CONVEXITY IN TOPOLOGICAL AFFINE PLANES

RAGHAVAN DHANDAPANI
Courant Institute, NYU
251 Mercer St.
New York, NY 10012
USA
e-mail: raghavan@cs.nyu.edu

JACOB E. GOODMAN *
Department of Mathematics
City College, CUNY
New York, NY 10031
USA
e-mail: jegcc@cunyvm.cuny.edu

ANDREAS HOLMSEN †
Department of Mathematics
University of Bergen
5008 Bergen
NORWAY
e-mail: andreash@mi.uib.no

RICHARD POLLACK ‡
Courant Institute, NYU
251 Mercer St.
New York, NY 10012
USA
e-mail: pollack@courant.nyu.edu

SHAKHAR SMORODINSKY
Courant Institute, NYU
251 Mercer St.
New York, NY 10012
USA
e-mail: shakhar@courant.nyu.edu

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Abstract

We extend to topological affine planes the standard theorems of convexity, among them the separation theorem, the anti-exchange theorem, Radon's, Helly's, Carathéodory's, and Kirchberger's theorems, and the Minkowski theorem on extreme points. In a few cases the proofs are obtained by adapting proofs of the original results in the Euclidean plane; in others it is necessary to devise new proofs that are valid in the more general setting considered here.

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1. Basic definitions

The notion of a *topological affine plane* (TAP) $\mathcal{A}$ is most simply defined by means of one of its standard models (see, e.g., [7]): Consider a circle C_∞ in the Euclidean plane; its interior, $\mathrm{int} C_\infty$, constitutes the set $|\mathcal{A}|$ of points of $\mathcal{A}$. For each pair of antipodal points on C_∞, the interior of a simple Jordan arc joining the points (and not meeting C_∞ anywhere else except at its endpoints) is called a *pseudoline*. Suppose we are given, for each pair x, y of points of $|\mathcal{A}|$, a unique pseudoline $\overleftrightarrow{xy}$ containing x and y and depending continuously on x and y in the Hausdorff metric. (Recall that two point sets lie within distance d of each other in the Hausdorff metric if each point of either lies within distance d of some point of the other.) Suppose further that any two of these pseudolines meet (and necessarily cross) at exactly one point, or else that they share their endpoints on C_∞, and that their intersection depends continuously on the two pseudolines. (It can be shown that this last condition follows as a consequence of the previous continuity assumption; see [10] for a more thorough discussion.) Thus any finite collection of these pseudolines forms what is commonly known as an *arrangement*; the TAP $\mathcal{A}$ can thus be thought of as a two-parameter "continuous arrangement" of pseudolines. If we work in the closed disk $|\mathcal{A}| \cup C_\infty$, rather than in the open disk $|\mathcal{A}|$, and identify antipodal points in C_∞, the same definitions give what is known as a *topological projective plane*.

In $\mathcal{A}$, the terms *pseudoray*, *pseudohalfspace*, *pseudotriangle*, etc. all have the obvious meaning.

We fix a parametrization of the circle C_∞ by a variable θ running from 0 to 2π. If a pseudoline in $\mathcal{A}$ is directed, we may then speak of its *direction* as the value of θ corresponding to its terminal point. The *angle* from one directed pseudoline to another is the length of the counterclockwise arc of C_∞ from the first endpoint to the second, and if x, y, z are three points in $|\mathcal{A}|$, $\angle xyz$ means the angle from $\overrightarrow{yx}$ to $\overrightarrow{yz}$. Two pseudolines are called *parallel* if their endpoints on C_∞ coincide. Given a direction θ, we may therefore speak of a *parallel sweep* having direction θ: choose any pseudoline *not* in direction θ, and join each point on it with the point on C_∞ having direction θ; by the continuity assumption, this gives a continuously varying family of parallel pseudolines. In the same way, we can speak of continuously *rotating* a pseudoline, or alternatively a pseudoray, about a point $x \in |\mathcal{A}|$.

If a directed pseudoline l meets a set Y and Y lies in the closed right pseudohalfplane determined by l, we call l a *left tangent* to Y; similarly for a *right tangent*.

The following facts about arrangements and topological affine planes will be used in the sequel.

1. Every arrangement of eight or fewer pseudolines in a TAP $\mathcal{A}$ is *stretchable*, i.e.,

there is a homeomorphism of $|\mathcal{A}|$ with the Euclidean plane that maps the pseudolines to straight lines [6]; this is not the case, in general, with arrangements of nine or more pseudolines. (This fact is not used here in an essential way, but we invoke it occasionally to make the situation easier to depict.)

