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EVERY SUPERATOMIC SUBALGEBRA OF AN INTERVAL ALGEBRA
IS EMBEDDABLE IN AN ORDINAL ALGEBRA

URI ABRAHAM AND ROBERT BONNET

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ABSTRACT. Let us recall that a Boolean algebra is superatomic if every subalgebra is atomic. So by the definition, every subalgebra of a superatomic algebra is superatomic. An obvious example of a superatomic algebra is the interval algebra generated by a well-ordered chain. In this work, we show that every superatomic subalgebra of an interval algebra is embeddable in an ordinal algebra, that is by definition, an interval algebra generated by a well-ordered chain. As a corollary, if $B$ is an infinite superatomic subalgebra of an interval algebra, then $B$ and the set $\text{At}(B)$ of atoms of $B$ have the same cardinality.

1. Survey of the results

In a Boolean algebra $B$ we denote by $0_B$ and $1_B$, respectively the smallest and the largest elements of $B$. For $x$ and $y$ in $B$, we denote by $x \cup y$ and $x \cap y$, respectively, the supremum and the infimum of $x$ and $y$ in $B$, by $-x$ the complement of $x$ in $B$, and $x - y = x \cap (-y)$. So $-x = 1_B - x - x$. Moreover, $x \subseteq y$ means $x \subseteq y$ and $x \neq y$.

For $0_B \neq a \in B$, we denote by $B \upharpoonright a$ the Boolean algebra induced by $B$ on the set $\{t \in B | t \subseteq a\}$. So $1_B \upharpoonright a = a$ and the complement of $t$ in $B \upharpoonright a$ is $a - t$.

For a Boolean algebra $B$ and a subset $D$ of $B$, we denote by $\text{At}(B)$ the set of atoms of $B$, and by $\text{cl}_B(D)$ the subalgebra of $B$ generated by $D$.

For example, if $I$ is an ideal of $B$, then $\text{cl}_B(I) = I \cup -I$, where $-I = \{-x \in B | x \in I\}$.

Definition. A Boolean algebra $B$ is said to be superatomic if every quotient of $B$ is atomic.
Day [3] (see also Koppelberg [5] and Roitman [7]) has shown the following result:

**Proposition.** Let $B$ be a Boolean algebra. The following properties are equivalent:

(i) $B$ is superatomic;

(ii) every subalgebra of $B$ is atomic; and

(iii) there is no embedding from the atomless countable algebra into $B$. □

If $E$ is a set, then $\wp(E)$, the power set of $E$, is regarded as a Boolean algebra.

Let $(C, \leq)$ be a partial ordered set. We say that $(C, \leq)$ is a chain if every pair of members of $C$ are comparable, and $(C, \leq)$ is well ordered if $(C, \leq)$ has no strictly decreasing sequence. Let $(C, \leq)$ be a chain with a first element denoted by $0_C$ (if $(C, \leq)$ has no first element, then we must add one). Let $C^+ \overset{def}{=} C \cup \{\infty_C\}$ be the chain, obtained by adding a greatest element $\infty_C$. We denote by $B(C)$ the subalgebra of $\wp(C)$ generated by the set of $\{a, b\}$ for $a \in C$ and $b \in C^+$, i.e. $\cl_{\wp(C)}(\{\{a, b\} | a \in C$ and $b \in C^+\})$. $B(C)$ is called the interval algebra on $C$ (see Koppelberg [5]).

**Theorem 1.** Let $B$ be a superatomic Boolean algebra. If $B$ is embeddable in an interval algebra, then $B$ is embeddable in an interval algebra generated by a well-ordered chain $C$. More precisely, $B$ is isomorphic to a subalgebra $B'$ of $B(C)$ and $\operatorname{At}(B') = \operatorname{At}(B(C))$.

Obviously Theorem 1 holds if $B$ is countable, since $B$ is isomorphic to an interval algebra (see Mayer and Pierce [6], Koppelberg [5]).

