Aronszajn trees

Uri Abraham
Department of Mathematics
Ben-Gurion University

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Abstract

This survey describes the contribution of N. Aronszajn to set theory: the construction of an Aronszajn tree. We would like to explain to the non-specialist in set theory what is an Aronszajn tree, how it is constructed, and why it occupies such a central place in set theory. In fact, we describe two constructions: the classical one, and a newer construction of a canonical Aronszajn tree due to Stevo Todorcevic.

1 Introduction

It is easy to describe the contribution of N. Aronszajn to set theory since it consists of a single article: the construction of what is now known as an Aronszajn tree. A rather simple construction by today's standards which is often taught in undergraduate set theory courses. Yet by some reason which we will try to explain, the notion of an Aronszajn tree is of prime importance in advanced set theory and it continues to contribute in essential ways to its development. (This contribution of Aronszajn was never published in a paper, and the closest reference is an article of Kurepa [**?**].)

The best way to introduce the notion of an Aronszajn tree is by contrasting it with the D. König's Lemma which says that any tree has an infinite branch if has infinitely many levels and each one is finite. This lemma is essentially about ω, the set of natural numbers, and an Aronszajn tree can be seen as a counterexample to the proposition that the lemma can be extended to ω_1, the first uncountable cardinal. It is this tension between the compactness at ω and the incompactness at ω_1, discovered by N. Aronszajn, that makes Aronszajn's construction and vision so important in set theory. In order to explain this, we must define first some basic concepts (such as trees). This will be done in the rest of our introduction. Then, in the second and third sections we will present two proofs of the existence of an Aronszajn tree. The second proof, due to S. Todorcevic, exemplify some of the method of today's combinatorial set theory.

The term "tree" often refers to graphs without cycles, but in set theory parlance (and in this paper) it refers to a certain type of partial orderings. So, in order to avoid any possible confusion, we shall begin with a definition of that notion. A *tree* is a partially ordered set $T = (A, <)$ so that for every node $a \in A$ the set of its predecessors $\{x \in A \mid x < a\}$ is well-ordered. So the set of predecessors of any node a is order isomorphic to some ordinal, denoted $ht(a)$, which is said to be the height of a. We often write $a \in T$ rather than $a \in A$. For any ordinal γ, the set of all nodes of T of height γ is denoted T_γ. For example, T_0 is the first level of the tree, consisting of all minimal members of T, T_n, for $n \in \omega$ is the set of nodes that have exactly n predecessors, etc. It is the costume to add to the definition of a tree the requirement that its first level consists of a single node, the root of the tree. So the root r of a tree T is such that $r < x$ for all other nodes x of the tree. The height of a tree T is the least ordinal η so that T has no nodes of height η. So $T_\eta = \emptyset$ but $T_\alpha \neq \emptyset$ for every $\alpha < \eta$. For example, a tree of height ω is one that has nodes at the n-th level for every finite n but no node at an infinite level.

Now a linearly ordered subset of a tree that is downwards closed is called a *branch*. That is, a branch b of a tree T is a subset of T such that for all $x, y \in b$ if $x \neq y$ then $x < y$ or $y < x$, and if $x < y \in b$ then $x \in b$ as well. A branch b can be bounded, if for some $x \in T$ all members of b are below x, or unbounded. König's lemma can be reformulated in terms of trees as follows.

Lemma 1.1 (König) *Any infinite tree has an infinite branch if all its levels are finite.*

A natural question arises: can this lemma be extended by replacing ω with ω_1? Namely, is it true that every tree of size ω_1 has an uncountable branch if all its levels are countable? Aronszajn answered this question in the negative by proving the existence of a tree T of height ω_1 with all levels countable and such that T has no uncountable branches. Such a tree is known today as an

Aronszajn tree. In today's terminology, what N. Aronszajn constructed was a *special* tree. For this notion we need the definition of an antichain. For a given tree T, we say that $X \subset T$ is an antichain if no two members of X are comparable in the ordering of the tree. That is, for every distinct $u, v \in X$ we have $u \not< v$ and $v \not< u$. A tree T is said to be *special* if and only if it is a countable union of antichains, that is there exist antichains A_i for $i \in \omega$ such that $T = \bigcup_{i\in\omega} A_i$. In this case the function taking $x \in T$ to the first i such that $x \in A_i$ is one-to-one on chains. This gives an equivalent definition: a tree is special iff there is a function f from the tree into ω that is one-to-one on chains. That is $x < y$ in the tree implies that $f(x) \neq f(y)$. We note in the following lemma that such a tree has no uncountable branch. The tree originally constructed by N. Aronszajn is a special tree.