2. There is a homeomorphism from the Euclidean plane to $|\mathcal{A}|$ taking each line to an arc of a circle passing through two antipodal points of C_∞. (The inverse of the mapping

$$(x,y) \mapsto \left(\frac{x}{1-x^2-y^2}, \frac{y}{1-x^2-y^2}\right)$$

does this, for example.) This shows that the Euclidean plane is, in particular, a TAP.

3. Every arrangement of pseudolines can be extended to a TAP [4]; see also [9].

For basic facts about pseudoline arrangements, see [5] or [7].

Given any two points $x, y \in |\mathcal{A}|$, we can speak unambiguously of the *pseudosegment* $\overline{xy}$. As in the Euclidean plane, we may therefore call a set $Y \subset |\mathcal{A}|$ *convex* if $x, y \in Y \Rightarrow \overline{xy} \subset Y$. Trivially, the intersection of convex sets is convex; this enables us to define, as usual, the convex hull $\operatorname{conv} S$ of a set S as the smallest convex set containing S, i.e., the intersection of all the convex sets containing S.

This notion of convexity enjoys the same basic properties with respect to the underlying set of pseudolines that defines our TAP as ordinary convexity does with respect to straight lines:

PROPOSITION 1.

1. $A \subset \operatorname{conv} A$
2. $A \subset B \Rightarrow \operatorname{conv} A \subset \operatorname{conv} B$
3. $\operatorname{conv} \operatorname{conv} A = \operatorname{conv} A$

We also have

PROPOSITION 2. *If $\mathcal{A}$ and $\mathcal{B}$ are TAPs and $f : |\mathcal{A}| \to |\mathcal{B}|$ is a homeomorphism that maps the pseudolines of $\mathcal{A}$ to those of $\mathcal{B}$, then*

1. X is convex in $\mathcal{A} \Rightarrow f(X)$ is convex in $\mathcal{B}$
2. $x \in \operatorname{conv} X$ in $\mathcal{A} \Leftrightarrow f(x) \in \operatorname{conv} f(X)$ in $\mathcal{B}$.

Proof. These both follow immediately from the definitions. □

We add that Cantwell [1] takes a synthetic-geometric approach to some of the same questions, and obtains several of the same results we do using different (and in some cases more difficult) proofs.

2. The separation theorem

We begin by establishing the following basic result.

THEOREM 3. *Given disjoint compact convex sets X, Y in a topological affine plane, there is a pseudoline l of the plane that separates X from Y.*

The proof proceeds by a sequence of auxiliary results.

LEMMA 4. *If Y is a compact convex set and $x \notin Y$, there is a pseudoline l through x that misses Y.*

Proof. Suppose every pseudoline through x met Y. Consider each pseudoray starting at x. Its line meets Y, either on the side of the pseudoray or on the opposite side, but not both (otherwise, by the convexity of Y, we would have $x \in Y$). This defines a function $f_x : C_\infty \to \{+,-\}$ which, for the same reason, cannot have the same value on a pair of antipodal points of C_∞. Hence for some $\theta_0 \in [0, 2\pi)$, we must have $f_x(\theta) = +$ for a sequence of directions arbitrarily close to θ_0, and $f_x(\theta) = -$ for another such sequence. This means that there is a sequence of points y_i with $\overrightarrow{xy_i}$ in directions θ_i arbitrarily close to θ_0, and another such sequence y'_i with $\overrightarrow{xy'_i}$ in directions arbitrarily close to $\theta_0 + \pi$. By the compactness of Y and the continuity of the pseudoline determined by a pair of points, we therefore get points $y_0, y'_0 \in Y$ with $\overrightarrow{xy_0}$ and $\overrightarrow{xy'_0}$ pointing in opposite directions, which again implies that $x \in Y$ by the convexity of Y, a contradiction. □

LEMMA 5. *If Y is a compact convex set and $x \notin Y$, there is a unique left tangent pseudoray $\overrightarrow{xy}$ from x to Y.*

Proof. Rotate a pseudoray x_θ in direction $\theta = 0$ to 2π around x. Then x_θ meets Y for some $\theta = \theta_1$ and, by Lemma 4, misses it for some $\theta = \theta_2$. As we rotate counterclockwise from θ_1 to θ_2, we reach (by the compactness of Y) a final direction θ_0 in which there exists a pseudoray xy_0, before we lose this property. Then x_{θ_0} is clearly a left tangent pseudoray from x to Y.

