The MST of Symmetric Disk Graphs (in Arbitrary Metric Spaces) is Light

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Abstract. Consider an n-point metric space $M = (V, \delta)$, and a transmission range assignment $r : V \to \mathbb{R}^+$ that maps each point $v \in V$ to the disk of radius $r(v)$ around it. The *symmetric disk graph* (henceforth, SDG) that corresponds to M and r is the undirected graph over V whose edge set includes an edge (u, v) if both $r(u)$ and $r(v)$ are no smaller than $\delta(u, v)$. SDGs are often used to model wireless communication networks.

Abu-Affash, Aschner, Carmi and Katz (SWAT 2010, [1]) showed that for any n-point *2-dimensional Euclidean* space M, the weight of the MST of every connected SDG for M is $O(\log n) \cdot w(MST(M))$, and that this bound is tight. However, the upper bound proof of [1] relies heavily on basic geometric properties of constant-dimensional Euclidean spaces, and does not extend to Euclidean spaces of super-constant dimension. A natural question that arises is whether this surprising upper bound of [1] can be generalized for wider families of metric spaces, such as high-dimensional Euclidean spaces.

In this paper we generalize the upper bound of Abu-Affash et al. [1] for Euclidean spaces of any dimension. Furthermore, our upper bound extends to *arbitrary metric spaces* and, in particular, it applies to any of the normed spaces ℓ_p. Specifically, we demonstrate that for *any* n-point metric space M, the weight of the MST of every connected SDG for M is $O(\log n) \cdot w(MST(M))$.

1 Introduction

1.1 The MST of Symmetric Disk Graphs

Consider a network that is represented as an (undirected) weighted graph $G = (V, E, w)$, and assume that we want to compute a spanning tree for G of *small weight*, i.e., of weight $w(G)$ that is close to the weight $w(MST(G))$ of the minimum spanning tree (MST) $MST(G)$ of G. (The weight of a graph $\mathcal{G}$, denoted $w(\mathcal{G})$, is defined as the sum of all edge weights in it.) However, due to some physical constraints (e.g., network faults) we are only given a connected spanning subgraph G' of G, rather than G itself. In this situation it is natural to use

* This research has been supported by the Clore Fellowship grant No. 81265410, by the BSF grant No. 2008430, and by the Lynn and William Frankel Center for CS.

F. Dehne, J. Iacono, and J.-R. Sack (Eds.): WADS 2011, LNCS 6844, pp. 692–703, 2011.

the MST $MST(G')$ of the given subgraph G'. The *weight-coefficient* of G' with respect to G is defined as the ratio between $w(MST(G'))$ and $w(MST(G))$. If the weight-coefficient of G' is small enough, we can use $MST(G')$ as a spanning tree for G of small weight.

The problem of computing spanning trees of small weight (especially the MST) is a fundamental one in Computer Science [19,17,7,26,13,10], and the above scenario arises naturally in many practical contexts (see, e.g., [30,12,35,23,24,25,11]). In particular, this scenario is motivated by wireless network design.

In this paper we focus on the *symmetric disk graph model* in wireless communication networks, which has been subject to considerable research. (See [16,14,15,31,5,33,1,22], and the references therein.) Let $M = (V, \delta)$ be an n-point metric space that is represented as a complete weighted graph $G(M) = (V, \binom{V}{2}, w)$ in which the weight $w(e)$ of each edge $e = (u, v)$ is equal to $\delta(u, v)$. Also, let $r : V \rightarrow \mathbb{R}^+$ be a transmission range assignment that maps each point $v \in V$ to the disk of radius $r(v)$ around it. The *symmetric disk graph* (henceforth, SDG) that corresponds to M and r, denoted $SDG(M, r)$,[1] is the undirected spanning subgraph of $G(M)$ whose edge set includes an edge $e = (u, v)$ if both $r(u)$ and $r(v)$ are no smaller than $w(e)$. Under the symmetric disk graph model we cannot use all the edges of $G(M)$, but rather only those that are present in $SDG(M, r)$. Clearly, if $r(v) \geq diam(M)$,[2] for each point $v \in V$, then $SDG(M, r)$ is simply the complete graph $G(M)$. However, the transmission ranges are usually significantly shorter than $diam(M)$, and many edges that belong to $G(M)$ may not be present in $SDG(M, r)$. Therefore, it is generally impossible to use the MST of M under the symmetric disk graph model, simply because some of the edges of $MST(M)$ are not present in $SDG(M, r)$ and thus cannot be accessed. Instead, assuming the weight-coefficient of $SDG(M, r)$ with respect to M is small enough, we can use $MST(SDG(M, r))$ as a spanning tree for M of small weight.

