

# Balancing Degree, Diameter and Weight in Euclidean Spanners <sup>\*</sup>

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## Abstract

In a seminal STOC'95 paper, Arya et al. [4] devised a construction that for any set  $S$  of  $n$  points in  $\mathbb{R}^d$  and any  $\epsilon > 0$ , provides a  $(1 + \epsilon)$ -spanner with diameter  $O(\log n)$ , weight  $O(\log^2 n) \cdot w(MST(S))$ , and constant maximum degree. Another construction of [4] provides a  $(1 + \epsilon)$ -spanner with  $O(n)$  edges and diameter  $O(\alpha(n))$ , where  $\alpha$  stands for the inverse Ackermann function. There are also a few other known constructions of  $(1 + \epsilon)$ -spanners. Das and Narasimhan [18] devised a construction with constant maximum degree and weight  $O(w(MST(S)))$ , but the diameter may be arbitrarily large. In another construction by Arya et al. [4] there is diameter  $O(\log n)$  and weight  $O(\log n) \cdot w(MST(S))$ , but it may have arbitrarily large maximum degree. While these constructions address some important practical scenarios, they fail to address situations in which we are prepared to compromise on one of the parameters, but cannot afford this parameter to be arbitrarily large.

In this paper we devise a novel *unified* construction that trades between the maximum degree, diameter and weight gracefully. For a positive integer  $k$ , our construction provides a  $(1 + \epsilon)$ -spanner with maximum degree  $O(k)$ , diameter  $O(\log_k n + \alpha(k))$ , weight  $O(k \cdot \log_k n \cdot \log n) \cdot w(MST(S))$ , and  $O(n)$  edges. Note that for  $k = O(1)$  this gives rise to maximum degree  $O(1)$ , diameter  $O(\log n)$  and weight  $O(\log^2 n) \cdot w(MST(S))$ , which is one of the aforementioned results of [4]. For  $k = n^{1/\alpha(n)}$  this gives rise to diameter  $O(\alpha(n))$ , weight  $O(n^{1/\alpha(n)} \cdot \log n \cdot \alpha(n)) \cdot w(MST(S))$  and maximum degree  $O(n^{1/\alpha(n)})$ . In the corresponding result from [4] the spanner has the same number of edges and diameter, but its weight and degree may be arbitrarily large. Our bound of  $O(\log_k n + \alpha(k))$  on the diameter is optimal under the constraints that the maximum degree is  $O(k)$  and the number of edges is  $O(n)$ . Similarly to the bound of Arya et al. [4], our bound on the weight is optimal up to a factor of  $\log n$ . Our construction also provides a similar tradeoff in the complementary range of parameters, i.e., when the weight should be smaller than  $\log^2 n$ , but the diameter is allowed to grow beyond  $\log n$ .

For random point sets in the  $d$ -dimensional unit cube, we “shave” a factor of  $\log n$  from the weight bound. Specifically, in this case our construction achieves maximum degree  $O(k)$ , diameter  $O(\log_k n + \alpha(k))$  and weight that is with high probability  $O(k \cdot \log_k n) \cdot w(MST(S))$ .

Finally, en route to these results we devise optimal constructions of 1-spanners for general tree metrics, which are of independent interest.

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# 1 Introduction

## 1.1 Euclidean Spanners

Consider the weighted complete graph  $\mathcal{S} = (S, \binom{S}{2})$  induced by a set  $S$  of  $n$  points in  $\mathbb{R}^d, d \geq 2$ . The weight of an edge  $(x, y) \in \binom{S}{2}$ , for a pair of distinct points  $x, y \in S$ , is defined to be the Euclidean distance  $\|x - y\|$  between  $x$  and  $y$ . Let  $G = (S, E)$  be a spanning subgraph of  $\mathcal{S}$ , with  $E \subseteq \binom{S}{2}$ , and assume that exactly as in  $\mathcal{S}$ , for any edge  $e = (x, y) \in E$ , its weight  $w(e)$  in  $G$  is defined to be  $\|x - y\|$ . For a parameter  $\epsilon > 0$ , the spanning subgraph  $G$  is called a  $(1 + \epsilon)$ -spanner for the point set  $S$  if for every pair  $x, y \in S$  of distinct points, the distance  $\text{dist}_G(x, y)$  between  $x$  and  $y$  in  $G$  is at most  $(1 + \epsilon) \cdot \|x - y\|$ . Euclidean spanners were introduced<sup>1</sup> in 1986 by Chew [14]. Since then they evolved into an important subarea of Computational Geometry [29, 17, 36, 30, 3, 16, 18, 4, 19, 5, 35, 1, 10, 20]. (See also the book by Narasimhan and Smid on Euclidean spanners [32], and the references therein.) Also, Euclidean spanners have numerous applications in geometric approximation algorithms [35, 24, 25], geometric distance oracles [24, 26, 25], Network Design [28, 31] and in other areas.

In many of these applications one is required to construct a  $(1 + \epsilon)$ -spanner  $G = (S, E)$  that satisfies a number of useful properties. First, the spanner should contain  $O(n)$  (or nearly  $O(n)$ ) edges. Second, its *weight*  $w(G) = \sum_{e \in E} w(e)$  should not be much greater than the weight  $w(MST(S))$  of the minimum spanning tree  $MST(S)$  of  $S$ . Third, its *diameter*  $\Lambda = \Lambda(G)$  should be small, i.e., for every pair of points  $x, y \in S$  there should exist a path  $P$  in  $G$  that contains at most  $\Lambda$  edges and has weight  $w(P) = \sum_{e \in E(P)} w(e) \leq (1 + \epsilon) \cdot \|x - y\|$ . Fourth, its *maximum degree* (henceforth, *degree*)  $\Delta(G)$  should be small.

In a seminal STOC'95 paper that culminated a long line of research, Arya et al. [4] devised a construction of  $(1 + \epsilon)$ -spanners with *lightness*<sup>2</sup>  $O(\log^2 n)$ , diameter  $O(\log n)$  and constant degree. They also devised a construction of  $(1 + \epsilon)$ -spanners with diameter  $O(\alpha(n))$  (respectively,  $O(1)$ ) and  $O(n)$  (resp.,  $O(n \cdot \log^* n)$ ) edges, where  $\alpha$  stands for the inverse Ackermann function [15, 40]. However, in the latter construction the resulting spanners may have arbitrarily large (i.e.,  $\Omega(n)$ ) lightness and degree. There are also a few other known constructions of  $(1 + \epsilon)$ -spanners. Das and Narasimhan [18] devised a construction with constant degree and lightness, but the diameter may be arbitrarily large. (See also [23] for a faster implementation of a spanner construction with constant degree and lightness.) There is also another construction by Arya et al. [4] that guarantees that both the diameter and the lightness are  $O(\log n)$ , but the degree may be arbitrarily large. While these constructions address some important practical scenarios, they certainly do not address all of them. In particular, they fail to address situations in which we are prepared to compromise on one of the parameters, but cannot afford this parameter to be arbitrarily large.

In this paper we devise a novel *unified* construction that trades between the degree, diameter and weight gracefully. For a positive integer  $k$ , our construction provides a  $(1 + \epsilon)$ -spanner with degree  $O(k)$ , diameter  $O(\log_k n + \alpha(k))$ , lightness  $O(k \cdot \log_k n \cdot \log n)$ , and  $O(n)$  edges. Also, we can improve the bound on the diameter from  $O(\log_k n + \alpha(k))$  to  $O(\log_k n)$ , at the expense of increasing the number of edges from  $O(n)$  to  $O(n \cdot \log^* n)$ . Note that for  $k = O(1)$  our tradeoff gives rise to degree  $O(1)$ , diameter  $O(\log n)$  and lightness  $O(\log^2 n)$ , which is one of the aforementioned results of [4]. Also, for  $k = n^{1/\alpha(n)}$  it gives rise to a spanner with degree  $O(n^{1/\alpha(n)})$ , diameter  $O(\alpha(n))$  and lightness  $O(n^{1/\alpha(n)} \cdot \log n \cdot \alpha(n))$ . In the corresponding result from [4] the spanner has the same number of edges and diameter, but its lightness and degree may be arbitrarily large.

In addition, we can achieve lightness  $o(\log^2 n)$  at the expense of increasing the diameter. Specifically, for a parameter  $k$  the second variant of our construction provides a  $(1 + \epsilon)$ -spanner with degree  $O(1)$ , diameter  $O(k \cdot \log_k n)$ , and lightness  $O(\log_k n \cdot \log n)$ . For example, for  $k = \log^\delta n$ , for an arbitrarily small constant  $\delta > 0$ , we get a  $(1 + \epsilon)$ -spanner with degree  $O(1)$ , diameter  $O(\log^{1+\delta} n)$  and lightness  $O(\frac{\log^2 n}{\log \log n})$ .

Our unified construction can be implemented in  $O(n \cdot \log n)$  time. This matches the state-of-the-art

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<sup>1</sup>The notion “spanner” was coined by Peleg and Ullman [33], who also introduced spanners for general graphs.

<sup>2</sup>For convenience, we will henceforth refer to the normalized notion of weight  $\Psi(G) = \frac{w(G)}{w(MST(S))}$ , which we call *lightness*.

	[4]	[4]	New	New	New	New	New	New
	I,II		I	I	I	II	II	II
$k$	1		$\log^\delta n$	$2^{\sqrt{\log n}}$	$n^{1/\alpha(n)}$	$\log^\delta n$	$2^{\sqrt{\log n}}$	$n^\zeta$
$\Delta$	1	$n$	$\log^\delta n$	$2^{\sqrt{\log n}}$	$n^{1/\alpha(n)}$	1	1	1
$\Lambda$	$\log n$	$\alpha(n)$	$\frac{\log n}{\log \log n}$	$\sqrt{\log n}$	$\alpha(n)$	$\log^{1+\delta} n$	$2^{O(\sqrt{\log n})}$	$n^\zeta$
$\Psi$	$\log^2 n$	$n$	$\log^{2+\delta} n$	$2^{O(\sqrt{\log n})}$	$n^{O(1/\alpha(n))}$	$\frac{\log^2 n}{\log \log n}$	$\log^{3/2} n$	$\log n$

Table 1: A concise comparison of previous and new results. Each column corresponds to a set of parameters that can be achieved simultaneously. For each column the first row indicates whether the result is new or due to [4]. (The first column is due to [4], but can also be achieved from both our tradeoffs.) For new results, the second row indicates whether it is obtained by the first (I) or the second (II) tradeoff. (The first tradeoff is degree  $O(k)$ , diameter  $O(\log_k n + \alpha(k))$ , and lightness  $O(k \cdot \log_k n \cdot \log n)$ . The second tradeoff is degree  $O(1)$ , diameter  $O(k \cdot \log_k n)$  and lightness  $O(\log_k n \cdot \log n)$ .) The third row indicates the value of  $k$  that is substituted in the corresponding tradeoff. The next three rows indicate the resulting degree ( $\Delta$ ), diameter ( $\Lambda$ ) and lightness ( $\Psi$ ). The number of edges used in all constructions is  $O(n)$ . To save space, the  $O$  notation is omitted everywhere except for the exponents. The letters  $\delta$  and  $\zeta$  stand for arbitrarily small positive constants.

running time of the aforementioned constructions [4, 23]. See Table 1 for a concise comparison of previous and new results.

To summarize, our construction improves existing constructions for some specific sets of parameters. For other sets of parameters it matches the current state-of-the-art bounds due to [4]. We believe that replacing a number of separate “ad-hoc” constructions by a single unified parameterized construction is an important step towards laying more solid theoretical foundation to the area of Euclidean spanners.

Note that in any construction of spanners with degree  $O(k)$ , the diameter is  $\Omega(\log_k n)$ . Also, Chan and Gupta [10] showed that any  $(1 + \epsilon)$ -spanner with  $O(n)$  edges must have diameter  $\Omega(\alpha(n))$ . Consequently, our upper bound of  $O(\log_k n + \alpha(k))$  on the diameter is tight under the constraints that the degree is  $O(k)$  and the number of edges is  $O(n)$ . If we allow  $O(n \cdot \log^* n)$  edges in the spanner, then our bound on the diameter is reduced to  $O(\log_k n)$ , which is again tight under the constraint that the degree is  $O(k)$ .

In addition, Dinitz et al. [20] showed that for any construction of spanners, if the diameter is at most  $O(\log_k n)$ , then the lightness is at least  $\Omega(k \cdot \log_k n)$  and vice versa, if the lightness is at most  $O(\log_k n)$ , then the diameter is at least  $\Omega(k \cdot \log_k n)$ . This lower bound implies that the bound on lightness in both our tradeoffs cannot possibly be improved by more than a factor of  $\log n$ . The same slack of  $\log n$  is present in the result of [4] that guarantees lightness  $O(\log^2 n)$ , diameter  $O(\log n)$  and constant degree.