The uniqueness follows from the fact that two pseudolines cannot meet twice. □

LEMMA 6. *If X and Y are disjoint compact convex subsets of $|\mathcal{A}|$ with $x_i \in X$ and $y_i \in Y$ for $i = 1,2$, then the pseudorays $\overrightarrow{x_1y_1}$ and $\overrightarrow{x_2y_2}$ cannot point in opposite directions.*

Proof. Stretching the four pseudolines gives the situation depicted in Figure 1. Since lines x_1y_1 and x_2y_2 are parallel, pseudosegments $\overline{x_1x_2}$ and $\overline{y_1y_2}$ would have to cross, with their intersection therefore lying in both X and Y, contradicting the disjointness of these sets. □

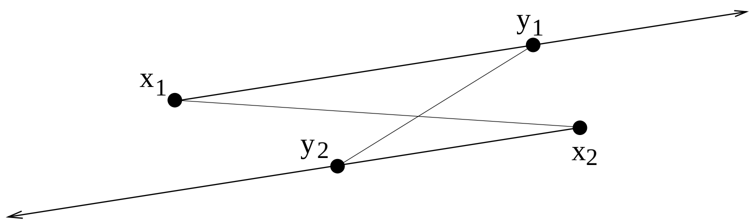


Figure 1: Opposite directions

LEMMA 7. *If a pseudoline l misses a compact convex set Y, and p is a point on l, we can rotate l slightly in both directions around p so that the resulting pseudolines still miss Y. The corresponding result holds if p is one of the endpoints of l on C_∞.*

Proof. Let $\delta = \inf\{\text{dist}(y,x) \mid y \in Y, x \in l\}$. Then $\delta > 0$ by the compactness of Y. If we rotate l in either direction around p so that the Hausdorff distance to l remains $< \delta$, the result follows. (The proof is unchanged if $x \in C_\infty$.) □

LEMMA 8. *If Y is a compact convex set and $x \notin Y$, there is a pseudotriangle containing x in its interior, all of whose sides (extended) miss Y, such that Y is contained in a region bounded by only two sides of the pseudotriangle, suitably extended.*

Proof. By Lemma 4, there is a pseudoline l through x missing Y. We may assume, without loss of generality, that l is directed so that Y lies on its right. Choose points u and v on l so that $v < x < u$, as in Figure 2. By Lemma 7, we can rotate l slightly around u, in the counterclockwise direction, so that the resulting directed pseudoline l_u still has Y on its right, and similarly we can rotate l slightly around v, in the clockwise direction, so that the resulting directed pseudoline l_v still has Y on its right. Let m be a pseudoline parallel to l and lying to its left. Let $p = l_u \cap l_v$, $q = l_u \cap m$, and $r = l_v \cap m$. Then pqr is the desired pseudotriangle. □

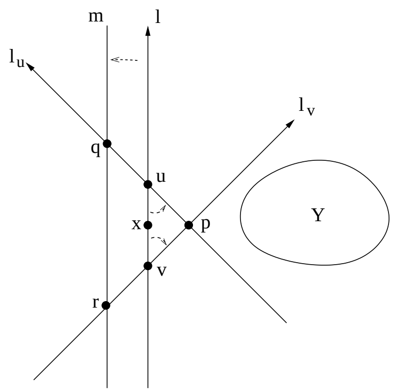


Figure 2: Modifying a pseudoline missing Y to a pseudotriangle

LEMMA 9. *Let Y be a compact convex set and l a pseudoline missing Y. For $x \in l$, let $f(x)$ be the direction of the pseudoray from x that is left tangent to Y. Then f is a monotone mapping from l to C_∞. The same conclusion holds if l passes through Y, provided f is restricted to the points of l lying on one side of Y. In both cases, the mapping f is continuous.*

Proof. Consider first the case where l misses the set Y altogether. Direct l so that Y is on its left, and direct C_∞ so that θ increases in the counterclockwise direction. We claim that $f : l \to C_\infty$ is then a monotone increasing function.

Suppose $x_1 < x_2$ on l, and let l_i be the left tangent from x_i to Y for $i = 1, 2$. Stretching l, l_1, and l_2 (Fact 1 above), we obtain the situation shown in Figure 3(a). If l_1 did not cross l_2 to the left of l, the portion of l_1 to the left of l would lie entirely below l_2, hence l_1 could not be tangent to Y. Therefore $f(x_1) < f(x_2)$.

A similar argument works in the case where l passes through Y and where $x_1 < x_2$ on the same side of Y, as in Figure 3(b) (where $x_1 < x_2$ are both below Y; the corresponding argument works if they are both above).

Finally, suppose $f(x_0) = \theta_0$, and suppose $\theta_1 < \theta_0 < \theta_2$, as in Figure 3(c). For $i = 1, 2$, let x_i be the intersecton of l with the (unique) right tangent l_i from θ_i to Y. (This makes l_i the *left* tangent from x_i to Y.) Then by the above, $f(x)$ lies between θ_1 and θ_2 for every x between x_1 and x_2 on l, so that $f : l \to C_\infty$ is continuous.