Abu-Affash et al. [1] showed that for any n-point *2-dimensional Euclidean* space M, the weight of the MST of every connected SDG for M is $O(\log n) \cdot w(MST(M))$. In other words, they proved that for any n-point 2-dimensional Euclidean space, the weight-coefficient of every connected SDG is $O(\log n)$. In addition, Abu-Affash et al. [1] provided a matching lower bound of $\Omega(\log n)$ on the weight-coefficient of connected SDGs that applies to a basic 1-dimensional Euclidean space. Notably, the upper bound proof of [1] relies heavily on basic geometric properties of constant-dimensional Euclidean spaces, and does not extend to Euclidean spaces of super-constant dimension. A natural question that arises is whether this surprising upper bound of [1] on the weight-coefficient of

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1 The definition of symmetric disk graph can be generalized in the obvious way for any (undirected) weighted graph. Specifically, the *symmetric disk graph* $SDG(G, r)$ that corresponds to a weighted graph $G = (V, E, w)$ and a transmission range assignment $r : V \rightarrow \mathbb{R}^+$ is the undirected spanning subgraph of G whose edge set includes an edge $e = (u, v) \in E$ if both $r(u)$ and $r(v)$ are no smaller than $w(e)$.

2 The *diameter* of a metric space M, denoted $diam(M)$, is defined as the largest pairwise distance in M.

connected SDGs can be generalized for wider families of metric spaces, such as high-dimensional Euclidean spaces.

In this paper we generalize the upper bound of Abu-Affash et al. [1] for Euclidean spaces of any dimension. Furthermore, our upper bound extends to *arbitrary metric spaces* and, in particular, it applies to any of the normed spaces ℓ_p. Specifically, we demonstrate that for *any* n-point metric space M, every connected SDG has weight-coefficient $O(\log n)$. In fact, our upper bound is even more general, applying to disconnected SDGs as well. That is, we show that the weight of the minimum spanning forest[3] (MSF) of every (possibly disconnected) SDG for M is $O(\log n) \cdot w(MST(M))$.

1.2 The Range Assignment Problem

Given a network $G = (V, E, w)$, a *(transmission) range assignment* for G is an assignment of transmission ranges to each of the vertices of G. A range assignment is called *complete* if the induced (directed) communication graph is strongly connected. In the *range assignment problem* the objective is to find a complete range assignment for which the total power consumption (henceforth, cost) is minimized. The power consumed by a vertex $v \in V$ is $r(v)^\alpha$, where $r(v) > 0$ is the range assigned to v and $\alpha \geq 1$ is some constant. Thus the cost of the range assignment is given by $\sum_{v \in V} r(v)^\alpha$. The range assignment problem was first studied by Kirousis et al. [18], who proved that the problem is NP-hard in 3-dimensional Euclidean spaces, assuming $\alpha = 2$, and also presented a 2-approximation algorithm. Subsequently, Clementi et al. [9] proved that the problem remains NP-hard in 2-dimensional Euclidean spaces.

We believe that it is more realistic to study the range assignment problem under the symmetric disk graph model. Specifically, the potential transmission range of a vertex v is bounded by some maximum range $r'(v)$, and any two vertices u, v can directly communicate with each other if and only if v lies within the range assigned to u and vice versa. Blough et al. [3] showed that this version of the range assignment problem is also NP-hard in 2-dimensional and 3-dimensional Euclidean spaces. Also, Calinescu et al. [4] devised a $(1+\frac{1}{2}\ln 3+\epsilon)$-approximation scheme and a more practical $(\frac{15}{8})$-approximation algorithm. Abu-Affash et al. [1] showed that, assuming $\alpha = 1$, the cost of an optimal range assignment with bounds on the ranges is greater by at most a logarithmic factor than the cost of an optimal range assignment without such bounds. This result of Abu-Affash et al. [1] is a simple corollary of their upper bound on the weight-coefficient of SDGs for 2-dimensional Euclidean spaces. Consequently, this result of [1] for the range assignment problem holds only in 2-dimensional Euclidean spaces. By applying our generalized upper bound on the weight-coefficient of SDGs, we extend this result of Abu-Affash et al. [1] to arbitrary metric spaces.

[3] The *minimum spanning forest* of a (possibly disconnected) weighted graph G is the union of the MSTs for its connected components. In other words, it is the maximal cycle-free spanning subgraph of G of minimum weight.

