r)$. To define the strategy $\sigma = \sigma^{m,\ell}_{\omega_n}$ for Remover we define for $A \in [\omega_n]^m$ and $s \in \omega$ the set $\sigma(A, s) \in [A]^\ell$ as follows. Find first a cycle q, phase p, and step number r so that $s = \mathit{stage}(q, p, r)$. Let $a = A(p)$ be the p-th member of A. We say that a is the "pivot" for A at stage s. Consider the strategy $\tau = \sigma^{p,\ell}_a$ in $\mathrm{MG}(a, p, \ell)$, and apply it to $A \cap a$ with move r. That is, define

$$\sigma(A, s) = \tau(A \cap a, r).$$

The strategies $\sigma^{m,\ell}_{\omega_n}$ are thus defined by induction on n for all m and $1 \leq \ell \leq m$, as are the strategies $\sigma^{m,\ell}_\gamma$ for every $n \in \omega$ and $\gamma \in \omega_n$.

We still need some preliminary definitions and lemmas before we can start the proof of Theorem 2.1. Consider a partial initial play $P = \langle X_s \mid 0 \leq s < s_0 \rangle$ in $\mathrm{MG}(\omega_n, m, \ell)$ in which Remover uses its strategy $\sigma = \sigma^{m,\ell}_{\omega_n}$ just defined. We make the following definitions.

1. Suppose that all indices of the q-th cycle,

$$C = [\mathit{stage}(q, m-1, 0), \ldots, \mathit{stage}(q, \ell, q-1)]$$

are below s_0. Then C is called the q-th cycle of P, and $\langle X_s \mid s \in C\rangle$ is called the play of the q-th cycle of P. An ordinal x is said to be *stable* in the q-th cycle of P if x is in each X_s for $s \in C$ and is never removed. That is, if i_0 is the first index in C then $x \in X_{i_0} \setminus \bigcup_{i \in C} \sigma(X_i, i)$.

2. Suppose that $S = [stage(q, p, 0), \ldots, stage(q, p, q-1)]$, the p-th phase of cycle q, is such that all indices of S are below s_0. The sequence $\langle X_s \mid s \in S\rangle$ is called the play of P of phase p in cycle q. We say that this play is "honest" iff, for some a, $X_s(p) = a$ for every $s \in S$.

3. Suppose that i and $i+1$ both belong to the same phase, p, in C. We say that the pair (X_i, X_{i+1}) is *honest* iff $X_i(p) = X_{i+1}(p) = a$.

The main lemma needed is the following stability property.

Lemma 2.3 *Let $P = \langle X_s \mid 0 \le s < s_0\rangle$ be a partial play in which Remover plays $\sigma = \sigma^{m,\ell}_{\omega_n}$. Let $R_i = \sigma(X_i, i)$ and $D_i = X_{i+1} \setminus (X_i \setminus R_i)$ be the sets removed and added. Suppose that the q-th cycle C is included in s_0 (so that X_s is defined for every s in C).*

1. *If i belongs to phase p in C and $i+1 < s_0$, then $X_i(p) \le X_{i+1}(p)$. If both of i and $i+1$ are in phase p of C, then the pair (X_i, X_{i+1}) is honest iff $D_i \subseteq X_i(p)$.*

2. *Suppose that i and $i+1$ belong to different phases in cycle C: say p and $p-1$ respectively. Then $X_{i+1}(p-1) \ge \max X_i(p) \cap (X_i \setminus R_i)$.*

3. *Suppose that x is a stable point of C. Then there exists a phase p in C that is a honest play with pivot x.*

Proof.

1. Say $a = X_i(p)$. The definition of σ is such that $R_i \subseteq a$. That is, Remover takes all its points below a, but Adder can add also above a. Thus the position of a can only decrease: $\lambda(a, X_i) \ge \lambda(a, X_{i+1})$. Hence $X_i(p) \le X_{i+1}(p)$. Equality $X_i(p) = X_{i+1}(p)$ holds exactly if $D_i \subseteq a$.