□

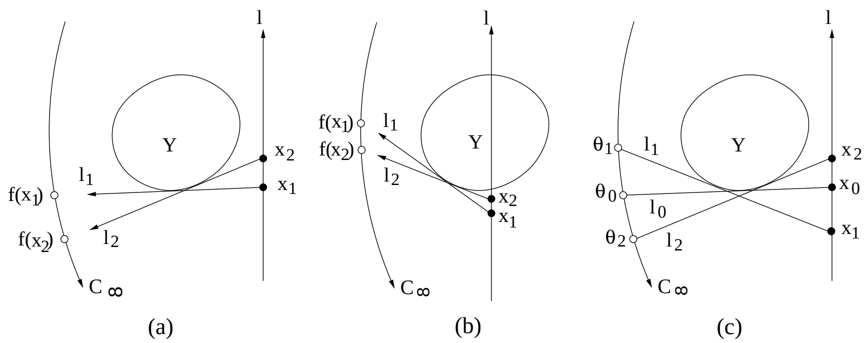


Figure 3: Monotonicity of tangent direction along a line

THEOREM 10. *Let Y be a compact convex set. For $x \notin Y$, let $f(x)$ be the direction of the pseudoray from x that is left tangent to Y. Then f is a continuous mapping from the complement of Y to C_∞.*

Proof. Suppose $x \notin Y$. By Lemma 8, there is a pseudotriangle pqr containing x whose sides, $\overleftrightarrow{pq}$, $\overleftrightarrow{pr}$, and $\overleftrightarrow{qr}$, suitably extended, miss Y, with Y contained in the region bounded by only (say) $\overleftrightarrow{pr}$ and $\overleftrightarrow{qr}$, as in Figure 4(a). Suppose $f(x) = \theta_0$, and suppose $\theta_1 < \theta_0 < \theta_2$. We must show that there is an open neighborhood of x that is mapped into (θ_1, θ_2) by f.

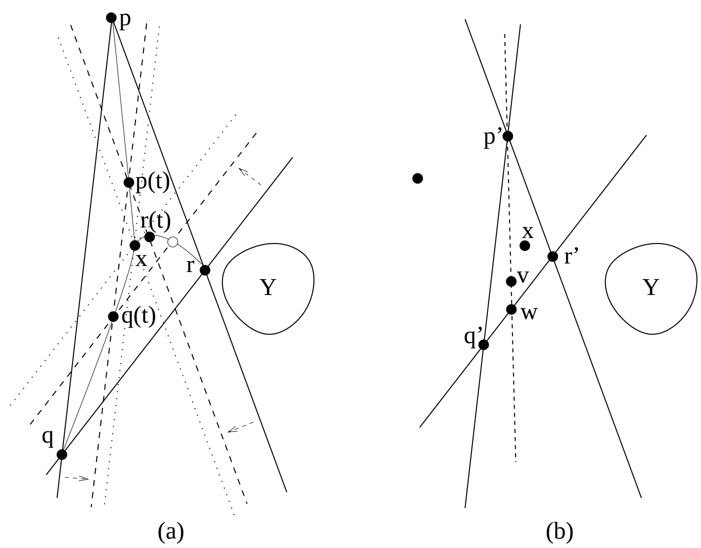


Figure 4: Continuity of the left tangent to a fixed set Y

Consider the pseudosegments $\overline{xp}$, $\overline{xq}$, and $\overline{xr}$. For a real parameter t going from 0 to 1, move a point $p(t)$ monotonically along segment $\overline{xp}$ from p to x, and — for each value of t — consider the parallel translate of pseudoline $\overleftrightarrow{pq}$ passing through $p(t)$. The latter intersects segment $\overline{xq}$ at some point: call it $q(t)$. Now the parallel translate of pseudoline $\overleftrightarrow{pr}$ and that of pseudoline $\overleftrightarrow{qr}$ passing through $p(t)$ and $q(t)$ respectively each meet the pseudosegment $\overline{xr}$ (but not necessarily at the same point: that would require Desargues's theorem, which is generally false in a TAP!), say at points $r_1(t)$ and $r_2(t)$, respectively. Let $r(t)$ be the one of the these two points which lies closer to x along $\overline{xr}$, as in Figure 4(a).