Lemma 1.2 *A special tree has no uncountable branch.*

Proof. Suppose T is special and $T = \bigcup_{i\in\omega} A_i$ is its representation as a countable union of antichains. If $b \subseteq T$ is any branch, then $b \cap A_i$ is a singleton since A_i is an antichain. Hence, b is countable. q.e.d

We end this introduction with some notational remarks concerning the ordinals. An ordinal represents a well-order type. In standard development of set-theory, any ordinal α is in fact the set of all ordinals of smaller order-type. In fact the membership relation $\in$ well-orders any ordinal, and we write either $\alpha \in \beta$ or $\alpha < \beta$ to denote the fact that α is smaller than β. We say that a set A with a well-ordering $<_A$ of A has order-type α (an ordinal) iff $(A, <_A)$ is order isomorphic to α with the membership ordering on α. An ordinal β is a successor ordinal if its well-ordering has a maximal member α. In this case we write $\beta = \alpha + 1$ and say that β is the successor of α (which is its predecessor). An ordinal β (greater than 0) is a limit ordinal if it is not a successor. An ordinal that is not equinumerous with a smaller ordinal is said to be a cardinal. If α is an ordinal then $A \subseteq \alpha$ is unbounded iff for all $x \in \alpha$ there exists $y \in A$ such that $x \leq y$. We also say in this case that A is cofinal in α. In case α is a successor ordinal, it suffices that A contains its predecessor in order to be unbounded, but the interesting case is to consider unbounded subsets of limit ordinals. The minimal order-type of an unbounded subset of α is called the cofinality of α. A countable limit ordinal has cofinality ω. An ordinal α that has cofinality α is said to be regular. It is necessarily a cardinal. The first uncountable ordinal ω_1 for example is regular.

2 Constructions of Aronszajn trees

It is easy to obtain a tree of height ω_1 with no uncountable branches: the tree consisting of all one-to-one countable functions from a countable ordinal into ω with the function extension ordering is such a tree. It is not an Aronszajn tree of course since already its ω level is uncountable (there are continuum many permutations of ω).

We begin by proving Aronszajn's theorem that a special Aronszajn tree exists.

Lemma 2.1 *There exists a special Aronszajn tree. That is, a tree of height ω_1 with all levels countable and with a decomposition into countably many antichains.*

Proof. By (transfinite) induction we define for every countable ordinal α a tree $T(\alpha)$ and a function $f(\alpha)$ from $T(\alpha)$ into ω. The following properties are required to hold for α:

1. The height of the tree $T(\alpha)$ is $\alpha + 1$ (the successor of α), and each of its levels is countable. The fact that the height of $T(\alpha)$ is a successor means that it has a "last" level. The function $f(\alpha)$ is such that the pre-image of $\{i\}$ (for every $i \in \omega$) is an antichain in $T(\alpha)$. We think of $T(\alpha)$ as the initial segment of the Aronszajn tree that we are about to construct, and $f(\alpha)$ is the corresponding specialization of that initial part.

2. For every $\eta < \alpha$, $T(\alpha)$ is an end-extension of $T(\eta)$. By this we mean that $T(\eta)$ is the initial segment of $T(\alpha)$ determined by its first $\eta + 1$ levels. (Namely $T(\eta) \subset T(\alpha)$, the ordering of $T(\eta)$ agrees with that of $T(\alpha)$ on the nodes of $T(\eta)$, and if $x \in T(\alpha)$ has height $\leq \eta$, then $x \in T(\eta)$.)

3. For every $\eta < \alpha$ the function $f(\eta)$ is the restriction of $f(\alpha)$ to $T(\eta)$.

4. For every node $y \in T(\alpha)$, let $b_y = \{x \in T(\alpha) \mid x \leq y\}$ be the branch consisting of y and its predecessors. Then the image of $f(\alpha)$ on b_y is co-infinite. That is, there are infinitely many $n \in \omega$ that are not in that image.