2. Suppose that i is in phase p, and $i+1$ in the following phase $p-1$. So i is the last stage of phase p, and $i+1$ is the first of phase $p-1$. Say $a = X_i(p)$. Let $x \in a \cap (X_i \setminus R_i)$ be any member of X_i below a that was not removed (since $a = X_i(p)$ and $p > \ell$ there is such x).

We have seen in the first item that $\lambda(a, X_{i+1}) \leq \lambda(a, X_i) = p$. Hence $\lambda(x, X_{i+1}) < \lambda(a, X_{i+1}) \leq p$ implies that $\lambda(x, X_{i+1}) < p$. Equivalently, $X_{i+1}(p-1) \geq x$.

Since x was an arbitrary member of $a \cap (X_i \setminus R_i)$, we can take it maximal and get $X_{i+1}(p-1) \geq \max X_i(p) \cap (X_i \setminus R_i)$.

3. This is the main item of the lemma. Suppose that x is stable in C, but there is no honest phase with x as a fixed pivot. We prove by induction on $p = m-1, \ldots, \ell$ that

   (a) for every i in phase p of C, $X_i(p) \geq x$, and

   (b) for the last stage i in phase p of C, $X_i(p) > x$.

   Suppose the claim holds for all phases greater than p and we prove it for p. Let i be the first index of phase p in C. In case $p = m-1$, $X_i(p) \geq x$ is obvious since $X_i(p) = \max X_i$. In case $p < m-1$, $p+1$ is the previous phase and $i-1$ is the last index of phase $p+1$. So $X_{i-1}(p+1) > x$ by the inductive assumption (item b) and as we proved that

   $$X_i(p) \geq \max X_{i-1}(p+1) \cap (X_{i-1} \setminus R_{i-1})$$

   $X_i(p) \geq x$ follows. Since

   $$X_i(p) \leq X_{i+1}(p) \leq \cdots \leq X_{i+q-1}(p)$$

   by item 1 of the lemma (where $i+q-1$ is the last index in phase p), we have only two possibilities. Either $X_{i+q-1}(p) = x$ in which case phase p is honest with pivot x, or else $X_{i+q-1}(p) > x$.

   But now a contradiction is derived when we reach the last phase $p = \ell$ of C. Because if j is the last index of this phase, then $X_j(\ell) > x$ by our inductive claim (item b). But then all ℓ points of $X_j \cap X_j(\ell)$ are removed at stage j, and thus x is not stable in C.

Now that Lemma 2.3 is proven, we can complete the proof of Theorem 2.1. We will prove by induction on $n < \omega$ the following claim for every

$0 < \ell < m$ and $\sigma = \sigma^{m,\ell}_{\omega_n}$:

For every $F \in [\omega_n]^{n+1}$ and $a \in \omega$ there exists $k \in \omega$ so that if $A_i \in [\omega_n]^m$ for $a \le i < a+k$ satisfy

1. $F \subseteq A_a$, and
2. $A_i \setminus \sigma(A_i, i) \subseteq A_{i+1}$ for every $a \le i \le a+k-2$, (2)

then

$$F \cap \bigcup_{a \le i < a+k} \sigma(A_i, i) \neq \emptyset.$$

This claim implies the theorem immediately. For $n = 0$ and $\sigma = \sigma_{\omega_0}$, it was proved in (1). Observe that if the claim holds for σ_{ω_n} then it holds for every σ_α where $\omega_n \le \alpha < \omega_{n+1}$. Observe also that if the claim holds, then it obviously holds for every $F \in [\omega_n]^r$ where $r \ge n+1$.