### 1.1.1 Euclidean Spanners for Random Point Sets

For random point sets in the  $d$ -dimensional unit cube (henceforth, unit cube), we “shave” a factor of  $\log n$  from the lightness bound in both our tradeoffs, and show that the first (respectively, second) variant of our construction achieves maximum degree  $O(k)$  (resp.,  $O(1)$ ), diameter  $O(\log_k n + \alpha(k))$  (resp.,  $O(k \cdot \log_k n)$ ) and lightness that is with high probability (henceforth, w.h.p.)  $O(k \cdot \log_k n)$  (resp.,  $O(\log_k n)$ ). Note that for  $k = O(1)$  both these tradeoffs give rise to degree  $O(1)$ , diameter  $O(\log n)$  and lightness (w.h.p.)  $O(\log n)$ .

### 1.1.2 Spanners for Doubling Metrics

The *doubling dimension* of a metric  $(X, \delta)$  is the smallest value  $\zeta$  such that every ball  $B$  in the metric can be covered by at most  $2^\zeta$  balls of half the radius of  $B$ . The metric  $(X, \delta)$  is called *doubling* if its doubling dimension  $\zeta$  is constant. Spanners for doubling metrics have received much attention in recent years (see, e.g., [11, 27, 10, 22]). In particular, Chan et al. [11] showed that for any doubling metric  $(X, \delta)$  there exists a  $(1 + \epsilon)$ -spanner with constant maximum degree, but this spanner may have arbitrarily large diameter. In addition, Chan and Gupta [10] devised a construction of  $(1 + \epsilon)$ -spanners for doubling metrics that achieves the optimal tradeoff between the number of edges and diameter, but these spanners may have arbitrarily large degree. We present a single construction of  $O(1)$ -spanners for

doubling metrics that achieves the optimal tradeoff between the degree, diameter and number of edges in the entire range of parameters. Specifically, for a parameter  $k$ , our construction provides an  $O(1)$ -spanner with maximum degree  $O(k)$ , diameter  $O(\log_k n + \alpha(k))$ , and  $O(n)$  edges. Also, we can improve the bound on the diameter from  $O(\log_k n + \alpha(k))$  to  $O(\log_k n)$ , at the expense of increasing the number of edges from  $O(n)$  to  $O(n \cdot \log^* n)$ . More generally, we can achieve the same optimal tradeoff between the number of edges and diameter as the spanners of [10] do, while also having the optimal maximum degree. The drawback is, however, that the stretch of our spanners is  $O(1)$  rather than  $1 + \epsilon$ .

## 1.2 Spanners for Tree Metrics

Let  $\vartheta_n$  be the metric induced by  $n$  points  $v_1, v_2, \dots, v_n$  lying on the  $x$ -axis with coordinates  $1, 2, \dots, n$ , respectively. In a classical STOC'82 paper [45], Yao showed that there exists a 1-spanner<sup>3</sup>  $G = (V, E)$  for  $\vartheta_n$  with diameter  $O(\alpha(n))$  and  $O(n)$  edges, and that this is tight. Chazelle [12] extended the result of [45] to arbitrary tree metrics. Other proofs of Chazelle's result appeared in [2, 9, 44, 38]. Thorup [44] also devised an efficient parallel algorithm for computing this 1-spanner. The problem was also studied for planar metrics [43], general metrics [42] and even for general graphs [8]. (See also Chapter 12 in [32] for an excellent survey on this problem.) The problem is also closely related to the extremely well-studied problem of computing partial sums. (See the papers of Tarjan [41], Yao [45], Chazelle and Rosenberg [13], Pătraşcu and Demaine [34], and the references therein.) For a discussion about the relationship between these two problems see the introduction of [1].

In all constructions [45, 12, 2, 9, 44, 38] of 1-spanners for tree metrics, the degree and lightness of the resulting spanner may be arbitrarily large. Moreover, the constraint that the diameter is  $O(\alpha(n))$  implies that the degree must be  $n^{\Omega(1/\alpha(n))}$ . A similar lower bound on the lightness follows from the result of [20].

En route to our tradeoffs for Euclidean spanners, we have extended the results of [45, 12, 2, 9, 44, 38] and devised a construction that achieves the *optimal* (up to constant factors) *tradeoff between all involved parameters*. Specifically, consider an  $n$ -vertex tree  $T$  of degree  $\Delta(T)$ , and let  $k$  be a positive integer. Our construction provides a 1-spanner for the tree metric  $M_T$  induced by  $T$  with degree  $O(\Delta(T) + k)$ , diameter  $O(\log_k n + \alpha(k))$ , lightness  $O(k \cdot \log_k n)$ , and  $O(n)$  edges. We can also get a spanner with diameter  $O(\log_k n)$ ,  $O(n \cdot \log^* n)$  edges, and the same degree and lightness as above. For the complementary range of diameter, the second variant of our construction provides a 1-spanner with degree  $O(\Delta(T))$ , diameter  $O(k \cdot \log_k n)$ , lightness  $O(\log_k n)$ , and  $O(n)$  edges. As was mentioned above, both these tradeoffs are *optimal up to constant factors*.

We show that this general tradeoff between various parameters of 1-spanners for tree metrics is useful for deriving new results (and improving existing results) in the context of Euclidean spanners and spanners for doubling metrics. We anticipate that this tradeoff would be found useful in the context of partial sums problems, for constructing transitive-closure spanners (see, e.g., [8, 7, 6]), and for other applications.

## 1.3 Our and Previous Techniques

The starting point for our construction is the construction of Arya et al. [4] that achieves diameter  $O(\log n)$ , lightness  $O(\log^2 n)$  and constant degree. The construction of [4] is built in two stages. First, a construction for the 1-dimensional case is devised. Then the 1-dimensional construction is extended to arbitrary constant dimension. For 1-dimensional spaces Arya et al. [4] start with devising a construction of 1-spanners with diameter, lightness and degree all bounded by  $O(\log n)$ . This construction is quite simple; it is essentially a flattened version of a deterministic skip-list. Next, by a more involved argument they show that the degree can be reduced to  $O(1)$ , at the expense of increasing the stretch parameter from 1 to  $1 + \epsilon$ . Finally, the generalization of their construction to point sets in the plane (or, more generally, to  $\mathbb{R}^d$ ) is far more involved. Specifically, to this end Arya et al. [4] employed two main tools.

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<sup>3</sup>The graph  $G$  is said to be a *1-spanner* for  $\vartheta_n$  if for every pair of distinct vertices  $v_i, v_j \in V$ , the distance between them in  $G$  is equal to their distance  $\|i - j\|$  in  $\vartheta_n$ . Yao stated this problem in terms of partial sums. However, the two statements of the problem are equivalent.

The first one is the *dumbbell trees*, the theory of which was developed by Arya et al. in the same paper [4]. (See also Chapter 11 of [32].) The second one is the bottom-up clustering technique that was developed by Frederickson [21] for topology trees. Roughly speaking, the *Dumbbell Theorem* of [4] states that for every point set  $S$ , one can construct a forest  $\mathcal{D}$  of  $O(1)$  dumbbell trees, in which there exists a tree  $T \in \mathcal{D}$  for every pair  $x, y$  of points from  $S$ , such that the distance  $dist_T(x, y)$  between  $x$  and  $y$  in  $T$  is at most  $(1 + \epsilon)$  times their Euclidean distance  $\|x - y\|$ . Arya et al. employ Frederickson’s clustering technique on each of these  $O(1)$  dumbbell trees to obtain their ultimate spanner.

Similarly to [4], we start with devising a construction of 1-spanners for the 1-dimensional case. However, our construction achieves both diameter and lightness at most  $O(\log n)$ , in conjunction with the *optimal degree*<sup>4</sup> of at most 3. (Note that [4] paid for decreasing the degree from  $O(\log n)$  to  $O(1)$  by increasing the stretch of the spanner from 1 to  $1 + \epsilon$ . Our construction achieves stretch 1 in conjunction with logarithmic diameter and lightness, and with the optimal degree 3.) Moreover, our construction is far more general, as it provides the *entire suite* of all possible values of diameter, lightness and degree, and it is *optimal up to constant factors* in the entire range of parameters. We then proceed to extending it to arbitrary tree metrics. Finally, we employ the dumbbell trees of Arya et al. [4]. Specifically, we construct our 1-spanners for the metrics induced by each of these dumbbell trees, and return their union as our ultimate spanner. As a result we obtain a *unified* construction of Euclidean spanners that achieves near-optimal tradeoffs in the entire range of parameters. We remark that it is unclear whether the construction of Arya et al. [4] can be extended to provide additional combinations between the diameter and lightness other than  $O(\log n)$  and  $O(\log^2 n)$ , respectively; roughly speaking, the logarithms there come from the number of levels in Frederickson’s topology trees. In particular, the construction of Arya et al. [4] that achieves diameter  $O(\alpha(n))$  and arbitrarily large lightness and degree is based on completely different ideas. On the other hand, our construction yields a stronger result (diameter  $O(\alpha(n))$ , lightness and degree  $n^{O(1/\alpha(n))}$ ), and this result is obtained by substituting a different parameter into one of our tradeoffs. Moreover, our construction is much simpler and more modular than that of [4]. In particular, it does not employ Frederickson’s bottom-up clustering technique, but rather constructs 1-spanners for dumbbell trees directly.

Also, our construction of 1-spanners for tree metrics (that we use for dumbbell trees) is fundamentally different from the previous constructions due to [45, 12, 2, 9, 44, 38]. In particular, the techniques of [12, 2, 9, 44, 38] for generalizing constructions of 1-spanners from 1-dimensional metrics to general tree metrics ensure that the diameter of the resulting spanners is not (much) greater than the diameter in the 1-dimensional case. However, the degree and/or lightness of spanners for tree metrics that are obtained by these techniques may be arbitrarily large. To overcome this obstacle we adapt the techniques of [12, 2, 9, 38] to our purposes. Next, we overview this adaptation. A central ingredient in the generalization techniques of [12, 2, 9, 38] is a tree decomposition procedure. Given an  $n$ -vertex rooted tree  $(T, rt)$  and a parameter  $k$ , this procedure computes a set  $C$  of  $O(k)$  *cut vertices*. This set has the property that removing all vertices of  $C$  from the tree  $T$  decomposes  $T$  into a collection  $\mathcal{F}$  of trees, so that each tree  $\tau \in \mathcal{F}$  contains  $O(n/k)$  vertices. This decomposition induces a tree  $\mathcal{Q} = \mathcal{Q}(\tau, C)$  over the vertex set  $C \cup \{rt\}$  in a natural way: a cut vertex  $w \in C$  is defined to be a child of its closest ancestor in  $T$  that belongs to  $C \cup \{rt\}$ . For our purposes, it is crucial that the degree of the tree  $\mathcal{Q}$  will not be (much) greater than the degree of  $T$ . In addition, it is essential that each tree  $\tau \in \mathcal{F}$  will be incident to at most  $O(1)$  cut vertices. We devise a novel decomposition procedure that guarantees these two basic properties. Intuitively, our decomposition procedure “slices” the tree in a “path-like” fashion. This path-like nature of our decomposition enables us to keep the degree and lightness of our construction for general tree metrics (essentially) as small as in the 1-dimensional case.

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<sup>4</sup>Observe that any graph (and, in particular, a 1-spanner) with maximum degree 2 must have diameter at least  $\frac{n-1}{2}$ .

## 1.4 Structure of the Paper

In Section 2 we describe our construction of 1-spanners for tree metrics. Therein we start (Section 2.1) with outlining our basic scheme. We proceed (Section 2.2) with describing our 1-dimensional construction. In Section 2.3 we extend this construction to general tree metrics. Our tree decomposition procedure (which is in the heart of this extension) is described in Section 2.3.1. In Section 3 we derive our results for Euclidean spanners and spanners for doubling metrics.

## 1.5 Preliminaries

An  $n$ -point metric space  $M = (V, dist)$  can be viewed as the complete graph  $G(M) = (V, \binom{V}{2}, dist)$  in which for every pair of points  $x, y \in V$ , the weight of the edge  $e = (x, y)$  in  $G(M)$  is defined by  $w(x, y) = dist(x, y)$ . Let  $G$  be a spanning subgraph of  $M$ . We say that  $G$  is a  $t$ -spanner for  $M$  if for every pair  $x, y \in V$  of distinct points, there exists a path in  $G$  between  $x$  and  $y$  whose weight (i.e., the sum of all edge weights in it) is at most  $t \cdot dist(x, y)$ . Such a path is called a  $t$ -spanner path. The stretch of  $G$  is the minimum number  $t$ , such that  $G$  is a  $t$ -spanner for  $M$ . Let  $T$  be an arbitrary tree, and denote by  $V(T)$  the vertex set of  $T$ . For any two vertices  $u, v$  in  $T$ , their (weighted) distance in  $T$  is denoted by  $dist_T(u, v)$ . The tree metric  $M_T$  induced by  $T$  is defined as  $M_T = (V(T), dist_T)$ . The size of  $T$ , denoted  $|T|$ , is the number of vertices in  $T$ . Finally, for a positive integer  $n$ , we denote the set  $\{1, 2, \dots, n\}$  by  $[n]$ .