5. For every node $y \in T(\alpha)$ such that $ht(y) < \alpha$ and for any finite set $E \subset \omega$ disjoint to the image under $f(\alpha)$ of b_y there is a node z above y

and of height α in $T(\alpha)$ such that the image under $f(\alpha)$ of b_z is disjoint to E. (Hence there is such a node z of hight α' when $ht(y) < \alpha' < \alpha$.)

6. For every node $y \in T(\alpha)$ such that $ht(y) < \alpha$ and for every $n \in \omega$ not in the image of b_y under $f(\alpha)$ there is an immediate successor z of s such that $f(\alpha)(z) = n$.

Item **??** is the most intriguing and is the one that contains the main idea of the proof. We shall see first that if this inductive construction can be carried on, then the resulting tree $T(\omega_1) = \bigcup_{\alpha \in \omega_1} T(\alpha)$ with the corresponding ordering is indeed a special Aronszajn tree. But this is clear, since items **??** and **??** ensure that the resulting tree is special of height ω_1, and the fact that $T(\alpha_1)$ is end-extended by $T(\alpha_2)$ when $\alpha_1 < \alpha_2$ ensures that all levels of $T(\omega_1)$ are countable.

We start with $T(0)$, a tree of height 1. For definiteness assume that it consists of a single node r which is going to be the root of the tree, and let $f(0)(r) \in \omega$ be arbitrarily chosen.

Let α be a countable ordinal. Assume that a sequence of trees $T(\alpha')$ and functions $f(\alpha')$ is given for $\alpha' < \alpha$ so that properties **??** to **??** hold, and we shall define $T(\alpha)$ and $f(\alpha)$. There are two cases: α a successor ordinal and α a limit. The successor case is easy. If α is the successor of α_0 and $T(\alpha_0)$ is defined then we obtain $T(\alpha)$ by adding to each node y of level α_0 in $T(\alpha_0)$ ω successors and allowing all possible values for the $f(\alpha)$ function on the new nodes. The fact that the range of $f(\alpha_0)$ on any b_y is not all of ω allows us to find a "free" value for the new nodes and to keep the requirement that the specializing function is one-to-one on branches.

The interesting case is when α is a limit (countable) ordinal. Let $\langle \alpha_i \mid i \in \omega \rangle$ be an increasing sequence cofinal in α. Let $T = \bigcup_{i \in \omega} T(\alpha_i)$ be the resulting tree formed as the union of all the trees along the cofinal sequence, and let $f = \bigcup_{i \in \omega} f(\alpha_i)$ be the specializing function. Then T is an end-extension of each $T(\alpha_i)$ and hence of each $T(\eta)$ for $\eta < \alpha$. Similarly, f extends all the functions $f(\alpha_i)$. We must add a last level, the α-th level, to T in order to obtain $T(\alpha)$, and extend f over that new level (in order to define $f(\alpha)$). We must define this α-th level and the function $f(\alpha)$ so that the inductive properties 4 and 5 hold (the other properties are evident).

A branch of T of order-type α is said to be an α branch. Such a branch contains nodes of every level $< \alpha$. We shall define a countable family B of α branches of T, and then define the α level of the tree $T(\alpha)$ by assigning a node above each branch in B.

For every node $y \in T$, and for every finite set $E \subset \omega$ disjoint to the image of b_y under f, we shall build a branch $b = b(y, E)$ that contains y and so that the image of b under f is co-infinite and disjoint to E. Viewing the pair (y, E) as a "mission" we can arrange all missions in a countable sequence, and deal with each one in turn. Given a mission (y, E) we shall define the branch $b(y, E)$ so that it differs from each of the (finitely many) branches constructed so far (to simplify the definition). Let $i_0 \in \omega$ be the first so that $ht(y) < \alpha_{i_0}$. We shall define an increasing sequence of nodes above y $\langle y_i \mid i \geq i_0 \rangle$, where y_i is of height α_i, and a sequence of finite subsets of ω, $\langle E_i \mid i \geq i_0 \rangle$ with $E \subset E_i \subset E_{i+1}$ so that the range of b_{y_i} under f is disjoint to E_i. There is no problem in constructing such a sequence using item **??** at each step. The resulting branch b is as required: its image under f is co-infinite since more and more values were excluded from this range. Namely, $\bigcup_{i \geq i_0} E_i$ is infinite and disjoint to the range of f applied to b.