Assume the claim for $n-1$ (and $a = 0$, this suffices for the induction). We want to prove the claim for $\sigma = \sigma^{m,\ell}_{\omega_n}$. Consider $F \in [\omega_n]^{n+1}$; let $x_0 = \max F$ and $F_0 = x_0 \cap F$. Since $x_0 < \omega_n$ we have the inductive assumption for games played on x_0, and so there is some $k_0 < \omega$ so that the claim holds for $F_0 \in [x_0]^n$ and for the strategies $\sigma^{m',\ell}_{x_0}$ (where $m' \le m$). That is, if $B_i \in [x_0]^{m'}$, where $m' \le m$, are such that $B_0, \ldots, B_{k-1}$ is a partial play according to $\sigma^{m',\ell}_{x_0}$ and $F_0 \subseteq B_0$ then $F_0 \cap \bigcup_{i<k} \sigma^{m',\ell}_{x_0}(B_i, i) \neq \emptyset$. Suppose that $a \in \omega$ is given. There is $k \in \omega$ such that the interval $[a, a+k)$ contains the q-th cycle for some $q \ge k_0$. Let C be that cycle. We will prove that k witnesses (2). So let $A_i \in [\omega_n]^m$ for $a \le i < a+k$ be given as in (2). If x_0 is not stable in C, then (2) obviously holds, and hence we may assume that x_0 is stable in that part of the given play. By the main Lemma 2.3, there is a phase p in C which is an honest play with pivot x_0. Let $[p_0, \ldots, p_0+q)$ be that phase, which contains q steps since it is in the q-th cycle. If $F_0 \not\subseteq A_{p_0}$ then (2) clearly holds. So we may assume that $F_0 \subseteq A_{p_0}$. The stability of x_0 as a pivot, implies that the sets $B_i = A_{p_0+i} \cap x_0$ for $i < q$ describe a play according to the strategy $\sigma^{m'\ell}_{x_0}$ (where $m' = |B_0|$). Now (2) follows from the inductive assumption.

3 Remover cannot ensure an empty set

In this section we prove for every $\ell < m$ in ω that $c(\omega_1, m, \ell) = 1$. We know by the previous section that $c(\omega_1, m, \ell) \le 1$, and it suffices to show that $c(\omega_1, m, \ell) \neq 0$ in order to conclude that $c(\omega_1, m, \ell) = 1$.

Theorem 3.1 *Assume that $1 \leq \ell < m$. For every strategy σ for Remover in the game $\mathrm{MG}(\omega_1, m, \ell)$, there is a play P such that $|fixed(P)| \geq 1$.*

Proof. Let σ be a strategy for Remover in $\mathrm{MG}(\omega_1, m, \ell)$. So for every $B \in [\omega_1]^m$ and $n \in \omega$, $\sigma(B, n) \in [B]^\ell$. An ordinal $\delta \in \omega_1$ is defined to be *good* iff for every $A_1 \subset \delta$ such that $|A_1| = m - \ell - 1$ and for every $n \geq 1$ there is a set $D \in [\omega_1 \setminus (A_1 \cup \{\delta\})]^\ell$ such that if we define $R = \sigma(A_1 \cup \{\delta\} \cup D, n)$ then $\delta \notin R$ and $D \setminus R \subset \delta$. (In this case, if we define $A_2 = (A_1 \cup D) \setminus R$, then $A_2 \subset \delta$ is of size $m - \ell - 1$, just like A_1.)

Lemma 3.2 *The set of good ordinals contains a closed unbounded subset of ω_1.*

Proof. Suppose for the sake of a contradiction that there is a stationary set $S \subseteq \omega_1$ of ordinals that are not good. For every $\delta \in S$ there are a set $A_1(\delta) \in [\delta]^{m-\ell-1}$ and a natural number $n(\delta) \geq 1$ such that for every $D \in [\omega_1 \setminus (A_1(\delta) \cup \{\delta\})]^\ell$ the set $R = \sigma(A_1(\delta) \cup \{\delta\} \cup D, n(\delta))$ does not satisfy the requirements for goodness in that either $\delta \in F$ or else $D \setminus R \not\subset \delta$.