## 2 1-Spanners for Tree Metrics

### 2.1 The Basic Scheme

Consider an arbitrary  $n$ -vertex (weighted) rooted tree  $(T, rt)$ , and let  $M_T$  be the tree metric induced by  $T$ . Clearly,  $T$  is both a 1-spanner and an MST of  $M_T$ , but its diameter may be arbitrarily large. We would like to reduce the diameter of this 1-spanner by adding to it some edges. On the other hand, the number of edges of the resulting spanner should still be linear in  $n$ . Moreover, the lightness and the maximum degree of the resulting spanner should also be reasonably small.

Let  $H$  be a spanning subgraph of  $M_T$ . The *monotone distance* between any two points  $u$  and  $v$  in  $H$  is defined as the minimum number of edges in a 1-spanner path in  $H$  connecting them. Two points in  $M_T$  are called *comparable* if one is an ancestor of the other in the underlying tree  $T$ . The *monotone diameter* (respectively, *comparable monotone diameter*) of  $H$ , denoted  $\Lambda(H)$  (resp.,  $\bar{\Lambda}(H)$ ), is defined as the maximum monotone distance in  $H$  between any two points (resp., any two comparable points) in  $M_T$ . Observe that if any two *comparable* points are connected via a 1-spanner path that consists of at most  $h$  edges, then any two *arbitrary* points are connected via a 1-spanner path that consists of at most  $2h$  edges. Consequently,  $\bar{\Lambda}(H) \leq \Lambda(H) \leq 2 \cdot \bar{\Lambda}(H)$ . We henceforth restrict the attention to comparable monotone diameter in the sequel.

Let  $k \geq 2$  be a fixed parameter. The first ingredient of the algorithm is to select a set of  $O(k)$  *cut vertices* whose removal from  $T$  partitions it into a collection of subtrees of size  $O(n/k)$  each. (As mentioned in the last paragraph of Section 1.3, we also require this set to satisfy several additional properties.) Having selected the cut vertices, the next step of the algorithm is to connect the cut vertices via  $O(k)$  edges, so that the monotone distance between any pair of comparable cut vertices will be small. (This phase does not involve a recursive call of the algorithm.) Finally, the algorithm calls itself recursively for each of the subtrees.

We insert all edges of the original tree  $T$  into our final spanner  $H$ . These edges connect between cut vertices and subtrees in the spanner. We remark that the spanner contains no other edges that connect between cut vertices and subtrees. Moreover, the spanner contains no edges that connect between different subtrees.

## 2.2 1-Dimensional Spaces

In this section we devise an optimal construction of 1-spanners for  $\vartheta_n$ . (Recall that  $\vartheta_n$  is the metric induced by  $n$  points  $v_1, v_2, \dots, v_n$  lying on the  $x$ -axis with coordinates  $1, 2, \dots, n$ , respectively.) Our argument extends easily to any 1-dimensional space.

Denote by  $P_n$  the path  $(v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n)$  that induces the metric  $\vartheta_n$ . We remark that the edges of  $P_n$  (henceforth, *path-edges*) belong to all spanners that we construct.

### 2.2.1 Selecting the Cut-Vertices

Let  $k \geq 2$  be a fixed parameter. The task of selecting the cut vertices in the 1-dimensional case is trivial. (We assume for simplicity that  $n$  is an integer power of  $k$ .) In addition to the two endpoints  $v_1$  and  $v_n$  of the path, we select the  $k - 1$  points  $r_1, r_2, \dots, r_{k-1}$  to be cut vertices, where for each  $i \in [k - 1]$ ,  $r_i = v_{i(n/k)}$ . Indeed, by removing the  $k + 1$  cut vertices  $r_0 = v_1, r_1, \dots, r_{k-1}, r_k = v_n$  from the path (along with their incident edges), we are left with  $k$  intervals  $I_1, I_2, \dots, I_k$  of length at most  $n/k$  each. The two endpoints  $v_1$  and  $v_n$  of the path are called the *sentinels*, and they play a special role in the construction. (See Figure 1 for an illustration for the case  $k = 2$ .)

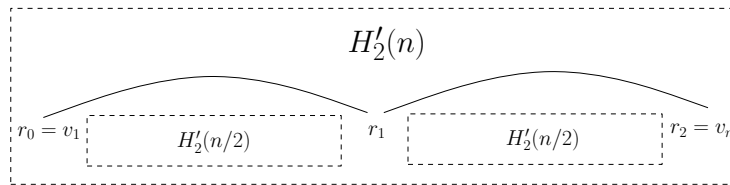


Figure 1: The construction for  $k = 2$ . Only the first level of the recursion is illustrated. (Path-edges are not depicted in the figure.) The cut vertex  $r_1 = v_{n/2}$  is connected via edges to the two sentinels  $v_1$  and  $v_n$ . The construction proceeds recursively for each of the two intervals  $I_1$  and  $I_2$ .

### 2.2.2 1-Spanners with Low Diameter

In this section we devise a construction  $H_k(n)$  of 1-spanners for  $\vartheta_n$  with comparable monotone diameter  $\bar{\Lambda}(n) = \bar{\Lambda}(H_k(n))$  in the range  $\Omega(\alpha(n)) = \bar{\Lambda}(n) = O(\log n)$ . In Section 2.2.3 we turn our attention to spanners with larger monotone diameter.

First, the algorithm connects the  $k + 1$  cut vertices  $r_0 = v_1, r_1, \dots, r_{k-1}, r_k = v_n$  via one of the aforementioned constructions of 1-spanners from [45, 12, 2, 9, 44, 38] (henceforth, *list-spanner*). In other words,  $O(k)$  edges between cut vertices are added to the spanner  $H_k(n)$  to guarantee that the monotone distance in the spanner between any two cut vertices<sup>5</sup> will be  $O(\alpha(k))$ . Then the algorithm adds to the spanner  $H_k(n)$  edges that connect each of the two sentinels to all other  $k$  cut vertices. Finally, the algorithm calls itself recursively for each of the intervals  $I_1, I_2, \dots, I_k$ . At the bottom level of the recursion, i.e., when  $n \leq k$ , the algorithm uses the list-spanner to connect all points, and, in addition, it adds to the spanner edges that connect each of the two sentinels  $v_1$  and  $v_n$  to all the other  $n - 1$  points. (See Figure 2 for an illustration.)

Denote by  $E(n)$  the number of edges in  $H_k(n)$ , excluding edges of  $P_n$ . Clearly,  $E(n)$  satisfies the recurrence  $E(n) \leq O(k) + k \cdot E(n/k)$ , with the base condition  $E(q) = O(q)$ , for all  $q \leq k$ , yielding  $E(n) = O(n)$ . Denote by  $\Delta(n)$  the maximum degree of a vertex in  $H_k(n)$ , excluding edges of  $P_n$ . Clearly,  $\Delta(n)$  satisfies the recurrence  $\Delta(n) \leq \max\{k, \Delta(n/k)\}$ , with the base condition  $\Delta(q) \leq q - 1$ , for all  $q \leq k$ , yielding  $\Delta(n) \leq k$ . Including edges of  $P_n$ , the number of edges increases by  $n - 1$  units, and the maximum degree increases by at most two units.

Denote by  $w(n)$  the weight of  $H_k(n)$ , excluding edges of  $P_n$ . Note that at most  $O(k)$  edges are added between cut vertices. Each of these edges has weight at most  $n - 1$ . The total weight of all edges within an

<sup>5</sup>In the 1-dimensional case any two points are comparable.

interval  $I_i$  is at most  $w(n/k)$ . Hence  $w(n)$  satisfies the recurrence  $w(n) \leq O(n \cdot k) + k \cdot w(n/k)$ , with the base condition  $w(q) = O(q^2)$ , for all  $q \leq k$ . It follows that  $w(n) = O(n \cdot k \cdot \log_k n) = O(k \cdot \log_k n) \cdot w(MST(\vartheta_n))$ . Including edges of  $P_n$ , the weight increases by  $w(P_n) = n - 1$  units.

Next, we show that the comparable monotone diameter  $\bar{\Lambda}(n)$  of  $H_k(n)$  is at most  $O(\log_k n + \alpha(k))$ . The *monotone radius*  $R(n)$  of  $H_k(n)$  is defined as the maximum monotone distance in  $H_k(n)$  between one of the sentinels (either  $v_1$  or  $v_n$ ) and some other point in  $\vartheta_n$ . Let  $v_j$  be a point in  $\vartheta_n$ , and let  $i$  be the index such that  $v_j \in \{r_i\} \cup I_{i+1}$ . (In other words,  $i$  is the index such that  $i(n/k) \leq j < (i+1)(n/k)$ .) We construct a 1-spanner path  $\Pi(v_1, v_j)$  in  $H_k(n)$  connecting the sentinel  $v_1$  and the point  $v_j$  recursively. If  $j = i(n/k)$  then  $v_j$  is the cut vertex  $r_i$ ; in this case the path  $\Pi(v_1, v_j)$  consists of the single edge  $(v_1, v_j)$ . Otherwise,  $j > i(n/k)$  and  $v_j \in I_{i+1}$ . In this case the path  $\Pi(v_1, v_j)$  starts with the two edges  $(v_1, v_{i(n/k)}) = (v_1, r_i)$ ,  $(v_{i(n/k)}) = (r_i, v_{i(n/k)+1})$ . The point  $v_{i(n/k)+1}$  is a sentinel of the  $(i+1)$ th interval  $I_{i+1}$ . Hence, the path  $\Pi(v_1, v_j)$  will continue recursively, from  $v_{i(n/k)+1}$  to  $v_j$ . It follows that the monotone radius  $R(n)$  satisfies the recurrence  $R(n) \leq 2 + R(n/k)$ , with the base condition  $R(q) = 1$ , for all  $q \leq k$ , yielding  $R(n) = O(\log_k n)$ . Having bounded the monotone radius  $R(n)$  of  $H_k(n)$ , we turn to bound the comparable monotone diameter  $\bar{\Lambda}(n)$  of  $H_k(n)$ . Consider an arbitrary pair  $v_j, v_{j'}$  of points in  $\vartheta_n$ , with  $j < j'$ , and let  $i, i'$  be the pair of indices such that  $v_j \in \{r_i\} \cup I_{i+1}, v_{j'} \in \{r_{i'}\} \cup I_{i'+1}$ . (In other words,  $i$  and  $i'$  are the indices such that  $i(n/k) \leq j < (i+1)(n/k), i'(n/k) \leq j' < (i'+1)(n/k)$ .) Suppose first that  $v_j$  and  $v_{j'}$  belong to the same interval  $I_{i+1} = I_{i'+1}$ . In this case it can be argued inductively that the monotone distance between  $v_j$  and  $v_{j'}$  is at most  $\bar{\Lambda}(n/k)$ . Otherwise, we construct a 1-spanner path  $\Pi(v_j, v_{j'})$  in  $H_k(n)$  connecting  $v_j$  and  $v_{j'}$  that consists of at most  $O(\alpha(k)) + 2R(n/k)$  edges. Let  $P = (v_j, v_{j_1}, \dots, v_{j'})$  be the sub-path of  $P_n$  between  $v_j$  and  $v_{j'}$ , and let  $c$  and  $c'$  denote the first and last cut vertices along  $P$ , respectively. The path  $\Pi(v_j, v_{j'})$  starts with a 1-spanner path  $\Pi(v_j, c)$  between  $v_j$  and  $c$ . If  $j = i(n/k)$  then  $v_j$  is the cut vertex  $r_i$ , and so  $c = v_j = r_i$ ; in this case  $\Pi(v_j, c)$  is defined as the empty path. Otherwise  $j > i(n/k)$  and  $v_j \in I_{i+1}$ , and so  $c = r_{i+1}$ . By the recursive nature of the construction,  $H_k(n)$  contains as a subgraph a 1-spanner for the  $(i+1)$ th interval  $I_{i+1}$ , having monotone radius at most  $R(n/k)$ ; hence, there is a 1-spanner path  $P^j$  between  $v_j$  and the (right) sentinel  $v_{(i+1)(n/k)-1}$  of  $I_{i+1}$  that consists of at most  $R(n/k)$  edges. The path  $\Pi(v_j, c)$  is defined as the concatenation of the path  $P^j$  and the edge  $(v_{(i+1)(n/k)-1}, v_{(i+1)(n/k)}) = (c, r_{i+1})$ ; note that  $\Pi(v_j, c)$  consists of at most  $R(n/k) + 1$  edges. The path  $\Pi(v_j, v_{j'})$  continues with a 1-spanner path  $\Pi(c, c')$  between the two cut vertices  $c$  and  $c'$  that consists of at most  $O(\alpha(k))$  edges; such a path is guaranteed by the list-spanner. The path  $\Pi(v_j, v_{j'})$  finishes with a 1-spanner path  $\Pi(c', v_{j'})$  between  $c'$  and  $v_{j'}$ . Notice that  $c' = r_{i'}$ . If  $j' = i'(n/k)$  then  $v_{j'}$  is the cut vertex  $r_{i'}$ , and so  $c' = r_{i'} = v_{j'}$ ; in this case  $\Pi(c', v_{j'})$  is defined as the empty path. Otherwise  $j' > i'(n/k)$  and  $v_{j'} \in I_{i'+1}$ , and so  $c' = r_{i'} \neq v_{j'}$ . By the recursive nature of the construction,  $H_k(n)$  contains as a subgraph a 1-spanner for the  $(i'+1)$ th interval  $I_{i'+1}$ , having monotone radius at most  $R(n/k)$ ; hence, there is a 1-spanner path  $P^{j'}$  between the (left) sentinel  $v_{i'(n/k)+1}$  of  $I_{i'+1}$  and  $v_{j'}$  that consists of at most  $R(n/k)$  edges. The path  $\Pi(c', v_{j'})$  is defined as the concatenation of the edge  $(v_{i'(n/k)+1}, v_{i'(n/k)+1}) = (c', r_{i'})$  and the path  $P^{j'}$ ; note that  $\Pi(c', v_{j'})$  consists of at most  $R(n/k) + 1$  edges. Thus,  $\Pi(v_j, v_{j'})$  is given as the concatenation of the three 1-spanner paths  $\Pi(v_j, c), \Pi(c, c')$  and  $\Pi(c', v_{j'})$ , i.e.,  $\Pi(v_j, v_{j'}) = \Pi(v_j, c) \circ \Pi(c, c') \circ \Pi(c', v_{j'})$ . It is easy to see that  $\Pi(v_j, v_{j'})$  is a 1-spanner path in  $H_k(n)$  between  $v_j$  and  $v_{j'}$  that consists of at most  $O(\alpha(k)) + 2R(n/k)$  edges. It follows that  $\bar{\Lambda}(n)$  satisfies the recurrence  $\bar{\Lambda}(n) \leq \max\{\bar{\Lambda}(n/k), O(\alpha(k)) + 2R(n/k)\}$ , with the base condition  $\bar{\Lambda}(q) = O(\alpha(q))$ , for all  $q \leq k$ . Hence  $\bar{\Lambda}(n) = O(\log_k n + \alpha(k))$ .