**Example and comment.** Let $B$ be a superatomic subalgebra of an interval algebra $B(C)$. The question is as follows: *Is there a superatomic interval algebra $A$ such that $B \subseteq A \subseteq B(C)$?* The answer is negative. Indeed, let $C = 2 \cdot \lambda$, where $\lambda$ denotes the chain of real numbers, and thus $C$ is the chain obtained from the chain $\lambda$ by replacing each real number by the 2-elements chain. Let $B \overset{def}{=} \cl_{B(C)}(\operatorname{At}(B(C)))$. Let $A \overset{def}{=} B(D)$ be a superatomic interval algebra, generated by a chain $D$, containing $B$. First, because $\operatorname{At}(B)$ is uncountable, the scattered chain $D$ is uncountable. By a theorem of Hausdorff (see Rosenstein [8, Theorem 5.28]), $D$ contains a copy of the chain $\omega_1$ or $\omega_1^\ast$. For a contradiction, let us suppose that $B(D) \subseteq B(2 \cdot \lambda)$. Because $B(D)$ satisfies: there is an uncountable family of pairwise disjoint elements such that each element contains infinitely many atoms, the algebra $B(2 \cdot \lambda)$ has the same property. Now, we obtain a contradiction with the fact that in $\lambda$, every family of pairwise disjoint nontrivial intervals is at most countable.

For an infinite well-ordered chain $C$, the sets $\operatorname{At}(B(C))$ and $B(C)$ have the same cardinality; hence Theorem 1 implies the following result, proved by M. Rubin and S. Shelah [1988]:

**Theorem 2.** Let $B$ be an infinite superatomic Boolean algebra, embeddable in an interval algebra. Then $\operatorname{At}(B)$ and $B$ have the same cardinality.

This result completes the different characterizations of superatomic subalgebras of an interval algebra, developed by R. Bonnet, M. Rubin, and H. Si-Kaddour [1988]. Let us give an application of the above theorem. A Boolean
algebra $B$ is said to be the thin-tall if $B$ is uncountable, and for each ordinal $\alpha$, the set $\text{At}(D_\alpha(B))$ is countable (the algebra $D_\alpha(B)$ is defined in 2.3). In particular $\text{At}(B)$ is countable. From Theorem 2, for example, it follows that a thin-tall Boolean algebra is not embeddable in an interval algebra (for an application, see R. Bonnet and S. Shelah [2]).

2. Notation and definition

2.1. Chains. Let $(C, \leq)$ be a chain, $u \in C$, and $D$ be a subset of $C$.

$(C, \leq)$ is a complete chain if every subset of $C$ has a supremum and an infimum.

$(C, \leq)$ is a relatively complete chain if every bounded nonempty subset $A$ of $C$ (i.e. there are $b$ and $c$ in $C$ such that $b \leq a \leq c$ for every $a \in A$) has a supremum and an infimum in $C$.

Assume that $C$ is complete. We denote by $c(D, C)$, or more simply $c(D)$, the closure of $D$ in $C$ by supremum and infimum. For the following result, see e.g. Rosenstein [8]:

Every chain $(C, \leq)$ is embeddable in a complete chain $(C^d, \leq)$; namely, its Dedekind completion, (completion by cuts), which satisfies: for every $c \in C^d$, if $c$ is not the first element of $C^d$ then $c = \sup\{p \in C|p \leq c\}$, and if $c$ is not the last element of $C^d$, then $c = \inf\{p \in C|p \geq c\}$.

$u$ is a predecessor in $C$ if there is an (unique) element $u^+ \in C$ such that $u^+ > u$ and $\{u, u^+\}$. We denote by $\text{Pred}(C)$ the set of predecessors of $C$.

$(C, \leq)$ is totally disconnected if for every $v < w$ in $C$, we have $[v, w] \cap \text{Pred}(C) \neq \emptyset$. Consequently $B(C)$ is atomic if and only if $C$ is totally disconnected. The word “totally disconnected” comes from the fact that a chain $C$ is totally disconnected if and only if $C$, endowed with the interval topology, is a totally disconnected space.