1.3 Proof Overview

As was mentioned above, the upper bound proof of [1] is very specific, and relies heavily on basic geometric properties of constant-dimensional Euclidean spaces. Hence, it does not apply to Euclidean spaces of super-constant dimension, let alone to arbitrary metric spaces. Our upper bound proof is based on completely different principles. In particular, it is independent of the geometry of the metric space and applies to every complete graph whose weight function satisfies the triangle inequality. In fact, at the heart of our proof is a lemma that applies to an even wider family of graphs, namely, the family of all traceable[4] weighted graphs. Specifically, let S and H be an SDG and a minimum-weight Hamiltonian path of some traceable weighted n-vertex graph G, respectively, and let F be the MSF of S. Our lemma states that there is a set $\tilde{E}$ of edges in F of weight at most $w(H)$, such that the graph $F \setminus \tilde{E}$ obtained by removing all edges of $\tilde{E}$ from F contains at least $\frac{1}{5} \cdot n$ isolated vertices. The proof of this lemma is based on a delicate combinatorial argument that does not assume either that the graph G is complete or that its weight function satisfies the triangle inequality. We believe that this lemma is of independent interest. (See Lemma 1 in Sect. 2.) By employing this lemma inductively, we are able to show that the weight of F is bounded above by $\log_{\frac{5}{4}} n \cdot w(H)$, which, by the triangle inequality, yields an upper bound of $2 \cdot \log_{\frac{5}{4}} n$ on the weight-coefficient of S with respect to G. Interestingly, our upper bound of $2 \cdot \log_{\frac{5}{4}} n$ on the weight-coefficient of SDGs for arbitrary metric spaces improves the corresponding upper bound of Abu-Affash et al. [1] (namely, $90 \cdot \log_{\frac{5}{4}} n + 1$), which holds only in 2-dimensional Euclidean spaces, by a multiplicative factor of 45.

1.4 Related Work on Disk Graphs

The symmetric disk graph model is a generalization of the extremely well-studied *unit disk graph model* (see, e.g., [8,23,20,25,21]). The *unit disk graph* of a metric space M, denoted $UDG(M)$, is the symmetric disk graph corresponding to M and the range assignment $r \equiv 1$ that maps each point to the unit disk around it. (It is usually assumed that M is a 2-dimensional Euclidean space.) Observe that in the case when $UDG(M)$ is connected, all edges of $MST(M)$ belong to $UDG(M)$, and so $MST(UDG(M)) = MST(M)$. Hence the weight-coefficient of connected unit disk graphs for arbitrary metric spaces is equal to 1. In the general case, it is easy to see that all edges of $MSF(UDG(M))$ belong to $MST(M)$, and so the weight-coefficient of (possibly disconnected) unit disk graphs for arbitrary metric spaces is at most 1.

Another model that has received much attention in the literature is the *asymmetric disk graph model* (see, e.g., [20,32,28,29,1]). The *asymmetric disk graph* corresponding to a metric space $M = (V, \delta)$ and a range assignment $r : V \rightarrow \mathbb{R}^+$ is the directed graph over V, where there is an arc (u, v) of weight $\delta(u, v)$ from u to v if $r(u) \geq \delta(u, v)$. On the negative side, Abu-Affash et al. [1] provided a lower bound of $\Omega(n)$ on the weight-coefficient of strongly connected asymmetric

[4] A graph is called *traceable* if it contains a Hamiltonian path.

disk graphs that applies to an n-point 2-dimensional Euclidean space. However, asymmetric communication models are generally considered to be impractical, because in such models many communication primitives become unacceptably complicated [27,34]. In particular, the asymmetric disk graph model is often viewed as less realistic than the symmetric disk graph model, where, as was mentioned above, we obtain a logarithmic upper bound on the weight-coefficient for arbitrary metric spaces.

1.5 Structure of the Paper

In Sect. 2 we obtain a logarithmic upper bound on the weight-coefficient of SDGs for arbitrary metric spaces. An application of this upper bound to the range assignment problem is given in Sect. 3.

1.6 Preliminaries

Given a (possibly weighted) graph G, its vertex set (respectively, edge set) is denoted by $V(G)$ (resp., $E(G)$). For an edge set $E' \subseteq E(G)$, we denote by $G \setminus E'$ the graph obtained by removing all edges of E' from G. Also, for an edge set E'' over the vertex set $V(G)$, we denote by $G \cup E''$ the graph obtained by adding all edges of E'' to G. The weight of an edge e in G is denoted by $w(e)$. For an edge set $E \subseteq E(G)$, its weight $w(E)$ is defined as the sum of all edge weights in it, i.e., $w(E) = \sum_{e \in E} w(e)$. The weight of G is defined as the weight of its edge set $E(G)$, namely, $w(G) = w(E(G))$. Finally, for a positive integer n, we denote the set $\{1, 2, \ldots, n\}$ by $[n]$.