By Fodor's lemma, there is a stationary set of ordinals $S' \subseteq S$ such that $n = n(\delta)$ and $A_1 = A_1(\delta)$ are fixed for all $\delta \in S'$. Let $\delta_0, \ldots, \delta_\ell$ be a set of $\ell+1$ ordinals from S'. Consider $A = A_1 \cup \{\delta_0, \ldots, \delta_\ell\}$. And define $R = \sigma(A, n)$. Then R has cardinality ℓ and so there is some δ_i not in R. Let δ be the highest indexed δ_i not in R. Then the set $D = \{\delta_0, \ldots, \delta_n\} \setminus \{\delta\}$ shows that $\delta \notin S'$, which is a contradiction. QED

We can conclude now the proof of our theorem, and prove for every $X_0 \in [\omega_1]^m$ that there is a play $P = \langle X_i \mid i \in \omega \rangle$ in which Remover employs its strategy σ and $|fixed(P)| \geq 1$.

Let $L = \{\delta_1, \ldots, \delta_\ell\}$ be a set of ℓ good ordinals, all above X_0. The first move of Remover produces $R_0 = \sigma(X_0, 0)$, to which Adder responds with $X_1 = (X_0 \setminus R_0) \cup L$. There are two cases now which determine the responses of Adder.

Case 1. Suppose that for every $n \geq 1$, $\sigma(X_1, n) = L$. In this case, Remover always removes L and Adder always puts L back and the resulting play is such that $X_i = X_1$ for all $i \geq 1$. Clearly the set of fixed points is X_0.

Case 2. Not Case 1. That is, for some $n \geq 1$, $L \setminus \sigma(X_1, n) \neq \emptyset$. Let $1 \leq n$ be the first such n, and let $\delta_i \in L \setminus \sigma(X_1, n)$ be with maximal possible index i. Then Adder plays with L in its first n moves until

$X_n = X_1$ is produced such that $\delta_i = \max\{X_n \setminus \sigma(X_n, n)\}$. From now on Adder can use the goodness of δ_i in defining X_m so that $\delta_i \in X_m$ is never removed and $\delta_i = \max(X_m \setminus R_m)$. This is clearly possible: after Remover subtracts R_m, Adder considers $A_1 = X_m \setminus (R_m \cup \{\delta_i\})$ and finds $D \in [\omega_1 \setminus (A_1 \cup \{\delta\})]^\ell$ by the goodness of δ. Then it defines $X_{m+1} = A_1 \cup \{\delta\} \cup D$. Since δ_i is never removed, the theorem is proved.

4 CH is equivalent to $c(\omega_2, 3, 1) = 2$

Our aim is to prove the theorem stated in the title of this section. There are two directions in the proof, and we first assume the continuum hypothesis (CH). We will use the Erdos-Rado theorem which states that $(2^{\aleph_0})^+ \rightarrow (\omega_1)^2_\omega$. That is, for every function $f : [(2^{\aleph_0})^+]^2 \rightarrow \omega$ there exists an homogeneous set of order-type ω_1 (a standard reference to infinitary combinatorics is [1]). If CH is assumed, we have $\omega_2 \rightarrow (\omega_1)^2_\omega$ (namely replace $(2^{\aleph_0})^+$ by ω_2).

Consider now the game $\mathrm{MG}(\omega_2, 3, 1)$ and let σ be a strategy for Remover. So for every $A \in [\omega_2]^3$ and $n \in \omega$, $\sigma(A, n) \in A$ (formally it is a singleton, but we refer to it as an ordinal). We will produce a play P such that $|\mathit{fixed}(P)| = 2$. Since $c(\omega_2, 3, 1) \leq 2$ is trivial, we conclude that $c(\omega_2, 3, 1) = 2$ if CH holds.