Let $\cong$ be an equivalence relation on $C$. For $a \in C$, we denote by $a/ \cong$ its equivalence class, i.e. $a/ \cong \overset{\text{def}}{=} \{a' \in C|a' \cong a\}$. We suppose that each equivalence class is an interval of $C$. For $\tilde{a}', \tilde{a}'' \in C/ \cong$ we set $\tilde{a}' < \tilde{a}''$ if for every $a' \in \tilde{a}'$ and $a'' \in \tilde{a}''$, we have $a' < a''$. If each equivalence class is an interval of $C$, then $(C/ \cong, \leq)$ is a chain. Moreover, if $(C, \leq)$ is a complete chain, then $(C/ \cong, \leq)$ is too.

2.2. Boolean algebras. An element $a$ of an interval algebra $B(C)$, different from $0_B$, has a unique decomposition (called the canonical decomposition), of the form: $a = \bigcup\{[a_{2i}, a_{2i+1})|i < n\}$ where $0 < n < \omega$, $0_C \leq a_0 < a_1 < a_2 < \cdots < a_{2n-1} \leq \infty_C$, and $a_k \in C^+ \overset{\text{def}}{=} C \cup \{\infty_C\}$, $(k = 0, 1, \ldots, 2n - 1)$. For such an element $a$, we set $\sigma(a) = \{a_k|k < 2n\} \subseteq C \cup \{\infty_C\}$. The integer $n$ is called the length of $a$, and is denoted by $l(a)$.

Every finite product of interval algebras is isomorphic to an interval algebra, and thus if $B$ is a subalgebra of a finite product of interval algebras, then $B$ is embeddable in an interval algebra.

More precisely, let us recall the following fact concerning chains and Boolean algebras (see [5, Proposition 15.11]). Let $C_1$ and $C_2$ be chains with first element $0_{C_1}$ and $0_{C_2}$ respectively. Let $C = C_1 \oplus C_2$ be the chain, lexicographic sum of $C_1$ and $C_2$ (so $c_1 < c_2$ for $c_1 \in C_1$ and $c_2 \in C_2$). Note that $C$ has
a first element, namely $0_{C_1}$. A canonical isomorphism $f$ from $B(C)$ onto $B(C_1) \times B(C_2)$ is obtained by letting: $f(c) = (c \cap C_1, c \cap C_2)$. Let us remark that we identified $\omega_{C_1}$ with $0_{C_1}$. $B(C_1), B(C_2)$ are factors of $B(C)$; and by identification, $B(C) \upharpoonright C_1 = B(C_1)$ and $B(C) \upharpoonright C_2 = B(C_2)$.

Let $D$ be a subset of $C$, containing $0_C$. Hence $D$ is a chain with a first element. We denote by $B_C(D)$ the subalgebra of $B(C)$ consisting of those elements $a$ such that $\sigma(a) \subseteq D \cup \{\omega_C\}$. Let us remark that the Boolean algebras $B_C(D)$ and $B(D)$ are isomorphic.

2.3. Let $B$ be a Boolean algebra. By induction, we define a sequence $(I_\alpha(B), D_\alpha(B), \pi^B_\alpha)$, with the conditions $D_\alpha(B) = B/I_\alpha(B)$ and $\pi^B_\alpha$ is the canonical homomorphism from $B$ onto $D_\alpha(B)$ (the algebra $D_\alpha(B)$ is called the $\alpha$-th Cantor Bendixson derivative of $B$). Let $I_0(B) = \{0\}$, and thus $D_0(B) = B$. Suppose that $I_\alpha(B)$ has been defined. Let $J_\alpha(B)$ be the ideal of $D_\alpha(B)$, generated by $\operatorname{At}(D_\alpha(B))$. Then $I_{\alpha+1}(B) \overset{\text{def}}{=} (\pi^B_\alpha)^{-1}(J_\alpha(B))$. Suppose that $\delta$ is a limit, and $I_\alpha(B)$ has been defined for every $\alpha < \delta$, then $I_\delta(B) = \bigcup_{\alpha < \delta} I_\alpha(B)$.

The following additional equivalences are well known and their proofs are straightforward (see Koppelberg [5]).