2 The MST of SDGs is Light

In this section we prove that the weight-coefficient of SDGs for arbitrary n-point metric spaces is $O(\log n)$.

We will use the following well-known fact in the sequel.

Fact 1. *Let G be a weighted graph in which ell edge weights are distinct. Then G has a unique MSF, and the edge of maximum weight in every cycle of G does not belong to the MSF of G.*

In what follows we assume for simplicity that all the distances in any metric space are distinct. This assumption does not lose generality, since any ties can be broken using, e.g., lexicographic rules. Given this assumption, Fact 1 implies that there is a unique MST for any metric space, and a unique MSF for every SDG of any metric space.

The following lemma is central in our upper bound proof.

Lemma 1. *Let $M = (V, \delta)$ be an n-point metric space and let $r : V \to \mathbb{R}^+$ be a range assignment. Also, let $F = (V, E_F)$ be the MSF of the symmetric disk graph $S = SDG(M, r)$ and let $H = (V, E_H)$ be a minimum-weight Hamiltonian path of M. Then there is an edge set $\tilde{E} \subseteq E_F$ of weight at most $w(H)$, such that the graph $F \setminus \tilde{E}$ contains at least $\frac{1}{5} \cdot n$ isolated vertices.*

Remark: This statement remains valid if instead of the metric space M we take an arbitrary traceable weighted graph.

Proof. First, we construct a bijection $f : E \to \tilde{E}$, where $E \subseteq E_H$ and $\tilde{E} \subseteq E_F$, that satisfies that $w(f(e)) \le w(e)$, for each edge $e \in E$. This would imply that $w(\tilde{E}) \le w(E) \le w(H)$. We then show that the graph $F \setminus \tilde{E}$ contains at least $\frac{1}{5} \cdot n$ isolated vertices, which concludes the proof of the lemma.

The edge set E (respectively, $\tilde{E}$) is defined as the union of three disjoint edge sets to be specified later, denoted E'_1, E'_2 and E''_3 (resp., $\tilde{E}_1, \tilde{E}_2$ and $\tilde{E}_3$); thus $E = E'_1 \cup E'_2 \cup E''_3$ and $\tilde{E} = \tilde{E}_1 \cup \tilde{E}_2 \cup \tilde{E}_3$. We will construct three bijections $f_1 : E'_1 \to \tilde{E}_1$, $f_2 : E'_2 \to \tilde{E}_2$ and $f_3 : E''_3 \to \tilde{E}_3$. The bijection f will be obtained as the extension of these functions to the domain E, that is, for an edge $e \in E$,

$$f(e) = \begin{cases} f_1(e), & \text{if } e \in E'_1; \\ f_2(e), & \text{if } e \in E'_2; \\ f_3(e), & \text{if } e \in E''_3. \end{cases}$$

In other words, the function f_1 (respectively, f_2; resp., f_3) defines the restriction of the function f to the domain E'_1 (resp., E'_2; resp., E''_3).

Denote by E' the set of all edges in H that belong to the SDG S, i.e., $E' = E_H \cap E(S)$, and let $E'' = E_H \setminus E'$ be the complementary edge set of E' in E_H. We define E'_1 as the set of all edges in E' that belong to the MSF F, i.e., $E'_1 = E' \cap E_F$, and $E'_2 = E' \setminus E'_1$ as the complementary edge set of E'_1 in E'. Note that (1) $E' \subseteq E(S)$, (2) $E'' \cap E(S) = \emptyset$, (3) $E'_1 \subseteq E_F$, and (4) $E'_2 \cap E_F = \emptyset$. Also, observe that, by definition, the edge set E contains the entire edge set $E' = E'_1 \cup E'_2$ and only a subset E''_3 of E''; thus $E = E'_1 \cup E'_2 \cup E''_3 \subseteq E' \cup E'' = E_H$.

The function f_1 is defined as the identity map, namely, for each edge $e \in E'_1$, we define $f_1(e) = e$. Also, define $\tilde{E}_1 = E'_1$. Observe that $\tilde{E}_1 \subseteq E_F$, and f_1 is a bijection from E'_1 to $\tilde{E}_1$.