Let B be the collection of all branches $b(y, E)$ constructed in this fashion, where $y \in T$ and E a finite subset of ω disjoint to the image of b_y under f. For every $b \in B$ add a new point to T that is above all the nodes of b. Together with the tree T this yields $T(\alpha)$. The specializing function $f(\alpha)$ is defined by extending f over these new points. For every new point x added above the branch b, we have $b = b(y, E)$ for some uniquely determined (y, E), and we define $f(\alpha)(x) \in \omega \setminus E$ as any value outside of the range of f over b (for example, pick the first value in this infinite set). Check now that all the inductive properties hold. q.e.d

2.1 A canonical construction of a special Aronszajn tree

In this subsection we present another construction of a special Aronszajn tree which is radically different from the former construction in that no induction is involved: the tree is defined "at once". This new proof is due to S. Todorcevic and it employs his miraculous theory of "walks over ordinals" ([**?**]). A theory that has allowed Todorcevic and other researchers to resolved many open problems and which continues to be a source of deep investigation. In presenting part of this theory here we restrict our attention to ω_1, but the theory applies to infinite successor cardinals in general.

Fix a sequence $\langle C_\alpha \mid 0 < \alpha \in \omega_1 \rangle$ such that, for every countable $\alpha > 0$,

1. C_α is an unbounded subset of α of order-type ω if α is a limit ordinal,

and

2. $C_\alpha = \{\alpha_0\}$ if α is the successor of α_0.

We say that $\langle C_\alpha \mid 0 < \alpha \in \omega_1 \rangle$ is a ladder system. Intuitively, a ladder system enables a short access to the countable ordinals: if $\alpha = \alpha_0 + 1$ is a successor ordinal then C_α is the singleton consisting of the maximal ordinal below α, but if α is a limit ordinal then C_α is a cofinal subset of α of the shortest possible length.

Let Q be the set of all finite sequences of members of ω. So $a \in Q$ iff for some $n \in \omega$ (called the length of a), a is a function from $\{0, \ldots, n-1\}$ into ω. We write $a = \langle a_0, \ldots, a_{n-1} \rangle$. If $a, b \in Q$ then $a^\frown b$ denotes the concatenation of a and b.

A lexicographical ordering $\lhd$ is defined on Q as follows. For $a, b \in Q$ where $a = \langle a_0, \ldots, a_{n-1} \rangle$ and $b = \langle b_0, \ldots, b_{m-1} \rangle$ we define $a \lhd b$ iff either

1. $n > m$ and for every $i < m$ we have $a_i = b_i$, or else

2. for some $i < \min(n, m)$, $a_i \neq b_i$ and for the least such i we have that $a_i < b_i$.

Observe that any sequence in Q is smaller than all of its proper initial segments; this is natural in our context. So the empty set is the maximum of Q. Omitting the empty set, we get a countable dense set without first or last element, and hence an ordering isomorphic to the rational.

We now define the walk from β down to α for every $\alpha \leq \beta < \omega_1$: $\mathrm{walk}(\alpha, \beta) = \langle \beta_0, \ldots, \beta_{n-1} \rangle$ so that $\beta_0 = \beta$, $\beta_0 > \beta_1 \ldots$ and $\beta_{n-1} = \alpha$. This walk will be instrumental in defining the trace of the walk, $\mathrm{trace}(\alpha, \beta)$ and its shadow, the famous $\rho_0(\alpha, \beta) \in Q$.

The definition of the ordinals $\beta_i \geq \alpha$ that constitute the walk from β down to α is by the following procedure. We start with $\beta_0 = \beta$. Suppose that $\beta_0 > \cdots > \beta_i$ have been defined. If $\beta_i = \alpha$ the procedure stops, but if $\beta_i > \alpha$ let β_{i+1} be the first ordinal in C_{β_i} that is not below α (there is such an ordinal since C_{β_i} is unbounded in β_i). In case β_i is a successor ordinal, β_{i+1} is the predecessor of β_i. The procedure must stop since there is no infinite descending sequence of ordinals, and hence we must arrive to some $\beta_i = \alpha$. The ordinals β_i are said to be "on the walk".