Denote the worst-case running time of the algorithm by  $t(n)$ , excluding the time needed to add the edges of  $P_n$  to the spanner. We remark that the list-spanner of [45, 12, 2, 9, 44, 38] can be implemented in linear time. By construction,  $t(n)$  satisfies the recurrence  $t(n) \leq O(k) + k \cdot t(n/k)$ , with the base condition  $t(q) = O(q)$ , for all  $q \leq k$ , yielding  $t(n) = O(n)$ . Hence, the overall running time of the algorithm is  $O(n)$ .

Finally, we present a simple transformation of this construction that reduces the maximum degree from  $k+2$  to  $k+1$ , without increasing any of the other parameters by more than a constant factor. Notice that the degree of the two sentinels  $r_0 = v_1$  and  $r_k = v_n$  in  $H_k(n)$  is at most  $k+1$ . Indeed, each of the two sentinels may be incident to all other  $k$  cut vertices; in addition,  $v_1$  (respectively,  $v_n$ )

is incident to its right neighbor  $v_2$  (resp., left neighbor  $v_{n-1}$ ) along the path  $P_n$ . On the other hand, a non-sentinel cut vertex  $r_i = v_{i(n/k)}$ ,  $i \in [k-1]$ , is incident to both its left neighbor  $v_{i(n/k)-1}$  and its right neighbor  $v_{i(n/k)+1}$  along  $P_n$ ; therefore, its degree may be as large as  $k+2$ . To decrease the degree of a non-sentinel cut vertex  $r_i$ ,  $i \in [k-1]$ , we replace it by a pair of consecutive (along  $P_n$ ) cut vertices  $r_i^L, r_i^R$ , where  $r_i^L = r_i = v_{i(n/k)}$ ,  $r_i^R = v_{i(n/k)+1}$ . The two sentinels  $v_1$  and  $v_n$  remain unchanged; for technical convenience, we define  $r_0^L = r_0^R = r_0 = v_1$ ,  $r_k^L = r_k^R = r_k = v_n$ . Thus, instead of having  $k+1$  cut vertices, we will now have  $2k$  cut vertices, namely,  $r_0^L = r_0^R = v_1$ ,  $r_1^L = v_{(n/k)}$ ,  $r_1^R = v_{(n/k)+1}$ ,  $\dots$ ,  $r_{k-1}^L = v_{(k-1)(n/k)}$ ,  $r_{k-1}^R = v_{(k-1)(n/k)+1}$ ,  $r_k^L = r_k^R = v_n$ . By removing these  $2k$  cut vertices from  $P_n$  (along with their incident edges), we are left with  $k$  intervals  $\tilde{I}_1, \tilde{I}_2, \dots, \tilde{I}_k$  of length at most  $n/k - 1$  each, where  $\tilde{I}_i = I_i \setminus \{r_{i-1}^R\}$ , for each index  $i \in [k]$ . We transform  $H_k(n)$  by replacing each edge  $(r_i, r_j)$  in it,  $0 \leq i < j \leq k$ , with the edge  $(r_i^R, r_j^L)$ ; this transformation is applied recursively for each of the intervals  $\tilde{I}_1, \tilde{I}_2, \dots, \tilde{I}_k$ , and is repeated as long as the number of points in an interval is greater than  $k$ . We denote the resulting graph by  $G_k(n)$ . Fix an arbitrary index  $\ell \in [k-1]$ . Clearly, the number of all edges  $(r_i, r_\ell)$  in  $H_k(n)$ , such that  $0 \leq i < \ell$ , is no greater than  $\ell$ ; also, the number of all edges  $(r_\ell, r_j)$  in  $H_k(n)$ , such that  $\ell < j \leq k$ , is no greater than  $k - \ell$ . Thus, by construction, the number of cut vertices that are incident to  $r_\ell^L$  (respectively,  $r_\ell^R$ ) in  $G_k(n)$  is at most  $\ell \leq k-1$  (resp., at most  $k - \ell \leq k-1$ ). Including edges of  $P_n$ , the degree of both  $r_\ell^L$  and  $r_\ell^R$  is at most  $k+1$ , for each index  $\ell \in [k-1]$ . Also, notice that the degree of the two sentinels  $v_1$  and  $v_n$  remains unchanged, and so it is at most  $k+1$  as well. Therefore, after the first level of recursion, each of the  $2k$  cut vertices  $r_0^L = r_0^R, r_1^L, r_1^R, \dots, r_{k-1}^L, r_{k-1}^R, r_k^L = r_k^R$  has degree at most  $k+1$  (including edges of  $P_n$ ), whereas all other vertices have degree zero (excluding edges of  $P_n$ ). In subsequent levels of recursion, the degree of these  $2k$  cut vertices remains unchanged, whereas the degree of the other vertices may increase. By employing this argument inductively, we conclude that the maximum degree of  $G_k(n)$  is at most  $k+1$ , as required. It is easy to see that, similarly to  $H_k(n)$ , the graph  $G_k(n)$  is a 1-spanner for  $\vartheta_n$ ; moreover, it has the same (up to constant factors) diameter, lightness and number of edges as those of  $H_k(n)$ .

The next theorem summarizes the properties of the graph  $G_k(n)$ .

**Theorem 2.1** *For any  $n$ -point 1-dimensional space and a parameter  $k \geq 2$ , there exists a 1-spanner with maximum degree at most  $k+1$ , diameter  $O(\log_k n + \alpha(k))$ , lightness  $O(k \cdot \log_k n)$ , and  $O(n)$  edges. The running time of this construction is  $O(n)$ .*

**Remark:** For  $k = 2$ , we get the optimal degree 3, in conjunction with logarithmic diameter and lightness. The same result also follows from Theorem 2.2 below.

### 2.2.3 1-Spanners with High Diameter

In this section we devise a construction  $H'_k(n)$  of 1-spanners for  $\vartheta_n$  with comparable monotone diameter  $\bar{\Lambda}'(n) = \bar{\Lambda}(H'_k(n))$  in the range  $\bar{\Lambda}'(n) = \Omega(\log n)$ .

The algorithm connects the  $k+1$  cut vertices  $r_0 = v_1, r_1, \dots, r_{k-1}, r_k = v_n$  via a path of length  $k$ , i.e., it adds the edges  $(r_0, r_1), (r_1, r_2), \dots, (r_{k-1}, r_k)$  into the spanner. In addition, it calls itself recursively for each of the intervals  $I_1, I_2, \dots, I_k$ . At the bottom level of the recursion, i.e., when  $n \leq k$ , the algorithm adds no additional edges to the spanner. (See Figures 1 and 2 for an illustration.)

Denote by  $\Delta'(n)$  the maximum degree of a vertex in  $H'_k(n)$ , excluding edges of  $P_n$ . Clearly,  $\Delta'(n)$  satisfies the recurrence  $\Delta'(n) \leq \max\{2, \Delta'(n/k)\}$ , with the base condition  $\Delta'(q) = 0$ , for all  $q \leq k$ , yielding  $\Delta'(n) \leq 2$ . Including edges of  $P_n$ , the maximum degree increases by at most two units, and so  $\Delta(H'_k(n)) \leq 4$ . Consequently, the number of edges in  $H'_k(n)$  is no greater than  $2n$ .

Denote by  $w'(n)$  the weight of  $H'_k(n)$ , excluding edges of  $P_n$ . Note that the weight of the path connecting all  $k+1$  cut vertices is equal to  $n-1$ . The total weight of all edges within an interval  $I_i$  is at most  $w'(n/k)$ . Hence  $w'(n)$  satisfies the recurrence  $w'(n) \leq n-1 + k \cdot w'(n/k)$ , with the base condition  $w'(q) \leq q-1$ , for all  $q \leq k$ . It follows that  $w'(n) = O(n \cdot \log_k n) = O(\log_k n) \cdot w(MST(\vartheta_n))$ . Including edges of  $P_n$ , the weight increases by  $w(P_n) = n-1$  units.

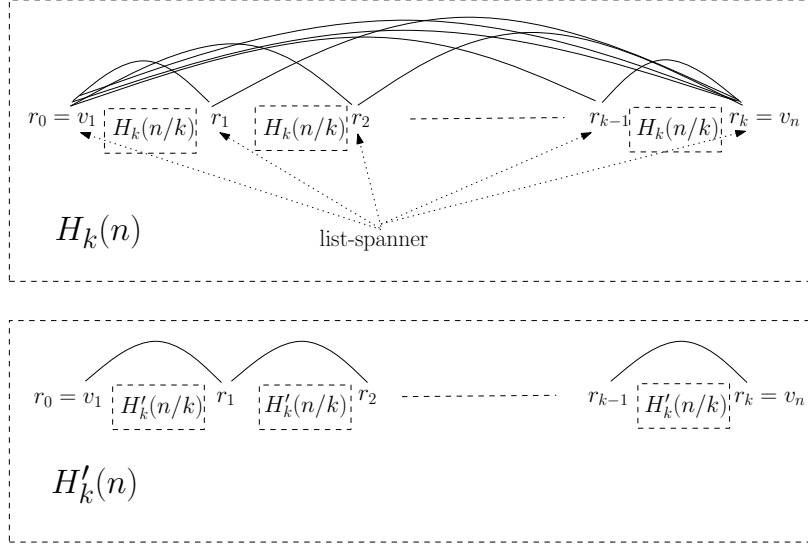


Figure 2: The constructions  $H_k(n)$  and  $H'_k(n)$  for a general parameter  $k, k \geq 2$ . Only the first level of the recursion is illustrated. (Path-edges are not depicted in the figure.) For  $H_k(n)$ , all the cut vertices are connected via the list-spanner, and, in addition, each of the two sentinels is connected to all other  $k$  cut vertices. For  $H'_k(n)$ , each cut vertex  $r_{i-1}$  is connected to the next cut vertex  $r_i$  in line,  $i \in [k]$ .