We proceed with constructing the function f_2.
Write $k = |E'_2|$, and let $e'_1, e'_2, \ldots, e'_k$ denote the edges of E'_2 by increasing order of weight. Next, we compute $k = |E'_2|$ spanning forests $F_1, F_2, \ldots, F_k$ of S, where each forest F_i contains a unique edge $\tilde{e}_i$ in $E_F \setminus E_H$ that satisfies that $w(\tilde{e}_i) \le w(e'_i)$; thus we can define $f_2(e'_i) = \tilde{e}_i$. The first forest F_1 is simply a copy of F. The rest of the forests $F_2, F_3, \ldots, F_k$ are computed iteratively as follows. For each index $i = 1, 2, \ldots, k$, the graph $F_i \cup \{e'_i\}$ obtained from F_i by adding to it the edge e'_i contains a unique cycle C_i. Since H is cycle-free, at least one edge of C_i does not belong to H; take $\tilde{e}_i$ to be an arbitrary such edge and define $f_2(e'_i) = \tilde{e}_i$. Finally, denote by $F_{i+1} = F_i \cup \{e'_i\} \setminus \{f_2(e'_i)\}$ the graph obtained from F_i by adding to it the edge e'_i and removing the edge $f_2(e'_i)$, for each $i \in [k-1]$. Define $\tilde{E}_2 = \{f_2(e'_i) \mid i \in [k]\}$. Observe that f_2 is a bijection from E'_2 to $\tilde{E}_2$.

Claim. (1) $\tilde{E}_2 \subseteq E_F \setminus \tilde{E}_1$. (2) For each index $i \in [k]$, $w(f_2(e'_i)) \le w(e'_i)$.

Proof. Fix an arbitrary index $i \in [k]$, and define $E'_{(i)} = \{e'_1, \ldots, e'_i\}$.

Note that the cycle C_i that is identified during the ith iteration of the above process is a subgraph of S. Moreover, $E(C_i) \subseteq E_F \cup E'_2$. Since $E'_2 \subseteq E_H$ and

$f_2(e'_i)$ is an edge of C_i that does not belong to H, it follows that $f_2(e'_i) \in E_F \setminus E_H$. This argument holds for any index $i \in [k]$, and so $\tilde{E}_2 = \{f_2(e'_i) \mid i \in [k]\} \subseteq E_F \setminus E_H \subseteq E_F \setminus \tilde{E}_1$. (The last inequality follows from the fact that $\tilde{E}_1 = E'_1 \subseteq E_H$.)

To prove the second assertion of the claim, notice that each edge of C_i that do not belong to F must belong to $E'_{(i)}$, i.e., $E(C_i) \setminus E_F \subseteq E'_{(i)}$. Fact 1 implies that the edge of maximum weight in C_i, denoted e^*_i, does not belong to F, hence $e^*_i \in E'_{(i)}$. Since e'_i is the edge of maximum weight in $E'_{(i)}$, it follows that $w(e^*_i) \leq w(e'_i)$. Also, as $f_2(e'_i)$ belongs to C_i, we have by definition $w(f_2(e'_i)) \leq w(e^*_i)$. Consequently, $w(f_2(e'_i)) \leq w(e^*_i) \leq w(e'_i)$, and we are done. □

Next, we construct the function f_3.

Denote by $H'' = H \setminus (E'_1 \cup E'_2)$ and $F'' = F \setminus (\tilde{E}_1 \cup \tilde{E}_2)$ the graphs obtained from H and F by removing all edges of $E' = E'_1 \cup E'_2$ and $\tilde{E}_1 \cup \tilde{E}_2$, respectively. By definition, $E(H'') = E''$. For an edge $e = (u, v)$, denote by $min(e)$ the endpoint of e with smaller radius, i.e., $min(e) = u$ if $r(u) < r(v)$, and $min(e) = v$ otherwise. The construction of the function f_3 is done in parallel to the computation of its domain E''_3; recall that E''_3 is the set of all edges in E'' that belong to E.

We start with initializing $E''_3 = \emptyset$. Then we examine the edges of E'' one after another in an arbitrary order. For each edge $e'' \in E''$, we check whether the vertex $min(e'')$ is isolated in F'' or not. If $min(e'')$ is isolated in F'', we leave H'', F'' and E''_3 intact. Otherwise, at least one edge is incident to $min(e'')$ in F''. Let $\tilde{e}$ be an arbitrary such edge, and define $f_3(e'') = \tilde{e}$. We remove the edge e'' from the graph H'' and add it to the edge set E''_3, and also remove the edge $f_3(e'')$ from the graph F''. This process is repeated iteratively until all edges of E'' have been examined. Define $\tilde{E}_3 = \{f_3(e'') \mid e'' \in E''_3\}$. At the end of this process, it holds that $H'' = H \setminus E = H \setminus (E'_1 \cup E'_2 \cup E''_3)$ and $F'' = F \setminus \tilde{E} = F \setminus (\tilde{E}_1 \cup \tilde{E}_2 \cup \tilde{E}_3)$. Observe that $\tilde{E}_3 \subseteq E_F \setminus (\tilde{E}_1 \cup \tilde{E}_2)$, and f_3 is a bijection from E''_3 to $\tilde{E}_3$.