If $\mathrm{walk}(\alpha, \beta) = \langle \beta_0, \ldots, \beta_{n-1} \rangle$ then (β_0, β_1) is the "first step", (β_1, β_2) the second step etc. So there are $n - 1$ steps and the trace of the walk is the sequence of length $n - 1$ of finite subsets of α defined as follows.

$$\text{trace}(\alpha,\beta) = \langle C_{\beta_i} \cap \alpha \mid 0 \leq i < n-1 \rangle. \tag{1}$$

If we collect the cardinalities of the finite sets appearing in the trace we obtain $\rho_0(\alpha, \beta)$:

$$\rho_0(\alpha,\beta) = \langle |C_{\beta_i} \cap \alpha| \mid 0 \leq i < n-1 \rangle.$$

Notice that for $i < n-1$, $\beta_i > \alpha$ so that $C_{\beta_i} \cap \alpha$ is finite and its cardinality $|C_{\beta_i} \cap \alpha|$ is in ω. So that $\rho_0(\alpha, \beta) \in Q$, namely it is a sequence of length $n-1$ of natural numbers.

A recursive definition of the walk, its trace and rho functions can be of help in a formal proof because it allows induction.

$$\text{walk}(\beta,\beta) = \langle \beta \rangle.$$

Correspondingly, in no step we get from β to β, $\text{trace}(\beta,\beta) = \emptyset$, and $\rho_0(\beta,\beta) = \emptyset$. For $\alpha < \beta$,

$$\text{walk}(\alpha,\beta) = \langle \beta \rangle^\frown \text{walk}(\alpha, \min(C_\beta \setminus \alpha)).$$

Correspondingly

$$\text{trace}(\alpha,\beta) = \langle C_\beta \cap \alpha \rangle^\frown \text{trace}(\alpha, \min(C_\beta \setminus \alpha)). \tag{2}$$

$$\rho_0(\alpha,\beta) = \langle |C_\beta \cap \alpha| \rangle^\frown \rho_0(\alpha, \min C_\beta \setminus \alpha)). \tag{3}$$

We first explain the notation employed in this definition, and then we shall clarify the definition itself. Here, $\langle \beta \rangle$ is the sequence of length 1 whose sole member is β. Assume that $\alpha < \beta$. Viewing the ordinal α as the set of all smaller ordinals, $C_\beta \setminus \alpha$ is the subset of C_β consisting of those members that are not below α. If $\alpha \in C_\beta$ then the first member of $C_\beta \setminus \alpha$ is α itself, but otherwise $\min(C_\beta \setminus \alpha)$ is the first member of C_β that is strictly above α. Anyhow, $C_\beta \cap \alpha$, which is the set of all members of C_β that are below α, is a finite set. If β is a successor ordinal then this set is empty, and if β is a limit ordinal then $C_\beta \cap \alpha$ is finite because C_β is of order type ω and unbounded in β and $\alpha < \beta$.

Suppose that $\text{walk}(\alpha, \beta) = \beta_0 > \cdots > \beta n - 1$. Then for every $0 \leq i \leq j \leq n-1$ we have $\text{walk}(\beta_j, \beta_i) = \beta_i > \cdots > \beta_j$. We rephrase this in the following lemma.

Lemma 2.2 *Suppose that $\alpha \le \beta \le \gamma$ are countable ordinals and β is on the walk from γ to α. Then* $\mathrm{walk}(\beta,\gamma)$ *and* $\mathrm{walk}(\alpha,\beta)$ *are an initial segment and (respectively) a final segment of* $\mathrm{walk}(\alpha,\gamma)$.

Proof. The proof is by induction on γ. If $\alpha = \gamma$ the proof is obvious. If $\alpha < \gamma$, then we have $\mathrm{walk}(\alpha,\gamma) = \langle\gamma\rangle^\frown \mathrm{walk}(\alpha, \min(C_\gamma \setminus \alpha))$. Case $\beta = \gamma$ is obvious, but otherwise β is on the walk from $\min(C_\gamma \setminus \alpha))$ to α and the inductive hypothesis can be used. q.e.d

Suppose now that $\delta \le \beta$ are countable ordinals and δ is a limit ordinal. Then $\mathrm{trace}(\delta,\beta)$ is bounded in δ. That is, for some $\alpha < \delta$ all sequences of $\mathrm{trace}(\delta,\beta)$ are contained in α. In this case, the following lemma applies.