Note that the monotone radius  $R'(n)$  of  $H'_k(n)$  satisfies the recurrence  $R'(n) \leq k + R'(n/k)$ , with the base condition  $R'(q) \leq q - 1$ , for all  $q \leq k$ . Hence,  $R'(n) = O(k \cdot \log_k n)$ . Using reasoning similar to that of Section 2.2.2, we get that the comparable monotone diameter  $\bar{\Lambda}'(n) = \bar{\Lambda}(H'_k(n))$  of  $H'_k(n)$  satisfies the recurrence  $\bar{\Lambda}'(n) \leq \max\{\bar{\Lambda}'(n/k), k + 2R'(n/k)\}$ , with the base condition  $\bar{\Lambda}'(q) \leq q - 1$ , for all  $q \leq k$ . It follows that  $\bar{\Lambda}'(n) = O(k \cdot \log_k n)$ .

We remark that the spanner  $H'_k(n)$  is a planar graph.

Denote the worst-case running time of the algorithm by  $t'(n)$ , excluding the time needed to add the edges of  $P_n$  to the spanner. It is easy to see that  $t'(n)$  satisfies the recurrence  $t'(n) \leq O(k) + k \cdot t'(n/k)$ , with the base condition  $t'(q) = O(1)$ , for all  $q \leq k$ , yielding  $t'(n) = O(n)$ . Hence, the overall running time of the algorithm is  $O(n)$ .

Finally, similarly to the construction of Section 2.2.2, the maximum degree of this construction can be reduced from 4 to 3, without increasing any of the other parameters by more than a constant factor.

**Theorem 2.2** *For any  $n$ -point 1-dimensional space and a parameter  $k$ , there exists a 1-spanner with maximum degree 3, diameter  $O(k \cdot \log_k n)$ , and lightness  $O(\log_k n)$ . Moreover, this 1-spanner is a planar graph. The running time of this construction is  $O(n)$ .*

## 2.3 General Tree Metrics

In this section we extend the constructions of Section 2.2 from line metrics to general tree metrics.

### 2.3.1 Selecting the Cut-Vertices

In this section we present a procedure for selecting, given a tree  $T$ , a subset of  $O(k)$  vertices whose removal from the tree partitions it into subtrees of size  $O(|T|/k)$  each. This subset will also satisfy several additional useful properties.

Let  $(T, rt)$  be a rooted tree. For an inner vertex  $v$  in  $T$  with  $ch(v)$  children, we denote its children, from left to right, by  $c_1(v), c_2(v), \dots, c_{ch(v)}(v)$ . Suppose without loss of generality that the size of the subtree  $T_{c_1(v)}$  of  $v$  is no smaller than the size of any other subtree of  $v$ , i.e.,  $|T_{c_1(v)}| \geq |T_{c_2(v)}|, |T_{c_3(v)}|, \dots, |T_{c_{ch(v)}(v)}|$ . (This assumption can be guaranteed by a straightforward procedure that runs in linear time.) We say

that the vertex  $c_1(v)$  is the *left-most* child of  $v$ . Also, an edge in  $T$  is called *left-most* if it connects a vertex  $v$  in  $T$  and its left-most child  $c_1(v)$ . We denote by  $P(v) = (v, c_1(v), \dots, l(v))$  the path of left-most edges leading down from  $v$  to some leaf  $l(v)$  in the subtree  $T_v$  of  $T$  rooted at  $v$ ; the leaf  $l(v)$  is referred to as the *left-most vertex* in  $T_v$ . Also, let  $l(T) = l(rt)$  denote the left-most vertex in the entire tree  $T$ . An inner vertex  $v$  in  $T$  is called *d-balanced*, for  $d \geq 1$ , or simply *balanced* if  $d$  is clear from the context, if  $|T_{c_1(v)}| \leq |T| - d$ . The first (i.e., closest to  $v$ ) balanced vertex along  $P(v)$  is denoted by  $b(v)$ ; if no vertex along  $P(v)$  is balanced, we write  $b(v) = \text{NULL}$ . Observe that for  $|T| \geq 2d$ , we have  $|T| - d \geq d \geq 1$ ; in this case the one-before-last vertex along  $P(v)$  (namely, the parent  $\pi(l(v))$  of  $l(v)$  in  $T$ ) is balanced. Hence, in this case  $b(v) \neq \text{NULL}$ .

Next, we present the Procedure *CV* (standing for cut vertices) that accepts as input a rooted tree  $(T, rt)$  and a parameter  $d \geq 1$ , and returns as output a subset of  $V(T)$ . If  $|T| < 2d$ , the procedure returns the empty set  $\emptyset$ . Otherwise  $|T| \geq 2d$ , and so the first balanced vertex  $b = b(rt)$  along  $P(rt)$  satisfies  $b \neq \text{NULL}$ . In this case for each child  $c_i(b)$  of  $b$ ,  $i \in [ch(b)]$ , the procedure recursively constructs the subset  $C_i = CV((T_{c_i(b)}, c_i(b)), d)$ , and then returns as output the vertex set  $\bigcup_{i=1}^{ch(b)} C_i \cup \{b\}$ . (See Figure 3 for an illustration.)

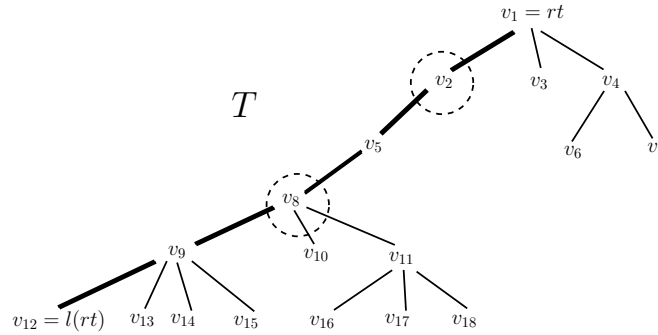


Figure 3: A rooted tree  $(T, rt)$  with  $n = |T| = 18$  vertices  $v_1 = rt, v_2, \dots, v_{18}$ . The edges of  $P(rt)$  are depicted by bold lines. The first 6-balanced vertex along  $P(rt)$  is  $v_2$ . The procedure *CV* on input  $(T, rt)$  and  $d = 6$  returns the subset  $\{v_2, v_8\}$ .

It is easy to see that the running time of this procedure is linear in  $|T|$ .

Let  $(T, rt)$  be an  $n$ -vertex rooted tree, and let  $d \geq 1$  be a fixed parameter. For convenience, we define  $n_i = |T_{c_i(b)}|$ , for each  $i \in [ch(b)]$ . Next, we analyze the properties of the set  $C = CV((T, rt), d)$  of *cut vertices*.

Observe that for  $n < 2d$ ,  $C = \emptyset$ , and for  $n \geq 2d$ ,  $C$  is non-empty.

Next, we provide an upper bound on  $|C|$  in the case  $n \geq 2d$ .

**Lemma 2.3** For  $n \geq 2d$ ,  $|C| \leq (n/d) - 1$ .

**Proof:** The proof is by induction on  $n = |T|$ .

*Basis:*  $2d \leq n < 3d$ . Fix an index  $i \in [ch(b)]$ . Since  $b$  is balanced, we have

$$n_i \leq n_1 \leq n - d < 2d,$$

implying that  $C_i = \emptyset$ . It follows that  $C = \bigcup_{i=1}^{ch(b)} C_i \cup \{b\} = \{b\}$ , and so  $|C| = 1 \leq (n/d) - 1$ .

*Induction Step:* We assume the correctness of the statement for all smaller values of  $n$ ,  $n \geq 3d$ , and prove it for  $n$ . Let  $I$  be the set of all indices  $i$  in  $[ch(b)]$  for which  $n_i \geq 2d$ . Observe that for each  $i \in [ch(b)] \setminus I$ ,  $C_i = \emptyset$ , and by the induction hypothesis, for each  $i \in I$ ,  $|C_i| \leq (n_i/d) - 1$ . By construction, the vertex sets  $C_1, C_2, \dots, C_{ch(b)}$  and  $\{b\}$  are pairwise disjoint, and  $C = \bigcup_{i=1}^{ch(b)} C_i \cup \{b\}$ . Hence

$$|C| = \sum_{i=1}^{ch(b)} |C_i| + 1 = \sum_{i \in I} |C_i| + 1 \leq \sum_{i \in I} ((n_i/d) - 1) + 1. \quad (1)$$

The analysis splits into three cases depending on the value of  $|I|$ .

*Case 1:*  $|I| = 0$ . Equation (1) yields  $|C| \leq 1 \leq (n/d) - 1$ .

*Case 2:*  $|I| = 1$ . By construction,  $n_1 \geq n_i$ , for each  $i \in [ch(b)]$ , implying that  $I = \{1\}$ . Since  $b$  is balanced,  $n_1 \leq n - d$ , and so (1) yields

$$|C| \leq (n_1/d) - 1 + 1 \leq (n - d)/d = (n/d) - 1.$$

*Case 3:*  $|I| \geq 2$ . Clearly,  $\sum_{i \in I} n_i \leq n - 1$ , and so (1) yields

$$|C| \leq \sum_{i \in I} ((n_i/d) - 1) + 1 = \sum_{i \in I} (n_i/d) - |I| + 1 \leq (n - 1)/d - 2 + 1 \leq (n/d) - 1.$$

□

Let  $b = b(rt)$ , and let  $\overline{T}_b$  be the subtree of  $T$  obtained by removing the subtree  $T_b$  from  $T$ . We use the following claim to prove Lemma 2.5.

**Claim 2.4**  $|\overline{T}_b| < d$ .

**Proof:** If  $b = rt$ , then  $\overline{T}_b$  is empty and the assertion of the claim is immediate. Otherwise, consider the parent  $\pi(b)$  of  $b$  in  $T$ . Since  $b$  is the first (i.e., closest to  $rt$ ) balanced vertex along  $P(rt)$ ,  $\pi(b)$  is non-balanced, and so  $|T_b| = |T_{c_1(\pi(b))}| > n - d$ . Hence  $|\overline{T}_b| = n - |T_b| < d$ , and we are done. □

For a subset  $U$  of  $V(T)$ , we denote by  $T \setminus U$  the forest obtained from  $T$  by removing all vertices in  $U$  along with the edges that are incident to them.

**Lemma 2.5** *The size of any subtree in the forest  $T \setminus C$  is smaller than  $2d$ .*

**Proof:** The proof is by induction on  $n = |T|$ . The basis  $n < 2d$  is trivial.

*Induction Step:* We assume the correctness of the statement for all smaller values of  $n$ ,  $n \geq 2d$ , and prove it for  $n$ . First, note that  $b = b(rt) \in C$ . Also, observe that for  $n \geq 2d$ ,

$$T \setminus C = \bigcup_{i=1}^{ch(b)} (T_{c_i(b)} \setminus C_i) \cup \{\overline{T}_b\}. \quad (2)$$

Consider a subtree  $T'$  in the forest  $T \setminus C$ . By (2), either  $T' = \overline{T}_b$ , or it belongs to the forest  $T_{c_i(b)} \setminus C_i$ , for some index  $i \in [ch(b)]$ . In the former case, the size bound follows from Claim 2.4, whereas in the latter case it follows from the induction hypothesis. □

Any subset  $U$  of  $V(T)$  induces a forest  $\mathcal{Q}(T, U)$  over  $U$  in the natural way: a vertex  $v \in U$  is defined to be a child of its closest ancestor in  $T$  that belongs to  $U$ . Define  $\mathcal{Q} = \mathcal{Q}(T, C)$ . Observe that for  $n < 2d$ ,  $C = \emptyset$ , and so  $\mathcal{Q} = \emptyset$ . Also, for  $n \geq 2d$ ,  $C$  is non-empty and  $b = b(rt) \neq NULL$ .

**Lemma 2.6** *For  $n \geq 2d$ ,  $\mathcal{Q}$  is a spanning tree of  $C$  rooted at  $b = b(rt)$ , such that for each vertex  $v$  in  $C$ , the number of children of  $v$  in  $\mathcal{Q}$ , denoted  $ch_{\mathcal{Q}}(v)$ , is no greater than the corresponding number  $ch(v)$  in  $T$ .*

**Remark:** This lemma implies that  $\Delta(\mathcal{Q}) \leq \Delta(T)$ .

**Proof:** The proof is by induction on  $n = |T|$ .

*Basis:*  $2d \leq n < 3d$ . In this case  $C = \{b\}$ , and so  $\mathcal{Q}$  consists of a single root vertex  $b$ .