Claim. For each edge $e'' \in E''_3$, $w(f_3(e'')) \leq w(e'')$.

Proof. Consider an arbitrary edge $e'' \in E''_3$ and the graph F'' just before the edge e'' was examined. Since no edge of E''_3 belongs to the SDG S, we have by definition $r(min(e'')) < w(e'')$. Also, since the graph F'' is a subgraph of S, the weight of every edge that is incident to $min(e'')$ in F'', including $f_3(e'') = \tilde{e}$, is no greater than $r(min(e''))$. Hence, $w(f_3(e'')) \leq r(min(e'')) < w(e'')$. □

We showed that the functions $f_1 : E'_1 \to \tilde{E}_1, f_2 : E'_2 \to \tilde{E}_2$ and $f_3 : E''_3 \to \tilde{E}_3$ are bijective, and that for each edge $e \in E'_1$ (respectively, $e \in E'_2$; resp., $e \in E''_3$), it holds that $w(f_1(e)) \leq w(e)$ (resp., $w(f_2(e)) \leq w(e)$; resp., $w(f_3(e)) \leq w(e)$). Furthermore, the domains E'_1, E'_2, E''_3 (respectively, images $\tilde{E}_1, \tilde{E}_2, \tilde{E}_3$) of these functions are pairwise disjoint subsets of E_H (resp., E_F). Hence, the extension f of these functions to the domain E is a bijection from $E = E'_1 \cup E'_2 \cup E''_3 \subseteq E_H$ to $\tilde{E} = \tilde{E}_1 \cup \tilde{E}_2 \cup \tilde{E}_3 \subseteq E_F$, such that $w(f(e)) \leq w(e)$, for each edge $e \in E$. It follows that

$$w(\tilde{E}) \ = \ \sum_{e \in \tilde{E}} w(e) \ = \ \sum_{e \in E} w(f(e)) \ \leq \ \sum_{e \in E} w(e) \ = \ w(E) \ \leq \ w(H).$$

To complete the proof of Lemma 1, we show that the graph $F'' = F \setminus \tilde{E}$ contains at least $\frac{1}{5} \cdot n$ isolated vertices. Denote by m_H (respectively, m_F) the number $|E(H'')|$ (resp., $|E(F'')|$) of edges in the graph H'' (resp., F'').

Suppose first that $m_F < \frac{2}{5} \cdot n$. Observe that in any n-vertex graph with m edges there are at least $n - 2m$ isolated vertices. Thus, the number of isolated vertices in F'' is bounded below by $n - 2m_F > n - \frac{4}{5} \cdot n = \frac{1}{5} \cdot n$, as required.

We henceforth assume that $m_F \geq \frac{2}{5} \cdot n$.

Since $H'' = H \setminus E$ and $E \subseteq E_H$, it holds that $|E(H'')| = |E_H| - |E|$. Similarly, we get that $|E(F'')| = |E_F| - |\tilde{E}|$. Also, observe that $|E_H| = n - 1 \geq |E_F|$ and $|E| = |\tilde{E}|$. Therefore,

$$m_H \;=\; |E(H'')| \;=\; |E_H| - |E| \;\geq\; |E_F| - |\tilde{E}| \;=\; |E(F'')| \;=\; m_F. \qquad (1)$$

Let $\mathcal{M}''$ be a maximal independent edge set (i.e., a maximal set of pairwise non-adjacent edges) in H''. Since H'' is a subgraph of the Hamiltonian path H, we conclude that at least half of the edges of H'' must belong to $\mathcal{M}''$. Consequently,

$$|\mathcal{M}''| \;\geq\; \frac{1}{2} \cdot |E(H'')| \;=\; \frac{1}{2} \cdot m_H \;\geq\; \frac{1}{2} \cdot m_F \;\geq\; \frac{1}{5} \cdot n.$$

(The second inequality follows from (1) whereas the third inequality follows from the above assumption.) By definition, for any pair e, e' of edges in $\mathcal{M}''$, $min(e) \neq min(e')$, hence the size of the vertex set $\mathcal{I}'' = \{min(e'') \mid e'' \in \mathcal{M}''\}$ satisfies $|\mathcal{I}''| = |\mathcal{M}''| \geq \frac{1}{5} \cdot n$. By construction, for each edge e'' in H'', the vertex $min(e'')$ is isolated in F''. In particular, all the vertices of $\mathcal{I}''$ are isolated in F''. Thus, the number of isolated vertices in F'' is bounded below by $|\mathcal{I}''| \geq \frac{1}{5} \cdot n$.