Lemma 2.3 *Suppose that $\alpha \le \delta \le \beta$ and* $\mathrm{trace}(\delta,\beta)$ *is contained in α. Then δ is on the walk from β to α, and* $\mathrm{trace}(\alpha,\beta) = \mathrm{trace}(\delta,\beta)^\frown \mathrm{trace}(\alpha,\delta)$.

The proof that δ is on the walk from β to α is by induction on β. The previous lemma then yields the decomposition of $\mathrm{trace}(\alpha,\beta)$ via δ q.e.d

For every $\beta \in \omega_1$ let $\lambda x \rho_0(x,\beta)$ be the function defined on $\{x \mid x \le \beta\}$ which takes x to $\rho_0(x,\beta) \in Q$.

Lemma 2.4 *For every $\beta \in \omega_1$, $\lambda x \rho_0(x,\beta)$ is order preserving from β into Q. That is, if $\alpha_1 < \alpha_2 \le \beta$ then $\rho_0(\alpha_1,\beta) \lhd \rho_0(\alpha_2,\beta)$.*

Proof. The proof is by induction on β. If $\alpha_2 = \beta$ then $\rho_0(\alpha_2,\beta) = \emptyset$, but $\rho_0(\alpha_1,\beta) \ne \emptyset$ and so the lemma follows in this case, because $\emptyset$ is the maximum of Q. Assume now that $\alpha_2 < \beta$. Consider the interval of ordinals $[\alpha_1,\alpha_2) = \{\xi \mid \alpha_1 \le \xi < \alpha_2\}$ and its intersection with C_β. There are two cases:

1. If $[\alpha_1,\alpha_2) \cap C_\beta = \emptyset$, then the first step (β,β_1) in the walk from β to α_1 is the same as the first step in the walk from β to α_2. The inductive assumption can be applied to $\alpha_1 < \alpha_2 \le \beta_1$ and the lemma can be concluded in this case. Namely, the induction yields $\rho_0(\alpha_1,\beta_1) \lhd \rho_0(\alpha_2,\beta_1)$, but then adding $|C_\beta \cap \alpha_1| = |C_\beta \cap \alpha_2|$ to the left of both sequences retains the ordering.

2. Otherwise, $[\alpha_1,\alpha_2) \cap C_\beta \ne \emptyset$. This implies that $C_\beta \cap \alpha_1$ has fewer members than $C_\beta \cap \alpha_2$, and hence that $\rho_0(\alpha_1,\beta) \lhd \rho_0(\alpha_2,\beta)$. q.e.d

Lemma 2.5 *Suppose that* $\alpha_1 \leq \beta^1$ *and* $\alpha_1 \leq \beta^2$ *are countable ordinals such that* $\text{trace}(\alpha_1, \beta^1) = \text{trace}(\alpha_1, \beta^2)$*. Then for every* $\alpha_0 \leq \alpha_1$ *we also have* $\text{trace}(\alpha_0, \beta^1) = \text{trace}(\alpha_0, \beta^2)$.

Hence, if $\text{trace}(\alpha_1, \beta^1) = \text{trace}(\alpha_1, \beta^2)$ *then* $\lambda x \rho_0(x, \beta^1) \restriction \alpha_1 = \lambda x \rho_0(x, \beta^2) \restriction \alpha_1$.

Proof. The lemma is proved by induction on $\max\{\beta^1, \beta^2\}$. If $\alpha_1 = \beta^1$ then $\text{trace}(\alpha_1, \beta^1) = \emptyset$ and hence $\text{trace}(\alpha_1, \beta^2) = \emptyset$, which implies that $\alpha_1 = \beta^2$ as well. So we may assume that $\alpha_1 < \beta^1, \beta^2$.

Let $\beta^1 > \beta^1_1$ be the first step in the walk from β^1 to α_1, and let $\beta^2 > \beta^2_1$ be the first step in the walk from β^2 to α_1. That is, $\beta^1_1 = \min(C_{\beta^1} \setminus \alpha_1)$, and $\beta^2_1 = \min(C_{\beta^2} \setminus \alpha_1)$.