**Proposition.** Let $B$ be a Boolean algebra. The following properties are equivalent:

(i) $B$ is superatomic, and

(ii) there is an ordinal $\gamma$ such that $1_B \in I_\gamma(B)$. □

Clearly the first ordinal $\gamma$ for which $1_B \in I_\gamma(B)$ is a successor ordinal, say $\alpha + 1$, and $\alpha$ is $\alpha$ is denoted by $\operatorname{rk}(B)$. Hence $1_B \in I_{\operatorname{rk}(B)+1}(B) - I_{\operatorname{rk}(B)}(B)$ and $D_{\operatorname{rk}(B)}(B)$ is a nontrivial finite algebra isomorphic to $\wp(n)$ ($n > 0$ integer), and if $n = 1$, then $I_{\operatorname{rk}(B)}(B)$ is a maximal ideal of $B$. Let $I(B)$ and $D(B)$ denote $I_{\operatorname{rk}(B)}(B)$ and $B/I(B)$ respectively.

For $b \in B$, $b \neq 0_B$ let $\operatorname{rk}_B(b)$ be the first ordinal $\alpha$ such that $b \notin I_\alpha(B)$. Hence $b \in I_{\operatorname{rk}_B(b)+1}(B) - I_{\operatorname{rk}_B(b)}(B)$. For instance $\operatorname{rk}_B(b) = 0$ for $b \in \operatorname{At}(B)$, and $\operatorname{rk}_B(1_B) = \operatorname{rk}(B)$.

**Notation 2.4.** Let $B$ be a subalgebra of an algebra $A$, and $c \in A$. We denote by $B \upharpoonright c$ the set of $b \cap c$ for $b \in B$. We regard $B \upharpoonright c$ as a subalgebra of the factor $A \upharpoonright c$ of $A$, and thus as a Boolean algebra. Remark that if $c \in B$, then $B \upharpoonright c$ is a factor of $B$.

By the definition, $B \upharpoonright c$ is an homomorphic image of $B$. From the fact that, for every superatomic Boolean algebra $A$ and every ideal $I$ of $A$, we have $\operatorname{rk}(A/I) \leq \operatorname{rk}(A)$, it follows that:

**Lemma 2.5.** Let $B$ be a subalgebra of a superatomic Boolean algebra $A$, and $c \in A$. Then $\operatorname{rk}(B \upharpoonright c) \leq \operatorname{rk}(B) = \operatorname{rk}_B(1_B)$. □

The following result is due to M. Rubin and S. Shelah [9], and is one of the ingredients of the original proof of Theorem 2.

**Proposition 2.6** (Rubin and Shelah). Let $B$ be an atomic subalgebra of an interval algebra. Then there are a totally disconnected complete chain $C$ and an embedding $\phi$ from $B$ into $B(C)$ such that $B(C)$ is an atomic algebra and $\operatorname{At}(\phi[B]) = \operatorname{At}(B(C))$.  

Note that the property of \( C \) implies that \( c(\bigcup \text{At}(B(C)), C) = C \). The proof of Proposition 2.6 needs some preliminary results. Let \( C \) be a chain such that \( B \subseteq B(C) \).

**Claim 2.7.** We can suppose that \( C \) satisfies (1): \( C \) is a complete chain, and (2): every atom of \( B \) is a finite subset of \( C \).

**Proof.** We can suppose that \( C \) is a complete chain (consider its Dedekind completion). Let \( \mathcal{C} = \bigcup \{\sigma(a) | a \in B\} \) be the set of endpoints of elements of \( B \). We set \( C^c = c(\mathcal{C}, C) \). The function \( \phi \) from \( B \) into the subalgebra \( B(C^c) \) of \( B(C) \) defined by \( \phi(b) = b \cap \mathcal{C} \) is trivially a one-to-one homomorphism, and \( C^c \) is as required (note that by the construction, (2) is satisfied).