Lemma 1 follows. □

Next, we employ Lemma 1 inductively to upper bound the weight of SDGs in terms of the weight of a minimum-weight Hamiltonian path of the metric space. The desired upper bound of $O(\log n)$ on the weight-coefficient of SDGs for arbitrary n-point metric spaces would immediately follow.

Lemma 2. *Let $M = (V, \delta)$ be an n-point metric space and let $r : V \to \mathbb{R}^+$ be a range assignment. Also, let $F = (V, E_F)$ be the MSF of the symmetric disk graph $S = SDG(M, r)$ and let $H = (V, E_H)$ be a minimum-weight Hamiltonian path of M. Then $w(F) \leq \log_{\frac{5}{4}} n \cdot w(H)$.*

Proof. The proof is by induction on the number n of points in V.
Basis: $n \leq 4$. The case $n = 1$ is trivial. Suppose next that $2 \leq n \leq 4$. In this case $\log_{\frac{5}{4}} n \geq \log_{\frac{5}{4}} 2 > 3$. Also, the MSF F of S contains at most 3 edges. By the triangle inequality, the weight of each edge of F is bounded above by the weight $w(H)$ of the Hamiltonian path H. Hence, $w(F) \leq 3 \cdot w(H) < \log_{\frac{5}{4}} n \cdot w(H)$.
Induction step: We assume that the statement holds for all smaller values of n, $n \geq 5$, and prove it for n. By Lemma 1, there is an edge set $\tilde{E} \subseteq E_F$ of weight at most $w(H)$, such that the set I of isolated vertices in the graph $F \setminus \tilde{E}$ satisfies $|I| \geq \frac{1}{5} \cdot n$. Consider the complementary edge set $\hat{E} = E_F \setminus \tilde{E}$ of edges in F. Observe that no edge of $\hat{E}$ is incident to a vertex of I. Let $\hat{M}$ be the sub-metric

of M induced by the point set of $\hat{V} = V \setminus I$, and let $\hat{r}$ be the restriction of the range assignment r to $\hat{V}$. Also, let $\hat{S} = SDG(\hat{M}, \hat{r})$ be the SDG corresponding to $\hat{M}$ and $\hat{r}$, and let $\hat{F} = (\hat{V}, E_{\hat{F}})$ be the MSF of $\hat{S}$. Notice that the induced subgraph of S over the vertex set $\hat{V}$ is equal to $\hat{S}$, implying that all edges of $\hat{E}$ belong to $\hat{S}$. Thus, since $\hat{F}$ is a spanning forest of $\hat{S}$, replacing the edge set $\hat{E}$ of F by the edge set $E_{\hat{F}}$ does not affect the connectivity of the graph, i.e., the graph $\bar{F} = F \setminus \hat{E} \cup E_{\hat{F}}$ that is obtained from F by removing the edge set $\hat{E}$ and adding the edge set $E_{\hat{F}}$ has exactly the same connected components as F. Consequently, by breaking all cycles in the graph $\bar{F}$, we get a spanning forest of S. The weight of this spanning forest is bounded above by the weight $w(\bar{F}) = w\left(F \setminus \hat{E} \cup E_{\hat{F}}\right)$ of $\bar{F}$, and is bounded below by the weight $w(F)$ of the MSF F of S. Hence $w(F) \leq w\left(F \setminus \hat{E} \cup E_{\hat{F}}\right)$, which implies that $w(\hat{E}) \leq w(E_{\hat{F}}) = w(\hat{F})$. Write $\hat{n} = |\hat{V}|$, and let $\hat{H} = (\hat{V}, E_{\hat{H}})$ be a minimum-weight Hamiltonian path of $\hat{M}$. Since $|I| \geq \frac{1}{5} \cdot n$, we have

$$\hat{n} \ = \ |\hat{V}| \ = \ |V \setminus I| \ \leq \ \frac{4}{5} \cdot n \ \leq \ n - 1.$$

(The last inequality holds for $n \geq 5$.) By the induction hypothesis for $\hat{n}$, $w(\hat{F}) \leq \log_{\frac{5}{4}} \hat{n} \cdot w(\hat{H})$. Also, the triangle inequality implies that $w(\hat{H}) \leq w(H)$. Hence,

$$\begin{aligned} w(\hat{E}) \ &\leq \ w(\hat{F}) \ \leq \ \log_{\frac{5}{4}} \hat{n} \cdot w(\hat{H}) \ \leq \ \log_{\frac{5}{4}} \left(\frac{4}{5} \cdot n\right) \cdot w(H) \\ &= \ \log_{\frac{5}{4}} n \cdot w(H) - w(H). \end{aligned}$$