Then by construction, and by the continuity both of a pseudoline as a function of a point pair and of the intersection of two pseudolines as a function of the pseudolines, for t sufficiently close to 1, the points $p(t), q(t), r(t)$ will reach positions p', q', r' with the following properties, as in Figure 4(b) :

1. $f(p'), f(q'), f(r') \in (\theta_1, \theta_2)$
2. x belongs to pseudotriangle $p'q'r'$
3. $p'q'$, $p'r'$, and $q'r'$ all miss Y, and Y lies in the region bounded by only $p'r'$ and $q'r'$ (see Figure 4(b)).

Then given any point v inside the pseudotriangle $p'q'r'$, there is a pseudoline containing v that passes through vertex p', crosses pseudosegment $\overleftrightarrow{q'r'}$ at a point w between vertices q' and r', and misses the region containing Y, hence misses Y itself. By Lemma 9, we have, first, $f(w) \in (\theta_1, \theta_2)$, and then, finally, $f(v) \in (\theta_1, \theta_2)$. □

We can now prove complete the proof of Theorem 3.

Proof. For each $x \in X$, consider the (unique) directed left tangent pseudoray l_x from x to Y (see Figure 5); it exists, by Lemma 5. This tangent has a direction θ_x. Since, by Lemma 6, the existence of such a pseudoray precludes the existence of another such pointing in the opposite direction, it follows that there is a direction θ_0, $0 < \theta_0 \leq 2\pi$, such that no l_x has direction θ_0. Let

$$\Theta = \{\theta\ ,\ \theta_0 - 2\pi \leq \theta < \theta_0 \mid l_x \text{ has direction } \theta \text{ for some } x \in X\},$$

and let $\theta' = \sup \Theta$. Then, by the compactness of X and the continuity of θ_x as a function of x (Theorem 10), there is an $x_0 \in X$ such that l_{x_0} has direction θ'. It follows that l_{x_0} is a right tangent to X, since the existence of a point of X to its right would (as a consequence of Lemma 9) contradict the fact that no direction in Θ exceeds θ'. This shows that X and Y have a "right-left XY tangent" l, i.e., a common tangent that meets X on the right before it meets Y on the left.

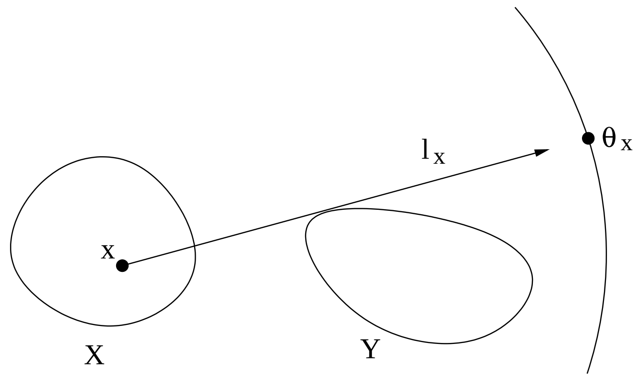


Figure 5: The left tangent from point x to set Y

Finally, to show the existence of a strict separator for X and Y, choose any point $p \in l$ strictly between $l \cap X$ and $l \cap Y$, as in Figure 6; such a point must exist by the compactness of X and Y. Rotating l slightly in the counterclockwise direction about p will then produce a strict separator, by Lemma 7. ☐

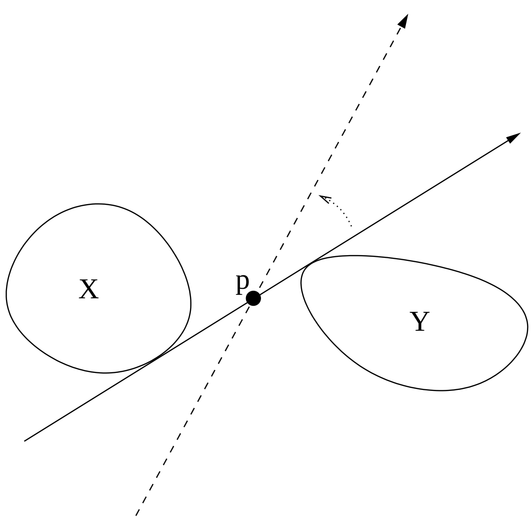


Figure 6: A strict separator

3. Some theorems of combinatorial convexity

In this section we generalize a number of standard results in combinatorial convexity to topological affine planes. An excellent single source for the original theorems is [3].

We begin with an extension of Radon's theorem to a TAP.