**Claim 2.8.** We can suppose that \( C \) satisfies (1) and (3): every atom of \( B \) is a singleton of \( C \).

**Proof.** For every \( a \in \text{At}(B) \), we have \( a = \bigcup_{i < l(a)} [a_{2i}, a_{2i+1}) = \bigcup_{i < l(a)} \{a_{2i}\} \) with \( a_k \in C^+ = C \cup \{\infty, C\} \), and \( a_{2i}, \ a_{2i+1} \) consecutive in \( C \). Let \( \sim \) be the equivalence on \( C \) defined by \( a_{2i} \sim a_{2i+1} \) for \( 0 < i < l(a) \) and \( a \in \text{At}(B) \). Let \( \mathcal{C} = C/\sim \). Hence \( \mathcal{C} \), with the induced linear order by \( C \), is a complete chain. Let \( \varphi \) be the function from \( B(C) \) into \( B(\mathcal{C}) \) defined as follows: if \( b = \bigcup_{i < l(b)} [b_{2i}, b_{2i+1}) \), then \( \varphi(b) = \bigcup_{i < l(b)} [b_{2i}/\sim, b_{2i+1}/\sim) \). Obviously \( \varphi \) is a homomorphism from \( B(C) \) onto \( B(\mathcal{C}) \). It suffices to show that \( \varphi(a) = [a_0/\sim, a_1/\sim) = \{a_0/\sim\} \neq 0 \) for \( a \in \text{At}(B) \), and \( \varphi \) restricted to \( B \) is one-to-one. But this is trivial.

**Proof of Proposition 2.6.** To prove the proposition, there is no loss in assuming that \( B \) and \( C \) satisfy the assumptions (1) and (3) of Claim 2.8. Let \( \equiv \) be the equivalence on \( C \) defined by \( x \equiv y \) if \( x = y \), or if \( x \leq y \) and \( [x, y) \) does not contain an atom of \( B \), or if \( y \leq x \) and \( [y, x) \) does not contain an atom of \( B \). Then the quotient chain \( C \equiv C/\equiv \) is complete and totally disconnected. Let \( \rho \) be the canonical increasing function from \( C \) onto \( C \). Note that if \( x < y \) in \( C \) are such that \( \rho(x) < \rho(y) \) in \( C \), then there is \( a \in \text{At}(B) \) such that \( a \subseteq [x, y) \). This shows that the function \( \phi \) from \( B \) into \( B(C) \) defined by \( \phi(a) = \bigcup \{[\rho(a_{2i}), \rho(a_{2i+1}))/i < l(a)\} \) for \( a = \bigcup \{[a_{2i}, a_{2i+1})i < l(a)\} \) in \( B \subseteq B(C) \), is as required, and satisfies \( \text{At}(\phi[B]) = \text{At}(B(C)) \). That finishes the proof of Proposition 2.6.

**3. Proof of Theorem 1**

3.1. To prove Theorem 1, there is no loss in assuming that (1): \( B \) satisfies both the premises and the conclusions of Proposition 2.6, and (2): \( I(B) = I_{\text{rk}(B)}(B) \) is a maximal ideal of \( B \). We denote by \( \equiv \) the relation on \( C \) defined by \( x \equiv y \) if \( x \leq y \) and there is \( b \in B \), with \( \text{rk}_B(b) < \text{rk}_B(1_B) \), containing \([x, y)\), or \( y \leq x \) and there is \( b \in B \), with \( \text{rk}_B(b) < \text{rk}_B(1_B) \), containing \([y, x)\). We show:

**Lemma 3.2.** Each equivalence class is an interval of \( C \). Let \( b \in B \) be such that \( \text{rk}_B(b) < \text{rk}_B(1_B) \). Then there is a finite subset \( a_0/\equiv, a_1/\equiv, \ldots, a_{n-1}/\equiv \) of equivalence classes such that \( b \subseteq \bigcup \{a_k/\equiv | k < n\} \).

**Proof.** The first part of the claim is trivial. Let us show the second one. Let
$b = \bigcup \{ b_{2i}, b_{2i+1} | i < n \}$. It suffices to show that $b_{2i} \cong b_{2i+1}$ for $i < n$. But this is a trivial consequence of the definition of $\cong$. □

The following two claims are obvious.