We conclude that

$$\begin{aligned} w(F) \ &= \ w(E_F) \ = \ w(\tilde{E}) + w(E_F \setminus \tilde{E}) \ = \ w(\tilde{E}) + w(\hat{E}) \\ &\leq \ w(H) + \log_{\frac{5}{4}} n \cdot w(H) - w(H) \ = \ \log_{\frac{5}{4}} n \cdot w(H). \end{aligned}$$

□

By the triangle inequality, the weight of a minimum-weight Hamiltonian path of any metric space is at most twice greater than the weight of the MST of that metric. We derive the main result of this paper as a corollary of Lemma 2.

Theorem 1. *For any n-point metric space $M = (V, \delta)$ and any range assignment $r : V \to \mathbb{R}^+$, $w(MSF(SDG(M, r))) = O(\log n) \cdot w(MST(M))$.*

3 The Range Assignment Problem

In this section we demonstrate that for any metric space, the cost of an optimal range assignment with bounds on the ranges is greater by at most a logarithmic factor than the cost of an optimal range assignment without such bounds. This result follows as a simple corollary of the upper bound given in Theorem 1.

Let $M = (V, \delta)$ be an n-point metric space, and assume that the n points of V, denoted by $v_1, v_2, \ldots, v_n$, represent transceivers. Also, let $r' : V \to \mathbb{R}^+$ be a *bounding range assignment* for V, i.e., a function that provides a maximum transmission range for each of the points of V, such that the SDG $SDG(M, r')$ corresponding to M and r' is connected. In the *bounded range assignment problem* the objective is to compute a range assignment $r : V \rightarrow \mathbb{R}^+$, such that (i) for each point $v_i \in V$, $r(v_i) \leq r'(v_i)$, (ii) the induced SDG (using the ranges $r(v_1), r(v_2), \ldots, r(v_n)$), namely $SDG(M, r)$, is connected, and (iii) $\sum_{i=1}^{n} r(v_i)$ is minimized. The sum $\sum_{i=1}^{n} r(v_i)$ is called the *cost* of the range assignment r. In the *unbounded range assignment problem* the maximum transmission range for each of the points of V is unbounded; that is, the unbounded range assignment problem is a special case of the bounded range assignment problem, where the bounding range assignment r' satisfies $r'(v_i) = diam(M)$, for each point $v_i \in V$.

Fix an arbitrary bounding range assignment $r' : V \rightarrow \mathbb{R}^+$. Denote by $OPT(M, r')$ the cost of an optimal solution for the bounded range assignment problem corresponding to M and r'. Also, denote by $OPT(M)$ the cost of an optimal solution for the unbounded range assignment problem corresponding to M. Notice that $OPT(M) \leq OPT(M, r')$. Next, we show that $OPT(M, r') = O(\log n) \cdot OPT(M)$.

Let $SDG(M, r')$ be the SDG corresponding to M and r', and let T be the MST of $SDG(M, r')$. We define r to be the range assignment that assigns $r(v_i)$ with the weight of the heaviest edge incident to v_i in T, for each point $v_i \in V$. By construction, $r(v_i) \leq r'(v_i)$, for each point $v_i \in V$. Also, notice that the SDG corresponding to M and r, namely $SDG(M, r)$, contains T and in thus connected. Hence, the range assignment r provides a feasible solution for the bounded range assignment problem corresponding to M and r', yielding $OPT(M, r') \leq \sum_{i=1}^{n} r(v_i)$. By a double counting argument, we get that $\sum_{i=1}^{n} r(v_i) \leq 2 \cdot w(T)$. Also, by Theorem 1, $w(T) = w(MST(SDG(M, r'))) = O(\log n) \cdot w(MST(M))$. Finally, it is easy to verify that $w(MST(M)) \leq OPT(M)$. Altogether,

$$\begin{aligned} OPT(M, r') &\leq \sum_{i=1}^{n} r(v_i) \leq 2 \cdot w(T) = 2 \cdot w(MST(SDG(M, r'))) \\ &= O(\log n) \cdot w(MST(M)) = O(\log n) \cdot OPT(M). \end{aligned}$$

Theorem 2. *For any n-point metric space $M = (V, \delta)$ and any bounding range assignment $r' : V \rightarrow \mathbb{R}^+$, $OPT(M, r') = O(\log n) \cdot OPT(M)$.*

Acknowledgments

The author thanks Rom Aschner, Michael Elkin and Matya Katz for helpful discussions.

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