Define an unordered pair $a, b \in [\omega_2]^2$ to be *good* iff for every $n \in \omega$ there is an ordinal $c \in \omega_2 \setminus \{a, b\}$ such that $\{c\} = \sigma(\{a, b, c\}, n)$. If there exists a good pair $\{a, b\}$, then Adder can find a play in which $\{a, b\}$ is the set of fixed points, and this proves the first direction of our theorem. So assume that no pair is good, and for every $\{a, b\} \in [\omega_2]^2$ there is $n \in \omega$ so that for every $c \in \omega_2 \setminus \{a, b\}$ $\sigma(\{a, b, c\}, n) \in \{a, b\}$. Define $f(\{a, b\})$ to be such a number n. By Erdos-Rado theorem there is an infinite homogeneous set, but clearly there cannot even be a homogeneous set of size 3.

Now we prove the other direction, namely that $\neg$CH implies that $c(\omega_2, 3, 1) = 1$. The argument is a variant of B. Model's proof of his theorem that $c(\Re, 3, 1) \leq 1$. It is a well-known observation that there is a function $g : [\Re]^2 \rightarrow \omega$ with no homogeneous triangle. That is, no set $\{a, b, c\}$ of three elements so that for some n $g(\{x, y\}) = n$ for all $x \neq y$ in $\{a, b, c\}$. (To see this, let $\{s_i \mid i \in \omega\}$ enumerate all rational numbers. Then define $g(\{a, b\}) = i$ iff i is the first index for which s_i is strictly between a and b.)

If $2^{\aleph_0} \geq \aleph_2$ then we may take our function g to be defined on $[\omega_2]^2$. So $g : [\omega_2]^2 \rightarrow \omega$ has no homogeneous triangles. This can be employed in defining a strategy for Remover in $\mathrm{MG}(\omega_2, 3, 1)$ as follows. Fix an enumeration of

ω, $\{n_i \mid i \in \omega\}$ in which each number is enumerated infinitely often. Let $A \in [\omega_2]^3$ be given at stage i of the play. Consider the color n_i. There are three cases:

1. There is no pair $\{x, y\} \subset A$ with $g(\{x, y\}) = n_i$. Then define $\sigma(A, i)$ as any ordinal in A.

2. There is a single pair $\{x, y\} \subset A$ with $g(\{x, y\}) = n_i$. Then define $\sigma(A, i) \in \{x, y\}$ (any value will do).

3. There are two pairs $\{x, y\}$ and $\{y, z\}$ taken from A with $g(\{x, y\}) = g(\{y, z\}) = n_i$. Observe that $g(\{x, z\}) \neq n_i$ since there is no homogeneous triangle. Hence the pairs $\{x, y\}$, $\{y, z\}$ are uniquely determined and we define $\sigma(A, i) = y$ (that is, y is the common point of the two pairs with color n_i).

This strategy ensures that there are never two fixed points. For suppose $\{a, b\} \in [\omega_2]^2$, and consider a play in which Remover uses this strategy σ. For any stage $k \in \omega$ we will find a later move in which one of a and b is removed (if present in X_n). Indeed at stage i with $n_i = g(\{a, b\})$, we ensured that $\sigma(A, i) \in \{a, b\}$.

5 On a game of Scheepers and Weiss

The game of D. Gale investigated by Ciesielski and Laver [2] resemble those of Model (they were found independently). We call the two player in Gale's games Adder and Coverer. Let X be an infinite ambient set. In its n-th move ($n \geq 0$), Adder chooses a finite subset X_n of X and Coverer chooses a single point x_n from X (or rather from $X_1 \cup \ldots \cup X_n \setminus \{x_1, \ldots, x_{n-1}\}$). The aim of Coverer is to achieve the equality

$$\bigcup_{n \in \omega} X_n = \{x_i \mid i \in \omega\}$$

in which case it wins the play. If even one point is not covered, Adder wins.