*Induction Step:* We assume the correctness of the statement for all smaller values of  $n$ ,  $n \geq 3d$ , and prove it for  $n$ . Let  $I$  be the set of all indices  $i$  in  $[ch(b)]$  for which  $n_i \geq 2d$ , and write  $I = \{i_1, i_2, \dots, i_{|I|}\}$ . Observe that for each index  $i \in [ch(b)] \setminus I$ ,  $C_i = \emptyset$ , and so  $\mathcal{Q}(T_{c_i(b)}, C_i)$  is an empty tree. By the induction

hypothesis, for each  $i \in I$ ,  $\mathcal{Q}_i = \mathcal{Q}(T_{c_i(b)}, C_i)$  is a spanning tree of  $C_i$  rooted at  $b_i = b(c_i(b)) \neq \text{NULL}$  in which the number of children of each vertex is no greater than the corresponding number in  $T_{c_i(b)}$ . By definition, the only children of  $b$  in  $\mathcal{Q}$  are the roots  $b_{i_1}, b_{i_2}, \dots, b_{i_{|I|}}$  of the non-empty trees  $\mathcal{Q}_{i_1}, \mathcal{Q}_{i_2}, \dots, \mathcal{Q}_{i_{|I|}}$ , respectively, and so  $\text{ch}_{\mathcal{Q}}(b) = |I| \leq \text{ch}(b)$ . In addition,  $b$  has no parent in  $\mathcal{Q}$ , and so it is the root of  $\mathcal{Q}$ .  $\square$

For a tree  $\tau$ , the root  $\text{rt}(\tau)$  of  $\tau$  and its left-most vertex  $l(\tau)$  are called the *sentinels* of  $\tau$ . The next lemma shows that each subtree in the forest  $T \setminus C$  is incident to at most two cut vertices. The proof of this lemma follows similar lines as those in the proof of Lemma 2.5, and is thus omitted.

**Lemma 2.7** *For any subtree  $T'$  in the forest  $T \setminus C$ , no other vertex in  $T'$  other than its two sentinels  $\text{rt}(T')$  and  $l(T')$  is incident to a vertex from  $C$ . Moreover, both  $\text{rt}(T')$  and  $l(T')$  are incident to at most one vertex from  $C$ ; specifically,  $\text{rt}(T')$  is incident to its parent in  $T$ , unless  $\text{rt}(T')$  is the root of  $T$ , and  $l(T')$  is incident to its left-most child in  $T$ , unless  $l(T')$  is a leaf in  $T$ .*

Similarly to the 1-dimensional case, we add the two sentinels  $\text{rt}(T)$  and  $l(T)$  of the original tree  $T$  to the set  $C$  of cut vertices. From now on we refer to the appended set  $\tilde{C} = C \cup \{\text{rt}(T), l(T)\}$  as the set of *cut vertices*. Intuitively, Lemma 2.7 shows that the Procedure *CV* “slices” the tree in a “path-like” fashion, i.e., in a way that is analogous to the decomposition of  $\vartheta_n$  into intervals described in Section 2.2.1. (See Figure 4 for an illustration.)

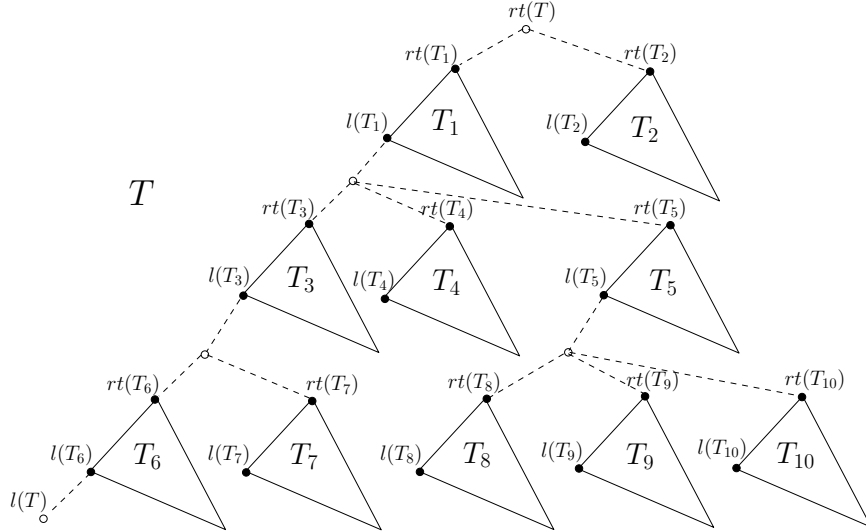


Figure 4: A “path-like” decomposition of the tree  $T$  into subtrees  $T_1, T_2, \dots, T_{10}$ . The 5 cut vertices of  $\tilde{C}$  (i.e., the 3 vertices of  $C$  and the 2 sentinels  $\text{rt}(T)$  and  $l(T)$  of  $T$ ) are depicted in the figure by empty dots, whereas the 20 sentinels of the subtrees  $T_1, T_2, \dots, T_{10}$  are depicted by filled dots. Similarly to the 1-dimensional case, each subtree  $T_i$  is incident to at most two cut vertices. Edges in  $T$  that connect sentinels of subtrees with cut vertices are depicted by dashed lines.

Lemmas 2.3, 2.5, 2.6 and 2.7 imply the following corollary, which summarizes the properties of the set  $\tilde{C}$  of cut vertices.

**Corollary 2.8** 1. For  $n \geq 2d$ ,  $|\tilde{C}| \leq (n/d) + 1$ .

2. The size of any subtree in the forest  $T \setminus \tilde{C}$  is smaller than  $2d$ .

3.  $\tilde{\mathcal{Q}} = \mathcal{Q}(T, \tilde{C})$  is a spanning tree of  $\tilde{C}$  rooted at  $\text{rt}(T)$ , with  $\Delta(\tilde{\mathcal{Q}}) \leq \Delta(T)$ .

4. For any subtree  $T'$  in the forest  $T \setminus \tilde{C}$ , only the two sentinels  $\text{rt}(T')$  and  $l(T')$  of  $T'$  are incident to a vertex from  $\tilde{C}$ . Moreover, both  $\text{rt}(T')$  and  $l(T')$  are incident to at most one vertex from  $\tilde{C}$ ; specifically,  $\text{rt}(T')$  is incident to its parent in  $T$ , and  $l(T')$  is incident to its left-most child in  $T$ , unless  $l(T')$  is a leaf in  $T$ .

**Remark:** The running time of the Procedure  $CV$  is  $O(n)$ . Hence the set  $\tilde{C}$  of cut vertices can be computed in linear time.

### 2.3.2 1-Spanners with Low Diameter

Consider an  $n$ -vertex (weighted) tree  $T$ , and let  $M_T$  be the tree metric induced by  $T$ . In this section we devise a construction  $\mathcal{H}_k(n)$  of 1-spanners for  $M_T$  with comparable monotone diameter  $\bar{\Lambda}(n) = \bar{\Lambda}(\mathcal{H}_k(n))$  in the range  $\Omega(\alpha(n)) = \bar{\Lambda}(n) = O(\log n)$ . Both in this construction and in the one of Section 2.3.3, all edges of the original tree  $T$  are added to the spanner.

Let  $k$  be a fixed parameter such that  $4 \leq k \leq n/2 - 1$ , and set  $d = n/k$ . (Observe that  $n \geq 2k + 2$  and  $d > 2$ .) To select the set  $\tilde{C}$  of cut vertices, we invoke the procedure  $CV$  on the input  $(T, rt)$  and  $d$ . Set  $C = CV((T, rt), d)$  and  $\tilde{C} = C \cup \{rt(T), l(T)\}$ . Since  $k \geq 4$ , it holds that  $2d = 2n/k < n$ . Denote the subtrees in the forest  $T \setminus \tilde{C}$  by  $T_1, T_2, \dots, T_p$ . By Corollary 2.8,  $|\tilde{C}| \leq (n/d) + 1 = k + 1$ , and each subtree  $T_i$  in  $T \setminus \tilde{C}$  has size less than  $2d = 2n/k$ . Observe that  $\sum_{i=1}^p |T_i| = n - |\tilde{C}| \geq n - k - 1$ , implying that the number  $p$  of subtrees in  $T \setminus \tilde{C}$  satisfies

$$p \geq \frac{n - k - 1}{2n/k} \geq k/4. \quad (3)$$

(The last inequality holds for  $k \leq n/2 - 1$ .)

To connect the set  $\tilde{C}$  of cut vertices, the algorithm first constructs the tree  $\tilde{Q} = \mathcal{Q}(T, \tilde{C})$ . Observe that  $\tilde{Q}$  inherits the tree structure of  $T$ , that is, for any two points  $u$  and  $v$  in  $\tilde{C}$ ,  $u$  is an ancestor of  $v$  in  $\tilde{Q}$  if and only if it is its ancestor in  $T$ . Consequently, any 1-spanner path for the tree metric  $M_{\tilde{Q}}$  induced by  $\tilde{Q}$  between two arbitrary comparable<sup>6</sup> points is also a 1-spanner path for the original tree metric  $M_T$ . The algorithm proceeds by building a 1-spanner for  $\tilde{Q}$  via one of the aforementioned generalized constructions from [12, 2, 44, 38] (henceforth, *tree-spanner*). In other words,  $O(k)$  edges between cut vertices are added to the spanner  $\mathcal{H}_k(n)$  to guarantee that the monotone distance in the spanner between any two comparable cut vertices will be  $O(\alpha(k))$ . Then the algorithm adds to the spanner  $\mathcal{H}_k(n)$  edges that connect each of the two sentinels to all other cut vertices. (In fact, the leaf  $l(T)$  needs not be connected to all cut vertices, but rather only to those which are its ancestors in  $T$ .) Finally, the algorithm calls itself recursively for each of the subtrees  $T_1, T_2, \dots, T_p$  of  $T$ . At the bottom level of the recursion, i.e., when  $n < 2k + 2$ , the algorithm uses the tree-spanner to connect all points, and, in addition, it adds to the spanner edges that connect each of the two sentinels  $rt(T)$  and  $l(T)$  to all the other  $n - 1$  points.

We denote by  $E(n)$  the number of edges in  $\mathcal{H}_k(n)$ , excluding edges of  $T$ . Clearly,  $E(n)$  satisfies the recurrence  $E(n) \leq O(k) + \sum_{i=1}^p E(|T_i|)$ , with the base condition  $E(q) = O(q)$ , for all  $q < 2k + 2$ . Recall that for each  $i \in [p]$ ,  $|T_i| \leq 2d = 2n/k < n$ , and by Equation (3), we have  $p \geq k/4$ . Also, since  $\tilde{C}$  is non-empty, it holds that  $\sum_{i=1}^p |T_i| = n - |\tilde{C}| \leq n - 1$ . Next, we prove by induction on  $n$  that  $E(n) \leq 4c(n - 1)$ , for a sufficiently large constant  $c$ . The basis  $n < 2k + 2$  is immediate. For  $n \geq 2k + 2$ , the induction hypothesis implies that

$$E(n) \leq c \cdot k + 4c \cdot \sum_{i=1}^p (|T_i| - 1) = c \cdot k - 4c \cdot p + 4c \cdot \sum_{i=1}^p |T_i| \leq c(k - 4p) + 4c(n - 1) \leq 4c(n - 1).$$

(The last inequality holds as  $p \geq k/4$ .)

Denote by  $\Delta(n)$  the maximum degree of a vertex in  $\mathcal{H}_k(n)$ , excluding edges of  $T$ . Since  $|\tilde{C}| \leq k + 1$ ,  $\Delta(n)$  satisfies the recurrence  $\Delta(n) \leq \max\{k, \Delta(2n/k)\}$ , with the base condition  $\Delta(q) \leq 2k$ , for all  $q < 2k + 2$ . It follows that  $\Delta(n) \leq 2k$ . Including edges of the tree  $T$ , the number of edges increases by at most  $n - 1$  units and the maximum degree increases by at most  $\Delta(T)$  units.

<sup>6</sup>This may not hold true for two points that are not comparable, as their least common ancestor may not belong to  $\tilde{Q}$ .