It follows from the recursive definition of the trace formula (**??**) that

$$\text{trace}(\alpha_1, \beta^1) = \langle C_{\beta^1} \cap \alpha_1 \rangle^\frown \text{trace}(\alpha_1, \beta^1_1).$$

$$\text{trace}(\alpha_1, \beta^2) = \langle C_{\beta^2} \cap \alpha_1 \rangle^\frown \text{trace}(\alpha_1, \beta^2_1).$$

Since these two traces are assumed to be equal, we get $C_{\beta^1} \cap \alpha_1 = C_{\beta^2} \cap \alpha_1$, and $\text{trace}(\alpha_1, \beta^2_1) = \text{trace}(\alpha_1, \beta^1_1)$. Since $\max\{\beta^1_1, \beta^2_1\} < \max\{\beta^1, \beta^2\}$, the inductive assumption implies that $\text{trace}(\alpha_0, \beta^2_1) = \text{trace}(\alpha_0, \beta^1_1)$. Let T_0 denote this common trace, and let X denote the common value $X = C_{\beta^1} \cap \alpha_1 = C_{\beta^2} \cap \alpha_1$. There are two cases now.

1. $X \cap [\alpha_0, \alpha_1) = \emptyset$. That is, $X \subseteq \alpha_0$. It follows that $C_{\beta^1} \cap \alpha_0 = C^2_\beta \cap \alpha_0 = X$, and the walks from β^2 and β^1 to α_0 lead to β^2_1 and to β^1_1 (respectively), and then $\text{trace}(\alpha_0, \beta^1) = \text{trace}(\alpha_0, \beta^2) = \langle X \rangle^\frown T_0$ follows as desired.

2. $X \cap [\alpha_0, \alpha_1) \neq \emptyset$ and we let β_1 be the first ordinal in this intersection. Then the first steps in the walks from both β^1 and β^2 to α_0 lead to β_1 (and continue from that place in unison). So that the conclusion of the lemma is obvious. q.e.d

The definition of the special Aronszajn tree of Todorcevic can now be given by the following formula:

$$T = \{\lambda x \rho_0(x, \beta) \restriction \alpha \mid \alpha \leq \beta < \omega_1\}.$$

The nodes of this tree are order preserving functions from countable ordinals into Q. The previous lemma implies immediately that every level of T is countable. Namely

$$T_\alpha = \{\lambda x \rho_0(x,\beta) \restriction \alpha \mid \alpha \leq \beta < \omega_1\}$$

is countable by Lemma **??** since the number of possible traces is countable. Since the nodes of T are order preserving functions into Q, T is clearly an Aronszajn tree. In fact, it is a special Aronszajn tree. To prove this fact, we define a specializing map $\tau : \bigcup\{T_\delta \mid \delta < \omega_1 \text{ limit}\} \to Q$ that is defined on the limit levels of the tree (a standard argument then shows that the tree is special). Given a node $t \in T_\alpha$, where α is a limit ordinal, we find $\beta \geq \alpha$ such that $t = \lambda x \rho_0(x,\beta) \restriction \alpha$. We define $\tau(t) = \rho_0(\alpha,\beta)$. We must prove that τ is well defined (its definition does not depend on the choice of β) and that τ is order-preserving.

So suppose that β_1 is another ordinal such that $t = \lambda x \rho_0(x,\beta_1) \restriction \alpha$. We prove $\rho_0(\alpha,\beta_1) = \rho_0(\alpha,\beta)$ by Lemma **??**. In details the argument runs as follows. Since α is a limit ordinal, we can pick an ordinal $\mu < \alpha$ that bounds both $\text{trace}(\alpha,\beta)$ and $\text{trace}(\alpha,\beta_1)$. Then Lemma **??** implies that $\text{trace}(\mu,\beta) = \text{trace}(\alpha,\beta)^\frown\text{trace}(\mu,\alpha)$, and $\text{trace}(\mu,\beta_1) = \text{trace}(\alpha,\beta_1)^\frown\text{trace}(\mu,\alpha)$. Since $\rho_0(\mu,\beta) = \rho_0(\mu,\beta_1)$, we get $\rho_0(\alpha,\beta) = \rho_0(\alpha,\beta_1)$ as required.

The fact that τ is order preserving is a direct consequence of Lemma **??**.