Scheepers and Weiss formulated in [7] a variant $\mathrm{SG}(\omega_1, k+1, k)$ of Gale's game and conjectured that Coverer does not have a remainder winning strategy for the game. They proved this conjecture for $k = 1, 2, 3$ and found additional results that strengthen the plausibility of their conjecture. The aim of this section is to prove that conjecture completely, for every k.

Fix some integer $k \geq 1$. The game $\mathrm{SG}(\omega_1, k+1, k)$ is defined as follows. The ambient set is $X = \omega_1$. In its first move, (indexed by 0), Adder chooses a set $O_0 \in [\omega_1]^{k+1}$, and Coverer chooses $T_0 \subset O_0$ of cardinality k. In its n-th move ($n \geq 1$) Adder chooses a set $O_n \subset \omega_1$ so that

1. $O_{n-1} \subset O_n$, and
2. $|O_n \setminus O_{n-1}| \leq k+1$.

Then Coverer chooses (in its n-th move) a set $T_n \subset O_n \setminus \bigcup_{0 \leq j < n} T_j$ such that $|T_n| \leq k$. The play is won by Coverer iff $\bigcup_{n \in \omega} O_n = \bigcup_{n \in \omega} T_n$.

A remainder strategy for Coverer is a function σ defined on the finite subsets A of ω_1 such that $\sigma(A) \subseteq A$ and $|\sigma(A)| \leq k$. A play $\{O_n, T_n \mid n \in \omega\}$ is played according to remainder strategy σ iff $O_0 \in [X]^{k+1}$, $O_{n-1} \subset O_n$, $|O_n \setminus O_{n-1}| \leq k+1$, $T_0 = \sigma(X_0)$ and $T_n = \sigma(O_n \setminus \bigcup_{0 \leq j < n} T_j)$.

The following was conjectured by Scheepers and Weiss [7].

Theorem 5.1 *For every $k \geq 1$ and for every remainder strategy for Coverer in the game $\mathrm{SG}(\omega_1, k+1, k)$ there exists a play $\{O_n, T_n \mid n \in \omega\}$ played according to σ in which Adder wins. In fact, except for the first set O_0 which has cardinality $k+1$, we can let Adder add only k ordinals at a time (that is $O_n \setminus O_{n-1}$ has cardinality k).*

Proof. An ordinal $\delta \in \omega_1$ is defined to be *good* iff for every finite $A \subset \delta + 1$ such that $\delta = \max A$ (so $\delta \in A$) and for every finite set $V \subset \omega_1$ such that $A \subseteq V$ there exists a set $B \subset \omega_1 \setminus V$ such that $|B| = k$ and

$$\delta = \max\{(A \cup B) \setminus \sigma(A \cup B)\}. \tag{3}$$

Lemma 5.2 *The set of good ordinals in ω_1 contains a closed unbounded set.*

Proof. Suppose that there exists a stationary set $S \subseteq \omega_1$ of ordinals that are not good. So for every $\delta \in S$ there exists a finite set $A(\delta) \subset \delta + 1$ such that $\delta = \max A(\delta)$, and there exists a finite set $V(\delta)$ such that $A(\delta) \subseteq V(\delta) \subset \omega_1$, and

$$\text{for every } \ B \in [\omega_1 \setminus V(\delta)]^k, \ (3) \ \text{ does not hold.} \tag{4}$$

Using Fodor's theorem, we can shrink S and obtain a stationary subset $S' \subseteq S$ such that, for some fixed A_0 and V_0, $A_0 = A(\delta) \cap \delta$ and $V_0 = V(\delta) \cap \delta$ for every $\delta \in S'$. So $A(\delta) = A_0 \cup \{\delta\}$ for every $\delta \in S'$. Shrinking S' even further, we may assume that if $\delta_1 < \delta_2$ are in S' then $V(\delta_1) \subset \delta_2$.