THEOREM 11. *Any set S of cardinality at least 4 in a TAP $\mathcal{A}$ can be partitioned into two subsets whose convex hulls meet.*

Proof. It is enough to prove this if $|S| = 4$. In that case, Fact 1 above allows us to straighten the arrangement of ≤ 6 pseudolines consisting of all the pseudolines joining the four points of S in pairs, and then an application of the standard version of Radon's theorem in $\mathbf{R}^2$ immediately yields the result, by Proposition 2(2) □

The usual proof of Helly's theorem from Radon's works in our situation:

THEOREM 12. *If $\{X_i\}_{i\in I}$ is a family of at least three convex sets in a TAP $\mathcal{A}$ with either I finite or each set X_i compact, and if every three of the sets have a nonempty intersection, then all the sets meet.*

Proof. Suppose first that the index set I is finite, say $I = \{1,\ldots,n\}$. We prove the result by induction. It clearly holds for $n = 3$. Supposing $n > 3$, we let $\Xi_j = \{X_i \mid i \neq j\}$. By induction hypothesis, the sets belonging to each family Ξ_j, $j = 1,\ldots,n$, contain a point x_j. For each j, we have $x_j \in X_i$ for all $i \neq j$. Since $n \geq 4$, we can apply Radon's theorem to the points $x_1,\ldots,x_n$: they can be partitioned into two sets $\{x_1,\ldots,x_k\} \cup \{x_{k+1},\ldots,x_n\}$ such that there is a point $x_0 \in \operatorname{conv}\{x_1,\ldots,x_k\} \cap \operatorname{conv}\{x_{k+1},\ldots,x_n\}$. But then by Proposition 1, we have $x_0 \in X_i$ for $i = k+1,\ldots,n$ and (respectively) $x_0 \in X_i$ for $i = 1,\ldots,k$, so that the conclusion follows.

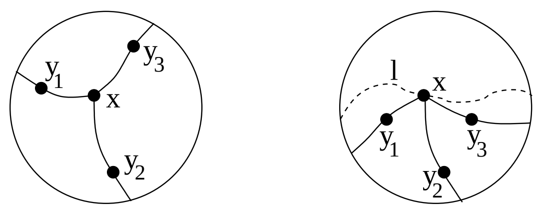
If I is infinite, but each X_i is compact, the result follows from the finite case, since any collection of compact sets has the finite intersection property (if every finite collection has a point in common, so does the entire set). □

The generalization of Carathéodory's theorem to TAPs requires a slightly more subtle argument:

LEMMA 13. *If x, y_1, y_2, y_3 are four distinct points in $|\mathcal{A}|$ such that the three angles $\angle y_1 x y_2$, $\angle y_2 x y_3$, and $\angle y_3 x y_1$ are all $\leq \pi$, then $x \in$ conv (y_1, y_2, y_3), and conversely.*

Proof. The situation is as shown in Figure 7(a). If, say, $\angle y_1xy_2 = \pi$, then y_1,x,y_2 are copseudolinear, in that order, so we are done. We may therefore assume that each of the three angles is $< \pi$. Suppose we had $x \notin$ conv (y_1,y_2,y_3). Then by Lemma 4, there would be a pseudoline l through x missing conv (y_1,y_2,y_3), as in Figure 7(b), but then one of the three angles in question ($\angle y_3xy_1$ in the figure) could not be $< \pi$.

For the converse, we need only observe that if, say, $\angle y_3xy_1 > \pi$, as in Figure 7(b), then there is a pseudoline l through x with all of the points y_i lying on the same side of l (in the figure, start with y_1x and rotate it slightly around x), so that x could not belong to conv (y_1,y_2,y_3). □



(a) (b)

Figure 7: $x \in \text{conv}(y_1,y_2,y_3)$

THEOREM 14. *If X is a set of points in a TAP and $y \in$ conv X, then there are points $x_1,x_2,x_3 \in X$ such that $y \in$ conv $\{x_1,x_2,x_3\}$.*

Proof. Since $y \in \text{conv}\, X$ we must have $X \neq \emptyset$. Choose any point $x \in X$. Let

$$\Theta_L = \{\theta, 0 \leq \theta < \pi \mid \text{there is some } z \in X \text{ with } z \text{ to the left of } \overrightarrow{xy} \text{ and } \angle zyx = \theta\}$$

and

$$\Theta_R = \{\theta, 0 \leq \theta < \pi \mid \text{there is some } z \in X \text{ with } z \text{ to the right of } \overrightarrow{xy} \text{ and } \angle xyz = \theta\}.$$

Let $\theta_L = \sup \Theta_L$ and $\theta_R = \sup \Theta_R$.

If $\theta_L + \theta_R > \pi$, we are done, by Lemma 13: see Figure 8.

If $\theta_L + \theta_R < \pi$, then by Lemma 13 (converse) we have a contradiction to the assumption that $y \in \text{conv}\, X$ (see Figure 9: the convex set shown there contains X but not y).