Next, we show that the lightness  $\Psi(\mathcal{H}_k(n))$  of the spanner  $\mathcal{H}_k(n)$  satisfies  $\Psi(\mathcal{H}_k(n)) = O(k \cdot \log_k n)$ . To this end, we extend the notion of *load* defined in [1]<sup>7</sup> for 1-dimensional spaces to general tree metrics. Consider an edge  $e' = (v, w)$  connecting two arbitrary points in  $M_T$ , and an edge  $e \in E(T)$ . The edge  $e'$  is said to *load*  $e$  if the unique path in  $T$  between the endpoints  $v$  and  $w$  of  $e'$  traverses  $e$ . For a spanning subgraph  $H$  of  $M_T$ , the number of edges  $e' \in E(H)$  that load an edge  $e \in E(T)$  is called the *load* of  $e$  by  $H$ , and is denoted by  $\chi(e) = \chi_H(e)$ . The *load* of  $H$  (with respect to  $T$ ),  $\chi(H) = \chi_T(H)$ , is the maximum load of an edge of  $T$  by  $H$ . By double-counting,

$$\begin{aligned} w(H) &= \sum_{e' \in E(H)} w(e') = \sum_{e' \in E(H)} \sum_{\{e \in E(T) : e \text{ loaded by } e'\}} w(e) = \sum_{e \in E(T)} \sum_{\{e' \in E(H) : e' \text{ loads } e\}} w(e) \\ &= \sum_{e \in E(T)} \chi_H(e) \cdot w(e) \leq \chi(H) \cdot \sum_{e \in E(T)} w(e) = \chi(H) \cdot w(T), \end{aligned} \quad (4)$$

implying that  $\Psi(H) = w(H)/w(T) \leq \chi(H)$ . Thus it suffices to provide an upper bound of  $O(k \cdot \log_k n)$  on the load  $\chi(\mathcal{H}_k(n))$  of  $\mathcal{H}_k(n)$ . Denote by  $\chi(n)$  the load of  $\mathcal{H}_k(n)$ , excluding edges of  $T$ . After the first level of recursion,  $\mathcal{H}_k(n)$  contains only  $O(k)$  edges that connect cut vertices. These edges contribute  $O(k)$  units of load to each edge of  $T$ . In particular, after the first level of recursion, each subtree in the forest  $T \setminus \tilde{C}$  is loaded by at most  $O(k)$  edges. Hence  $\chi(n)$  satisfies the recurrence  $\chi(n) \leq O(k) + \chi(2n/k)$ , with the base condition  $\chi(q) = O(q)$ , for all  $q < 2k + 2$ , yielding  $\chi(n) = O(k \cdot \log_k n)$ . Including edges of the tree  $T$ , the load increases by one unit, and we are done.

Next, we show that  $\bar{\Lambda}(n) = \bar{\Lambda}(\mathcal{H}_k(n)) = O(\log_k n + \alpha(k))$ . The *leaf radius*  $\hat{R}(n)$  of  $\mathcal{H}_k(n)$  is defined as the maximum monotone distance between the left-most vertex  $l(T)$  in  $T$  and one of its ancestors in  $T$ . By Corollary 2.8, similarly to the 1-dimensional case,  $\hat{R}(n)$  satisfies the recurrence  $\hat{R}(n) \leq 2 + \hat{R}(2n/k)$ , with the base condition  $\hat{R}(q) = 1$ , for all  $q < 2k + 2$ . Hence,  $\hat{R}(n) = O(\log_k n)$ . Similarly, we define the *root radius*  $\check{R}(n)$  as the maximum monotone distance between the root  $rt(T)$  of  $T$  and some other point in  $T$ . By the same argument we get  $\check{R}(n) = O(\log_k n)$ . Applying again Corollary 2.8 and reasoning similar to the 1-dimensional case, we get that  $\bar{\Lambda}(n)$  satisfies the recurrence  $\bar{\Lambda}(n) \leq \max\{\bar{\Lambda}(2n/k), O(\alpha(k)) + \check{R}(2n/k) + \hat{R}(2n/k)\}$ , with the base condition  $\bar{\Lambda}(q) = O(\alpha(q))$ , for all  $q < 2k + 2$ . It follows that  $\bar{\Lambda}(n) = O(\log_k n + \alpha(k))$ .

Finally, we argue that the worst-case running time of the algorithm, denoted  $t(n)$ , is  $O(n \cdot \log_k n)$ . The algorithm starts by invoking the decomposition procedure for selecting the set  $\tilde{C}$  of cut vertices. As was mentioned above, this step requires  $O(n)$  time. Next, the algorithm builds the tree  $\tilde{Q}$ , which can be carried out in time  $O(|\tilde{Q}|) = O(k)$ . The algorithm proceeds by building the tree-spanner for  $\tilde{Q}$ . The tree-spanner of [12, 2, 44, 38] can be built within linear time. Hence, building the tree-spanner for  $\tilde{Q}$  requires  $O(k)$  time. Next, the algorithm adds to the spanner edges that connect each of the two sentinels to all other cut vertices, which can be carried out within time  $O(k)$  as well. Finally, the algorithm calls itself recursively for each of the subtrees  $T_1, T_2, \dots, T_p$  of  $T$ , which requires at most  $\sum_{i=1}^p t(|T_i|)$  time. At the bottom level of the recursion, i.e., when  $n < 2k + 2$ , the algorithm uses the tree-spanner to connect all points, and in addition, it adds to the spanner edges that connect each of the two sentinels of the tree to all the other  $n - 1$  points. Hence, the running time of the algorithm at the bottom level of the recursion is  $O(n)$ . It follows that  $t(n)$  satisfies the recurrence  $t(n) \leq O(n) + \sum_{i=1}^p t(|T_i|)$ , with the base condition  $t(q) = O(q)$ , for all  $q < 2k + 2$ . Recall that  $k \geq 4$ , and for each  $i \in [p]$ ,  $|T_i| \leq 2d = 2n/k < n$ . We conclude that  $t(n) = O(n \cdot \log_k n)$ .

**Theorem 2.9** *For any tree metric  $M_T$  and a parameter  $k$ , there exists a 1-spanner with maximum degree at most  $\Delta(T) + 2k$ , diameter  $O(\log_k n + \alpha(k))$ , lightness  $O(k \cdot \log_k n)$ , and  $O(n)$  edges. The running time of this construction is  $O(n \cdot \log_k n)$ .*

We remark that the maximum degree  $\Delta(\mathcal{H})$  of the spanner  $\mathcal{H} = \mathcal{H}_k(n)$  cannot be in general smaller than the maximum degree  $\Delta(T)$  of the original tree. Indeed, consider a unit weight star  $T$  with edge

<sup>7</sup>Agarwal et al. [1] used a slightly different notion which they called *covering*. The notion of load as defined above was introduced in [20], but the two notions are very close.

set  $\{(rt, v_1), (rt, v_2), \dots, (rt, v_{n-1})\}$ . Obviously, any spanner  $\mathcal{H}$  for  $M_T$  with  $\Delta(\mathcal{H}) < n - 1$  distorts the distance between the root  $rt$  and some other vertex.

### 2.3.3 1-Spanners with High Diameter

In this section we devise a construction  $\mathcal{H}'_k(n)$  of 1-spanners for  $M_T$  with comparable monotone diameter  $\bar{\Lambda}'(n) = \bar{\Lambda}(\mathcal{H}'_k(n))$  in the range  $\bar{\Lambda}'(n) = \Omega(\log n)$ .

The algorithm starts with constructing the tree  $\tilde{\mathcal{Q}} = \mathcal{Q}(T, \tilde{C})$  that spans the set  $\tilde{C}$  of cut vertices. All edges of  $\tilde{\mathcal{Q}}$  are inserted into  $\mathcal{H}'_k(n)$ . (This step is analogous to taking the edges  $(r_0, r_1), (r_1, r_2), \dots, (r_{k-1}, r_k)$  in the 1-dimensional construction of Section 2.2.3.) Observe that the depth of  $\tilde{\mathcal{Q}}$  is at most  $k$ , implying that any two comparable cut vertices are connected via a 1-spanner path in  $\tilde{\mathcal{Q}}$  that consists of at most  $k$  edges; since  $\tilde{\mathcal{Q}}$  inherits the tree structure of  $T$ , this path is also a 1-spanner path for the original tree metric  $M_T$ . Then the algorithm calls itself recursively for each of the subtrees  $T_1, T_2, \dots, T_p$  of  $T$ . At the bottom level of the recursion, i.e., when  $n < 2k + 2$ , the algorithm adds no additional edges to the spanner.

Similarly to Section 2.3.2 it follows that the number of edges in  $\mathcal{H}'_k(n)$  is  $O(n)$ . Next, we analyze the maximum degree of this construction. Denote by  $\Delta'(n)$  the maximum degree of a vertex in  $\mathcal{H}'_k(n)$ , excluding edges of  $T$ , and let  $\Delta_0 = \Delta(T)$  denote the maximum degree of the original tree  $T$ . By the third assertion of Corollary 2.8,  $\Delta(\tilde{\mathcal{Q}}) \leq \Delta_0$ , and so  $\Delta'(n)$  satisfies the recurrence  $\Delta'(n) \leq \max\{\Delta_0, \Delta'(2n/k)\}$ , with the base condition  $\Delta'(q) = 0$ , for all  $q < 2k + 2$ , yielding  $\Delta'(n) \leq \Delta_0$ . It follows that the maximum degree  $\Delta(\mathcal{H}'_k(n))$  of  $\mathcal{H}'_k(n)$  is at most  $2 \cdot \Delta_0 = 2 \cdot \Delta(T)$ .

Next, we show that the load  $\chi(\mathcal{H}'_k(n))$  of  $\mathcal{H}'_k(n)$  is  $O(\log_k n)$ , which, by (4), implies that  $\Psi(\mathcal{H}'_k(n)) = O(\log_k n)$ . Denote by  $\chi'(n)$  the load of  $\mathcal{H}'_k(n)$ , excluding edges of  $T$ . After the first level of recursion,  $\mathcal{H}'_k(n)$  contains just the edges of the tree  $\tilde{\mathcal{Q}}$ . We argue that each edge  $e = (u, v)$  of  $T$  is loaded by at most one edge of  $\tilde{\mathcal{Q}}$ . Indeed, if both  $u$  and  $v$  are cut vertices, then  $e$  is also an edge of  $\tilde{\mathcal{Q}}$ , and so it is loaded by itself. Otherwise, either  $u$  or  $v$  (or both of them) belongs to some subtree  $T_i$  in the forest  $T \setminus \tilde{C}$ . In this case, the fourth assertion of Corollary 2.8 implies that  $e$  is loaded by at most one edge in  $\tilde{\mathcal{Q}}$ , namely, the edge connecting the parent of  $rt(T_i)$  in  $T$  and the left-most child of  $l(T_i)$  in  $T$ , if exists. In particular, after the first level of recursion, each subtree in the forest  $T \setminus \tilde{C}$  is loaded by at most one edge. Hence  $\chi'(n)$  satisfies the recurrence  $\chi'(n) \leq 1 + \chi'(2n/k)$ , with the base condition  $\chi'(q) = 0$ , for all  $q < 2k + 2$ . It follows that  $\chi'(n) = O(\log_k n)$ . Including edges of  $T$ , the load increases by one unit, and we are done.

By Corollary 2.8, similarly to the 1-dimensional case, the leaf radius  $\hat{R}'(n)$  of  $\mathcal{H}'_k(n)$  satisfies the recurrence  $\hat{R}'(n) \leq k + \hat{R}'(2n/k)$ , with the base condition  $\hat{R}'(q) \leq q - 1$ , for all  $q < 2k + 2$ , yielding  $\hat{R}'(n) = O(k \cdot \log_k n)$ . Similarly, we get that  $\check{R}'(n) = O(k \cdot \log_k n)$ . Applying Corollary 2.8 and reasoning similar to the 1-dimensional case, we get that the comparable monotone diameter  $\bar{\Lambda}'(n) = \bar{\Lambda}(\mathcal{H}'_k(n))$  of  $\mathcal{H}'_k(n)$  satisfies the recurrence  $\bar{\Lambda}'(n) \leq \max\{\bar{\Lambda}'(2n/k), k + \check{R}'(2n/k) + \hat{R}'(2n/k)\}$ , with the base condition  $\bar{\Lambda}'(q) \leq q - 1$ , for all  $q < 2k + 2$ . It follows that  $\bar{\Lambda}'(n) = O(k \cdot \log_k n)$ .

We remark that  $\mathcal{H}'_k(n)$  is a planar graph.

Finally, by employing an argument very similar to the one used in Section 2.3.2, we get that the worst-case running time of the algorithm is  $O(n \cdot \log_k n)$ .

**Theorem 2.10** *For any tree metric  $M_T$  and a parameter  $k$ , there exists a 1-spanner with maximum degree at most  $2 \cdot \Delta(T)$ , diameter  $O(k \cdot \log_k n)$ , lightness  $O(\log_k n)$ , and  $O(n)$  edges. Moreover, this 1-spanner is a planar graph. The running time of this construction is  $O(n \cdot \log_k n)$ .*

## 3 Euclidean Spanners

In this section we demonstrate that our 1-spanners for tree metrics can be used for constructing Euclidean spanners and spanners for doubling metrics.

We start with employing the Dumbbell Theorem of [4] in conjunction with our 1-spanners for tree metrics to construct Euclidean spanners.