Now let L be any set of $k+1$ ordinals from S'. Define $T = \sigma(A_0 \cup L)$. Since T has at most k ordinals, $L \setminus T \neq \emptyset$, and we let $\delta = \max(L \setminus T)$. Now $A(\delta) = A_0 \cup \{\delta\}$, and if we define $B = L \setminus \{\delta\}$, then B is disjoint to $V(\delta)$ and $\delta = \max[A(\delta) \cup B \setminus \sigma(A(\delta) \cup B)]$ which contradicts (4). QED

Now we describe how Adder plays against the strategy σ of Coverer in the SG$(\omega_1, k+1, k)$ game. The first set O_0 is any set of size $k+1$ of good ordinals. Coverer then plays $T_0 = \sigma(O_0)$ of size at most k. Let δ be the maximal ordinal in $O_0 \setminus T_0$. The play that Adder develops is such that δ is never covered, and hence Coverer fails. The sets O_n played by Adder in its n-th stage, and the sets $R_n = T_0 \cup \ldots \cup T_n$ that were removed by Coverer in its moves up-to and including stage n are such that $\delta = \max\{O_n \setminus R_n\}$.

Let's see that Adder can keep holding this inductive property and thus ensuring that δ is never covered by Coverer. Suppose the n-th moves O_n and T_n were played by Adder and Coverer. Then Adder defines $A = O_n \setminus R_n$, and $V = O_n$. By assumption $\delta = \max A$. Since δ is good, there exists a set $B \in [\omega_1 \setminus V]^k$ such that

$$\delta = \max\{A \cup B \setminus \sigma(A \cup B)\}. \tag{5}$$

Then Adder plays with $O_{n+1} = O_n \cup B$. Coverer responds with $T_{n+1} = \sigma(O_{n+1} \setminus R_n)$, and we must check that $\delta = \max O_{n+1} \setminus R_{n+1}$. But this follows from (5) since $A \cup B = O_{n+1} \setminus R_n$.

6 Conclusion

We do not know the answers to the following questions which seem interesting to us.

1. As we have said, Model proved with CH that for every $m \in \omega$, $c(\Re, m, 1) \leq 1$. Namely there is a strategy for Remover which leaves no pair of ordinals fixed. We observed that CH is not needed if the ambient set is ω_1, and extended this result to ω_n. Yet, one may insist on playing with reals and ask whether $c(\Re, m, 1) \leq 1$ can be proved without the aid of CH. Model [4] proved this for $m = 3$ (we reported a variant of the proof in section 4), and the simplest open question is to prove that $c(\Re, 4, 1) \leq 1$.

2. For every $n \in \omega$ we have a strategy for Remover that ensures no more than n fixed points in any play of MG$(\omega_n, m, 1)$. That is, $c(\omega_n, m, 1) \leq$

n. When is it true that $c(\omega_n, m, 1) = n$? This may depend on m: for $m = 3$ we have proved in section 4 that $c(\omega_2, 3, 1) = 2$ is equivalent to CH, but what about $m = 4$?

3. Model also considered games in which Adder is allowed to add more points than Remover can remove. For example, consider a game on ω_1 in which Adder can add two ordinals in a move, and Remover can take only one. A strategy for Remover is a choice function, that is a function f defined on the non-empty finite subsets of ω_1 such that $f(A) \in A$. The proof of section 3 can be adapted to show that Adder can always secure at least one fixed point, but we suspect that it can do better. We conjecture that Adder can secure infinitely many fixed points. Is it true that for every order type $\alpha < \omega_1$ Adder can secure a set of fixed points of order type at least α?

4. In proving the conjecture of Scheepers and Weiss we showed that for every strategy for Coverer in $\mathrm{SG}(\omega_1, k+1, k)$ there is a play in which Coverer fails by omitting one point. Can Adder force a failure in which at least two points are never removed? At the end of their paper [7], Scheepers and Weiss remark that it is probably true that for every strategy for Coverer in $\mathrm{SG}(\omega_1, k+1, k)$ (for any $k \in \omega$) and for every $\alpha \in \omega$ there is a play in which the order-type of the points not taken is at least α.

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