Claim 3.3. If $a \in \text{At}(B)$, then $a$ is contained in an equivalence class. □

Claim 3.4. Let $\bar{a}$ be an equivalence class. If $\bar{a}$ has a last element $v$, then $v \notin \text{Pred}(C)$. □

Definition 3.5. Let $C$ be a complete totally disconnected chain, $B$ a superatomic subalgebra of $B(C)$, $\lambda$ an ordinal, and $\psi$ a function from $B$ into $B(\lambda)$. We say that $(B, C, \lambda, \psi)$ is a good system if $\text{At}(B) = \text{At}(B(C))$, $\psi$ is a one-to-one homomorphism from $B$ into the interval algebra $B(\lambda)$, and the restriction $\psi \restriction \text{At}(B)$ of $\psi$ on $\text{At}(B)$ is a one-to-one function from $\text{At}(B)$ onto $\text{At}(B(\lambda))$. We say that there is a good system for $(B, C)$ if there are $\lambda$ and $\psi$ such that $(B, C, \lambda, \psi)$ is a good system.

Note that $\text{At}(\psi(B)) = \psi[\text{At}(B)]$. Equivalently a good system is the $(B, C, \lambda, \psi_0)$, where $\text{At}(B) = \text{At}(B(C))$, and $\psi_0$ is a one-to-one function from $\text{Pred}(C)$ into the chain $\lambda$ such that the function $\psi_0$ from $\text{At}(B(C))$ into $\text{At}(B(\lambda))$ defined by $\psi_0([u, u + 1)) = [\psi_0(u), \psi_0(u + 1)]$ for $u \in \text{Pred}(C)$ can be extended in an embedding $\psi$ from $B$ into $B(\lambda)$.

We prove by induction on $\alpha$, that the following statement $\text{Th}(\alpha)$ holds:

for every chain $C$ and for every superatomic subalgebra $B$ of $B(C)$, such that $\text{rk}(B) \leq \alpha$ and $\text{At}(B) = \text{At}(B(C))$, there is a good system for $(B, C)$.

$\text{Th}(0)$ and $\text{Th}(1)$ hold. Indeed $B$ is isomorphic to the Boolean algebra $F_{\text{C}}(X)$ of finite or cofinite subsets of a set $X$, where $X = \text{At}(B(C))$ (since $I(B) = I_{\text{rk}(B)}(B)$ is a maximal ideal of $B$). Consider $\lambda$ be the (initial) ordinal corresponding to the cardinality of the set $\text{At}(B(C))$. In what follows, we suppose that $\text{rk}(B) \geq 2$.

Claim 3.6. Let $\bar{a}/\text{cong}$ be an equivalence class, and

$$
\bar{a} \overset{\text{def}}{=} ((a/ \cong \cup \{ \inf(a/ \cong) \}) - \{ \max(a/ \cong) \}) = [\inf(a/ \cong), \sup(a/ \cong)].
$$

There is a good system for for $(B \upharpoonright \bar{a}, \bar{a})$.

Proof. By induction. If $\bar{a} = \{a\}$, then it is trivial. Assume $|\bar{a}| \geq 2$. Let $c \in \text{Pred}(\bar{a})$, $\bar{a}^+ = \{ x \in \bar{a} | x > c \}$, and $\bar{a}^- = \{ x \in \bar{a} | x \leq c \}$. Note that $\bar{a}$ is the lexicographic sum $\bar{a}^- + \bar{a}^+$, and $\bar{a}^+$ has a first element (the successor of $c$). If $\bar{a}^-$ has no first element, then we must add one, namely $\inf(a/ \cong)$. Suppose that $(B \upharpoonright \bar{a}^+, \bar{a}^+, \lambda^+, \psi^+)$ and $(B \upharpoonright \bar{a}^-, \bar{a}^-, \lambda^-, \psi^-)$ are good. Let $(B \upharpoonright \bar{a}, \bar{a}, \lambda^+ + \lambda^+, \psi)$, where $\psi$ is defined in the following way: for $b \in B \upharpoonright \bar{a}$, we have $b = (\bar{b}^-, \bar{b}^+) \in B(\bar{a}^-) \times B(\bar{a}^+)$ and we set $\psi(b) = \langle \psi^-(\bar{b}^-), \psi^+(\bar{b}^+) \rangle \in B(\lambda^-) \times B(\lambda^+)$ (that is identified with $B(\lambda^+ + \lambda^-)$). Trivially, $(B \upharpoonright \bar{a}, \bar{a}, \lambda^+ + \lambda^+, \psi)$ is as required. So, it suffices to prove that Claim 3.6, whenever $\bar{a}^+ = \bar{a}$ or $\bar{a}^- = \bar{a}$. We prove the case $\bar{a}^+ = \bar{a}$. The case $\bar{a}^- = \bar{a}$ is similar: but note that if $a/ \cong$ has no first element, then $\bar{a} = (a/ \cong) \cup \{ \inf(a/ \cong) \}$, and the algebras