**Theorem 3.1** (“Dumbbell Theorem”, Theorem 2 in [4]) *Given a set  $S$  of  $n$  points in  $\mathbb{R}^d$  and a parameter  $\epsilon > 0$ , a forest  $\mathcal{D}$  consisting of  $O(1)$  rooted binary trees of size  $O(n)$  each can be built in time  $O(n \cdot \log n)$ , having the following properties:*

1. *For each tree in  $\mathcal{D}$ , there is a 1-1 correspondence between the leaves of this tree and the points of  $S$ .*
2. *Each internal vertex in the tree has a unique representative point, which can be selected arbitrarily from the points in any of its descendant leaves.*
3. *Given any two points  $u, v \in S$ , there is a tree in  $\mathcal{D}$ , so that the path formed by walking from representative to representative along the unique path in that tree between these vertices, is a  $(1 + \epsilon)$ -spanner path for  $u$  and  $v$ .*

For each dumbbell tree in  $\mathcal{D}$ , we use the following representative assignment from [4]. Leaf labels are propagated up the tree. An internal vertex chooses to itself one of the propagated labels and propagates the other one up the tree. Each label is used at most twice, once at a leaf and once at an internal vertex. Any label assignment induces a weight function over the edges of the dumbbell tree in the obvious way. (The weight of an edge is set to be the Euclidean distance between the representatives corresponding to the two endpoints of that edge.) Arya et al. [4] proved that the lightness of dumbbell trees is always  $O(\log n)$ , regardless of which representative assignment is chosen for the internal vertices.

Next, we describe our construction of Euclidean spanners with diameter in the range  $\Omega(\alpha(n)) = \Lambda = O(\log n)$ .

We remark that each dumbbell tree has size  $O(n)$ . For each (weighted) dumbbell tree  $DT_i \in \mathcal{D}$ , denote by  $M_i$  the  $O(n)$ -point tree metric induced by  $DT_i$ . To obtain our construction of  $(1 + \epsilon)$ -spanners with low diameter, we set  $k = n^{1/\Lambda}$ , and build the 1-spanner construction  $\mathcal{H}^i = \mathcal{H}_k^i(O(n))$  that is guaranteed by Theorem 2.9 for each of the tree metrics  $M_i$ . Then we translate each  $\mathcal{H}^i$  to be a spanning subgraph  $\check{\mathcal{H}}^i$  of  $S$  in the obvious way. (Each edge in  $\mathcal{H}^i$  is replaced with an edge that connects the representatives corresponding to the endpoints of that edge.) Finally, let  $\mathcal{E}_k(n)$  be the spanner obtained from the union of all the graphs  $\check{\mathcal{H}}^i$ .

Theorem 2.9 implies that each graph  $\check{\mathcal{H}}^i$  contains only  $O(n)$  edges. By the Dumbbell Theorem,  $\mathcal{E}_k(n)$  is the union of a constant number of such graphs. Thus the total number of edges in  $\mathcal{E}_k(n)$  is  $O(n)$ .

We proceed by showing that  $\mathcal{E}_k(n)$  is a  $(1 + \epsilon)$ -spanner for  $S$  with diameter  $\Lambda = \Lambda(\mathcal{E}_k(n))$  at most  $O(\log_k n + \alpha(k))$ . By the Dumbbell Theorem, for any pair  $u, v$  of points in  $S$ , there is a dumbbell tree  $DT_i$ , so that the unique path  $P_{u,v}$  connecting  $u$  and  $v$  in  $DT_i$  is a  $(1 + \epsilon)$ -spanner path for them. Theorem 2.9 implies that there is a 1-spanner path  $P$  in  $\mathcal{H}^i$  between  $u$  and  $v$  that consists of at most  $O(\log_k n + \alpha(k))$  edges. By the triangle inequality, the weight of the corresponding translated path  $\check{P}$  in  $\check{\mathcal{H}}^i$  is no greater than the weight of  $P_{u,v}$ . Hence,  $\check{P}$  is a  $(1 + \epsilon)$ -spanner path for  $u$  and  $v$  that consists of at most  $O(\log_k n + \alpha(k))$  edges.

Next, we show that the maximum degree  $\Delta(\mathcal{E}_k(n))$  of  $\mathcal{E}_k(n)$  is  $O(k)$ . Since each dumbbell tree  $DT_i$  is binary, Theorem 2.9 implies that  $\Delta(\mathcal{H}^i) = O(k)$ . Recall that each label is used at most twice in  $DT_i$ , and so  $\Delta(\check{\mathcal{H}}^i) \leq 2 \cdot \Delta(\mathcal{H}^i) = O(k)$ . The union of  $O(1)$  such graphs will also have maximum degree  $O(k)$ .

We argue that the lightness  $\Psi(\mathcal{E}_k(n))$  of  $\mathcal{E}_k(n)$  is  $O(k \cdot \log_k n \cdot \log n)$ . Consider an arbitrary dumbbell tree  $DT_i$ . Recall that the lightness of all dumbbell trees is  $O(\log n)$ , and so  $w(DT_i) = O(\log n) \cdot w(MST(S))$ . By Theorem 2.9, the weight  $w(\mathcal{H}^i)$  of  $\mathcal{H}^i$  is at most  $O(k \cdot \log_k n) \cdot w(DT_i) = O(k \cdot \log_k n \cdot \log n) \cdot w(MST(S))$ . By the triangle inequality, the weight of each edge in  $\check{\mathcal{H}}^i$  is no greater than the corresponding weight in  $\mathcal{H}^i$ , implying that the weight  $w(\check{\mathcal{H}}^i)$  of the graph  $\check{\mathcal{H}}^i$  satisfies  $w(\check{\mathcal{H}}^i) \leq w(\mathcal{H}^i) = O(k \cdot \log_k n \cdot \log n) \cdot w(MST(S))$ . The union of  $O(1)$  such graphs will also have weight  $O(k \cdot \log_k n \cdot \log n) \cdot w(MST(S))$ .

Finally, we bound the running time of this construction. By the Dumbbell Theorem, the forest  $\mathcal{D}$  of dumbbell trees can be built in  $O(n \cdot \log n)$  time. Theorem 2.9 implies that we can compute each of the graphs  $\mathcal{H}^i$  in time  $O(n \cdot \log_k n) = O(n \cdot \log n)$ . Moreover, as each graph  $\mathcal{H}^i$  contains only  $O(n)$  edges, translating it into a graph  $\check{\mathcal{H}}^i$  as described above can be carried out in  $O(n)$  time. Since there is

a constant number of such graphs, it follows that the overall time needed to compute our construction  $\mathcal{E}_k(n)$  of Euclidean spanners is  $O(n \cdot \log n)$ .

To obtain our construction of Euclidean spanners for the complementary range  $\Lambda = \Omega(\log n)$ , we use our 1-spanners for tree metrics from Theorem 2.10 instead of Theorem 2.9.

**Corollary 3.2** *For any set  $S$  of  $n$  points in  $\mathbb{R}^d$ , any  $\epsilon > 0$  and a parameter  $k$ , there exists a  $(1 + \epsilon)$ -spanner with maximum degree  $O(k)$ , diameter  $O(\log_k n + \alpha(k))$ , lightness  $O(k \cdot \log_k n \cdot \log n)$ , and  $O(n)$  edges. There also exists a  $(1 + \epsilon)$ -spanner with maximum degree  $O(1)$ , diameter  $O(k \cdot \log_k n)$ , and lightness  $O(\log_k n \cdot \log n)$ . Both these constructions can be implemented in time  $O(n \cdot \log n)$ .*

We can show that the lightness of dumbbell trees for random point sets in the unit cube is (w.h.p.)  $O(1)$ . The same bound was obtained independently of us by Smid [37]. Our proof of this bound can be found in the technical report version of this paper [39]. We derive the following result as a corollary.

**Corollary 3.3** *For any set  $S$  of  $n$  points that are chosen independently and uniformly at random from the unit cube, any  $\epsilon > 0$  and a parameter  $k$ , there exists a  $(1 + \epsilon)$ -spanner with maximum degree  $O(k)$ , diameter  $O(\log_k n + \alpha(k))$ , lightness (w.h.p.)  $O(k \cdot \log_k n)$ , and  $O(n)$  edges. There also exists a  $(1 + \epsilon)$ -spanner with maximum degree  $O(1)$ , diameter  $O(k \cdot \log_k n)$ , and lightness (w.h.p.)  $O(\log_k n)$ . Both these constructions can be implemented in time  $O(n \cdot \log n)$ .*

Chan et al. [11] showed that for any doubling metric  $(X, \delta)$  there exists a  $(1 + \epsilon)$ -spanner with constant maximum degree. On the way to this result they proved the following lemma, which we employ in conjunction with our 1-spanners for tree metrics to construct spanners for doubling metrics.

**Lemma 3.4 (Lemma 3.1 in [11])** *For any doubling metric  $(X, \delta)$ , there exists a collection  $\mathcal{T}$  of  $m = O(1)$  spanning trees for  $(X, \delta)$ ,  $\mathcal{T} = \{\tau_1, \tau_2, \dots, \tau_m\}$ , that satisfies the following two properties:*

1. *For each index  $i \in [m]$ , the maximum degree  $\Delta(\tau_i)$  of the tree  $\tau_i$  is constant, i.e.,  $\Delta(\tau_i) = O(1)$ .*
2. *For each pair of points  $x, y \in X$  there exists an index  $i \in [m]$ , such that  $\text{dist}_{\tau_i}(x, y) = O(1) \cdot \delta(x, y)$ .*

To obtain our spanners for doubling metrics we start with constructing the collection  $\mathcal{T} = \{\tau_1, \tau_2, \dots, \tau_m\}$  of spanning trees with properties listed in Lemma 3.4. Next, we apply Theorem 2.9 with some parameter  $k$  to construct a 1-spanner  $\mathcal{Z}^i = \mathcal{Z}_k^i(n)$  for the tree metric induced by the  $i$ th tree  $\tau_i$  in  $\mathcal{T}$ , for each  $i \in [m]$ . Notice that, in general, edge weights in the graphs  $\mathcal{Z}^i$ ,  $i \in [m]$ , may be greater than the corresponding metric distances; for each  $i \in [m]$ , let  $\check{\mathcal{Z}}^i$  be the graph obtained from  $\mathcal{Z}^i$ , by assigning weight  $\delta(x, y)$  to each edge  $(x, y) \in \mathcal{Z}^i$ . Our spanner  $\mathcal{Z}$  is set to be the union of all the graphs  $\check{\mathcal{Z}}^i$ , i.e.,  $\mathcal{Z} = \bigcup_{i=1}^m \check{\mathcal{Z}}^i$ .

By Theorem 2.9, each of the graphs  $\check{\mathcal{Z}}^i$  contains only  $O(n)$  edges. Hence, the number of edges in  $\mathcal{Z}$  is at most  $m \cdot O(n) = O(n)$ .

To argue that  $\mathcal{Z}$  is an  $O(1)$ -spanner for  $(X, \delta)$  consider a pair of points  $x, y \in X$ . By Lemma 3.4, there exists an index  $i \in [m]$ , such that  $\text{dist}_{\tau_i}(x, y) = O(1) \cdot \delta(x, y)$ . Since  $\mathcal{Z}^i$  is a 1-spanner for the metric induced by  $\tau_i$ , it follows that  $\text{dist}_{\mathcal{Z}^i}(x, y) = \text{dist}_{\tau_i}(x, y)$ . Also, we have  $\text{dist}_{\check{\mathcal{Z}}^i}(x, y) \leq \text{dist}_{\mathcal{Z}^i}(x, y)$ . Finally, since  $\check{\mathcal{Z}}^i \subseteq \mathcal{Z}$ , we conclude that  $\text{dist}_{\mathcal{Z}}(x, y) \leq \text{dist}_{\check{\mathcal{Z}}^i}(x, y) \leq \text{dist}_{\mathcal{Z}^i}(x, y) = \text{dist}_{\tau_i}(x, y) = O(1) \cdot \delta(x, y)$ . Observe also that  $\Lambda(\mathcal{Z}^i) = O(\log_k n + \alpha(k))$ , and so there is a path between  $x$  and  $y$  in  $\mathcal{Z}^i$  that consists of at most  $\Lambda(\mathcal{Z}^i)$  edges and has length at most  $\text{dist}_{\mathcal{Z}^i}(x, y)$ . Consequently,  $\Lambda(\mathcal{Z}) = O(\log_k n + \alpha(k))$ .

By Theorem 2.9, the maximum degree of each graph  $\check{\mathcal{Z}}^i$  satisfies  $\Delta(\check{\mathcal{Z}}^i) \leq \Delta(\tau_i) + 2k$ . By Lemma 3.4, for each index  $i \in [m]$ ,  $\Delta(\tau_i) = O(1)$ . Hence  $\Delta(\check{\mathcal{Z}}^i) = O(k)$ . Since  $m = O(1)$ , it follows that  $\Delta(\mathcal{Z}) \leq \sum_{i=1}^m \Delta(\check{\mathcal{Z}}^i) \leq m \cdot O(k) = O(k)$ , and we are done.

**Corollary 3.5** *For any  $n$ -point doubling metric  $(X, \delta)$  and a parameter  $k$ , there exists an  $O(1)$ -spanner with maximum degree  $O(k)$ , diameter  $O(\log_k n + \alpha(k))$ , and  $O(n)$  edges.*

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