3 Todorčević theorem

Theorem 3.1 (Todorcevic) $\omega_1 \to [\omega_1]^2_{\omega_1}$. *That is, there exists a function $f : [\omega_1]2 \to \omega_1$ such that for every uncountable $A \subseteq \omega_1$ $f[A] = \omega_1$.*

In fact, we will find a function $g : [\omega_1]^2 \to \omega_1$ such that, for every $A \in [\omega_1]^{\aleph_1}$, $g[A]$ contains a club set. This entails the theorem by partitioning $\omega_1 = \bigcup_{i\in\omega_1} S_i$ into ω_1 pairwise disjoint stationary sets, and then defining $f(a,b) = i$ iff $g(a,b) \in S_i$.

Let T be a special Aronszajn tree, with $s : T \to \omega$ as its specializing function (one-to-one on chains). Let $\{a_\alpha \mid \alpha \in \omega_1\}$ be an uncountable antichain in T such that for $\alpha < \beta$ the level of α is below the level of β. Given $\alpha < \beta$ define $g(\alpha,\beta)$ as follows. Let $p \in T$ be the splitting point of a_α and a_β and suppose that $s(p) = k$. Let x be the first point below a_β (if there is any) such that the level of x is above the level of a_α and $s(x) < k$. Let

δ be the level of x on the tree, and define $g(\alpha, \beta) = \delta$ (an arbitrary value is chosen if there is no such x).

Let $A \subseteq \omega_1$ be an arbitrary uncountable set. We must prove that $g[A]$ meets every stationary set S. For this, take $M \prec H(\aleph_2)$ be a countable elementary substructure containing T, s, and A and such that $M \cap \omega_1 = \delta \in S$. Pick some $\beta \in A$ with $\beta > \delta$. Then $a_\beta \notin M$ and the level of a_β is above δ. Let $x < a_\beta$ be that node at level δ below a_β, and suppose that $s(x) = n$. We shall find $a_\alpha \in M$ such that $g(\alpha, \beta) = \delta$. On the branch determined by x, $\{y \in T \mid y < x\}$, the function s is one-to-one and there is a finite number of y's with $s(y) < n$. Let $y_0 < x$ be above all of those y's. Since $a_\beta \in A$ is above y_0 and is not in M, the set $\{a \in A \mid y_0 < a\} \in M$ is uncountable. Let's say that $y \in T$ is "heavy" if $y_0 \leq y$ and $\{a \in A \mid y < a\} \in M$ is uncountable. If y is heavy and $y_0 \leq y' \leq y$ then y' is heavy too. The collection of heavy nodes is in M and is uncountable (because every level of T is uncountable). Since there are no uncountable chains in T we can find two heavy nodes that are incomparable and hence there is a heavy node incomparable with x and we can have it in M. So let $y > y_0$ be some heavy node in M and so that y and x split, and let p be their splitting node (which is evidently the splitting node of y and a_β as well). Say $s(p) = k$. By the choice of y_0 (and as $p < x$) $k > n$. Now as y is heavy, we can find $a_\alpha \in A$ above y and in M such that the level of a_α is above all the (finitely many) nodes below x whose s value is below k. It is now easy to check that x is the first point below a_β such that the level of x is above the level of a_α and $s(x) < k$, and thus $g(\alpha, \beta) = \delta$ as required. q.e.d

Concluding remarks. The bibliography reported below is a partial list of papers that either directly investigate Aronszajn trees or employ this notion in substantial ways. The reader can get an effective impression that this notion continues to be an important driving force in set theory since the days of its invention by N. Aronszajn. The interest in Aronszajn trees stem from the tension between the compactness and incompactness phenomenon. Generalizing the question on Aronszajn trees to arbitrary cardinals, [**?**] defines that an infinite cardinal κ has the *tree property* iff every tree of height κ and with all levels of size $< \kappa$ necessarily has a branch of length κ. So, for example, ω has the tree property but $\aleph_1$ has not. Investigating the question of which uncountable regular cardinals have the tree property is an important direction and a continuing challenge in set theory which is directly connected with Aronszajn's work. (See [**?**] for more information on large cardinals and weakly compact cardinals.)

Investigations of the combinatorics of ω_1 is a main issue in set theory and Aronszajn trees and their linearizations are instrumental in these investigations (see for example recent work of J. T. Moore).

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