$$
B \upharpoonright (a/ \cong) \quad \text{and} \quad B \upharpoonright ((a/ \cong) \cup \{ \inf(a/ \cong) \})
$$


are isomorphic. \( \hat{a}^+ = \hat{a} \) satisfies: \( \hat{a} \) has a first element, denoted by \( e \), and for every element \( x \) of \( \hat{a} \), we have \( x \cong e \).

Case 1. \( a \rangle / \cong \) has a last element \( e^+ \). Hence \( e^+ / \cong e \) and \( \hat{a} \leq \{a, e^+\} \) (that is the case of the example which follows from Theorem 1). Let \( b \in B \) be such that \( \hat{a} \leq b \), and \( \text{rk}_{B | b}(1_{B | b}) = \text{rk}_{B}(b) < \text{rk}_{B}(1_{B}) \). Let \( B \upharpoonright \hat{a} \equiv \{c \cap \hat{a} | c \in B\} \). Note that \( B \upharpoonright \hat{a} = (B \upharpoonright b) \upharpoonright \hat{a} \). We regard \( B \upharpoonright \hat{a} \) as a Boolean algebra. By the definition, \( B \upharpoonright \hat{a} \) is a homomorphic image of \( B \upharpoonright b \). From the fact that for every superatomic Boolean algebra \( A \) and every ideal \( I \) of \( A \), we have \( \text{rk}(A/I) \leq \text{rk}(A) \), it follows that \( \text{rk}(B \upharpoonright \hat{a}) \leq \text{rk}(B \upharpoonright b) = \text{rk}_{B}(b) < \text{rk}_{B}(1_{B}) = \text{rk}_{B}(B) \). By the induction hypothesis there is a good system for \( (B \upharpoonright \hat{a}, \hat{a}) \).

Case 2. \( a \rangle / \cong \) has no last element. Hence \( \hat{a} = a \rangle / \cong \). Let \( (e_{\alpha})_{\alpha < \sigma} \) be a stricly increasing sequence, cofinal in \( \hat{a} \). We can suppose that \( e_0 = e \), and \( e_\beta = \text{sup}(e_\alpha | \alpha < \beta) \) for every limit ordinal \( \beta < \sigma \) (because \( \hat{a} \) is relatively complete). Let \( \alpha < \sigma \) be given. Let \( b_\alpha \in B \) be such that \( \text{rk}_{B}(b_\alpha) = \text{rk}_{B}(1_{B}) \) and \( (e_\alpha, e_{\alpha+1}) \subseteq b_\alpha \). We set \( B_\alpha \equiv (B \upharpoonright e_\alpha, e_{\alpha+1}) \). Since Lemma 2.5, we have \( \text{rk}(B_\alpha) \leq \text{rk}(B \upharpoonright b_\alpha) = \text{rk}_{B}(b_\alpha) < \text{rk}_{B}(1_{B}) = \text{rk}(B) \). Applying the induction hypothesis to \( (B_\alpha, (e_\alpha, e_{\alpha+1})) \), there is a good system \( (B_\alpha, (e_\alpha, e_{\alpha+1}), \mu_\alpha, \psi_\alpha) \). Hence \( \psi_\alpha[(e_\alpha, e_{\alpha+1})] = \mu_\alpha \). Let \( \mu = \sum_{\alpha < \sigma} \mu_\alpha \), and \( \psi = \bigcup \{\psi_\alpha | \alpha < \sigma\} \). We have \( \text{Pred}(\mu) = \mu \). We extend \( \psi \) in an one-to-one homomorphism \( \psi \) from \( B \upharpoonright \hat{a} \) into \( B(\mu) \): let \( b \in B \). We set:

\[
\psi(b) = \bigcup \{\mu_\alpha[(e_\alpha, e_{\alpha+1})] \subseteq b \} \cup \{\psi_\alpha(b \cap [e_\alpha, e_{\alpha+1}]) | b \cap [e_\alpha, e_{\alpha+1}] \neq 0_{B_\alpha}, 1_{B_\alpha}\}.
\]

We must remark that \( \psi \) is well defined, because \( b \) is a finite union of half-open intervals and thus \( \{\alpha < \sigma | (e_\alpha, e_{\alpha+1}) \subseteq b \} \in B(\sigma) \), and the set \( \{\alpha < \sigma | b \cap [e_\alpha, e_{\alpha+1}] \neq 0_{B_\alpha}, 1_{B_\alpha}\} \) is finite. Consequently \( \psi(b) \) is a finite union of half-open intervals of \( \mu \), and thus \( (B \upharpoonright \hat{a}, \hat{a}, \psi, \psi) \) is a good system. That finishes the proof of Claim 3.6. \( \square \)

End of the proof of Theorem 1. Let \( (a\zeta \rangle / \cong)_{\zeta \lessdot \theta} \) be an enumeration of the set of equivalence classes. By Claim 3.6, for \( \zeta < \theta \), let \( (B \upharpoonright \hat{a}\zeta, \hat{a}\zeta, \lambda\zeta, \psi_{\hat{a}\zeta}) \) be a good system. Let \( \lambda = \sum_{\zeta \lessdot \theta} \lambda_{\hat{a}\zeta} \). Each \( \lambda_{\hat{a}\zeta} \) is an interval of \( \lambda \). Now, let \( \psi \) be the function from \( \text{At}(B) \) into \( \lambda \) defined by \( \psi(a) = \psi_{\hat{a}\zeta}(a) \) where \( \hat{a}\zeta \) is the unique class such that \( a \in \text{At}(B) \cap B \upharpoonright \hat{a}\zeta \). Let \( b \in B \). First, suppose that \( \text{rk}_{B}(b) < \text{rk}_{B}(1_{B}) \). There is a finite subset \( \{\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_{n-1}\} \) of equivalence classes such that \( b \subseteq \bigcup_{\hat{a}_k | k < n} \), follows from Lemma 3.2. We set \( \psi(b) = \bigcup \{\psi_{\hat{a}_k}(b \cap \hat{a}_k) | k < n\} \). Now, because \( I(B) = I_{\text{rk}_{B}(B)}(B) \) is a maximal ideal of \( B \), if \( \text{rk}_{B}(b) = \text{rk}_{B}(1_{B}) \), then \( \text{rk}_{B}(-b) < \text{rk}_{B}(1_{B}) \), and we set \( \psi(b) = -\psi(-b) \). The fact that \( \psi \) is a one-to-one homomorphism from \( B \) into \( B(\lambda) \) is a consequence of the following obvious result:

Claim 3.7. Let \( B' \) and \( B'' \) be two atomic algebras and \( \psi \) be a one-to-one function from \( \text{At}(B') \) onto \( \text{At}(B'') \). We suppose that, for each \( b \in B' \), there is an unique element of \( B'' \), denoted by \( \psi(b) \), such that for every \( a \in \text{At}(B') \), we have \( a \subseteq b \) if and only if \( \psi(a) \subseteq \psi(b) \). Then \( \psi \) is a one-to-one homomorphism from \( B' \) into \( B'' \), extending \( \psi \). This finishes the proof of Theorem 1. \( \square \)
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