The only remaining case is where $\theta_L + \theta_R = \pi$. Notice first that if θ_L or θ_R *itself* $= \pi$, say the former, then either we are done if there a point $z \in X$ to the right of

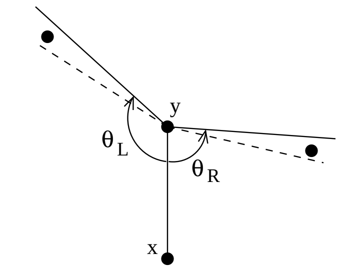


Figure 8: $\theta_L + \theta_R > \pi$

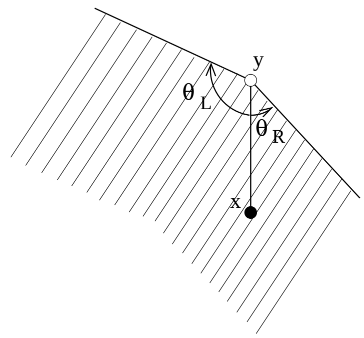


Figure 9: $\theta_L + \theta_R < \pi$

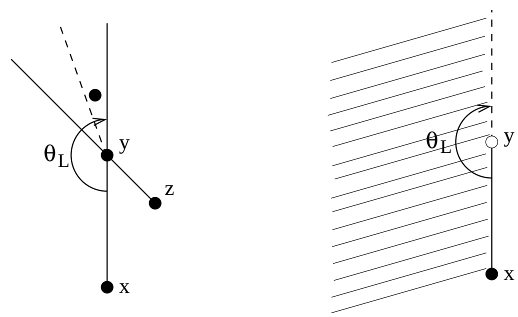
$\overrightarrow{xy}$, as in Figure 10(a), or else we have a contradiction to the fact that $y \in \operatorname{conv} X$, as in Figure 10(b), where the convex set shown contains X but not y.

So suppose finally that $\theta_L + \theta_R = \pi$, with both θ_L and $\theta_R > 0$. Let l be the line formed by the two rays in those directions. If there are points $z_1, z_2 \in X$ on line l lying on opposite sides of y, we are done. If not, say there is no point $z \in X$ on one side of y on l, as in Figure 11. Then y is not contained in the convex set shown, but this set contains X, which again contradicts the hypothesis. □

Notice that a similar argument gives us the existence of a supporting pseudoline at any boundary point of a convex set in a TAP:

THEOREM 15. *If X is a convex set in a TAP $\mathcal{A}$, and x is a boundary point of X, then there is a pseudoline l through x with no point of X on one side of it.*

Proof. We may suppose X contains points other than x. Let y be such a point, and consider the pseudoline $\overleftrightarrow{xy}$. If there were a point $z \in X \cap \overleftrightarrow{xy}$ lying on the other side of x from y, as in Figure 12(a), then there could not be points of X on both sides



(a) (b)

Figure 10: $\theta_L = \pi$

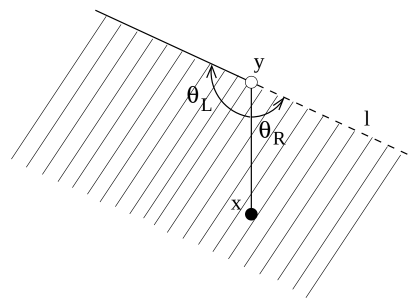


Figure 11: $\theta_L + \theta_R = \pi$, $\theta_L > 0$, $\theta_R > 0$

of $\overleftrightarrow{xy}$, since otherwise x would be an interior point of X; hence in this case we are done. Let us suppose, then that there is no point of $X \cap \overleftrightarrow{xy}$ lying on the other side of x from y, as in Figure 12(b). Then, as in the proof of Theorem 14, let

$$\Theta_L = \{\theta, 0 \leq \theta < \pi \mid \text{there is some } z \in X \text{ with } z \text{ to the left of } \overrightarrow{yx} \text{ and } \angle zxy = \theta\}$$

and

$$\Theta_L = \{\theta, 0 \leq \theta < \pi \mid \text{there is some } z \in X \text{ with } z \text{ to the right of } \overrightarrow{yx} \text{ and } \angle yxz = \theta\}.$$

Let $\theta_L = \sup \Theta_L$ and $\theta_R = \sup \Theta_R$. We cannot have $\theta_L + \theta_R > \pi$, since that would make x an interior point of X. It follows that the pseudoline through x in direction (say) θ_L is a supporting pseudoline for X. □

As a corollary of Carathéodory's theorem, we obtain yet another basic fact about convex sets in the setting of a TAP:

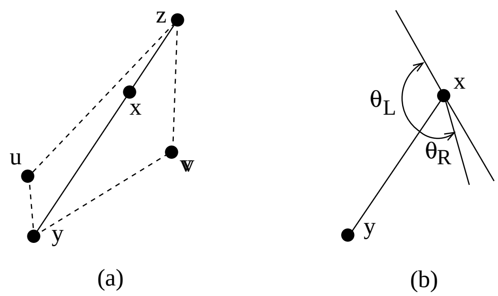


Figure 12: Existence of a supporting pseudoline

COROLLARY 16. *If X is compact, then* conv X *is also compact.*

Proof. Since X is bounded (i.e., bounded away from C_∞), a parallel sweep in three suitable directions yields a pseudotriangle T containing X. But the interior of T is convex; this shows that conv X, which is inside T, is also bounded.

To show that conv X is closed, we argue as follows. Suppose $\lim_{n\to\infty} x_n = x$ and $x_n \in \text{conv}\, X$ for every n. By Theorem 14, for every n, there are points $x_n^1, x_n^2, x_n^3 \in X$ such that $x_n \in \text{conv}\,(x_n^1, x_n^2, x_n^3)$. Since all the points x_n^1 belong to the compact set X, there is a subsequence $(x_{n_i}^1)$ of (x_n^1) converging to a point x^1 of X, then there is a subsequence $(x_{n_{i_j}}^2)$ of $(x_{n_i}^2)$ converging to a point x^2 of X, and finally there is a subsequence $(x_{n_{i_{j_k}}}^3)$ of $(x_{n_{i_j}}^3)$ converging to a point x^3 of X. Since $x_n \in \text{conv}\,(x_{n_{i_{j_k}}}^1, x_{n_{i_{j_k}}}^2, x_{n_{i_{j_k}}}^3)$ for every n, we must have $x \in \text{conv}(x^1, x^2, x^3)$, otherwise by the separation theorem for a point and a convex set in a TAP, there would be a pseudoline strictly separating x from conv (x^1, x^2, x^3), and we could not have $x_n \in \text{conv}\,(x_{n_{i_{j_k}}}^1, x_{n_{i_{j_k}}}^2, x_{n_{i_{j_k}}}^3)$ for all n. □

A second corollary is a generalization of Kirchberger's theorem to TAPs:

THEOREM 17. *If X and Y are compact sets of points in a TAP $\mathcal{A}$ with the property that given any four points in their union there is a pseudoline of $\mathcal{A}$ separating those in X from those in Y, then there is a pseudoline of $\mathcal{A}$ separating all of X from all of Y.*

Proof. Suppose not. Since conv X and conv Y are compact by Corollary 16, it follows from Theorem 3 that there is a point $z \in \text{conv}\, X \cap \text{conv}\, Y$. By Theorem 14, there are points $x_1, x_2, x_3 \in X$ and $y_1, y_2, y_3 \in Y$ such that $z \in \text{conv}\,(x_1, x_2, x_3)$ and $z \in \text{conv}\,(y_1, y_2, y_3)$. For convenience, straighten the six pseudolines $\overleftrightarrow{x_1x_2}, \overleftrightarrow{x_1x_3}, \overleftrightarrow{x_2x_3}$,

$\overleftrightarrow{y_1y_2}, \overleftrightarrow{y_1y_3}, \overleftrightarrow{y_2y_3}$. This yields two intersecting triangles with vertices x_1, x_2, x_3 and y_1, y_2, y_3 respectively. If any x_i were contained in triangle $y_1y_2y_3$ or any y_i in triangle $x_1x_2x_3$, as in Figure 13(a), this would contradict the hypothesis. Since the interiors of the two triangles meet, however, it follows that a side of one must intersect a side of the other, as in Figure 13(b), again contradicting the hypothesis. □

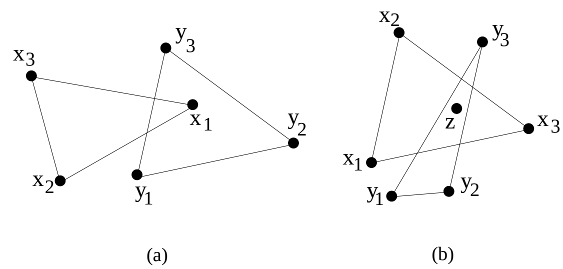


Figure 13: Kirchberger's theorem

Yet another corollary is the following result, which allows us to define extreme points of convex sets in a TAP, and to generalize to TAPs the planar case of the Minkowski theorem on extreme points (see, e.g., [11], p.276).

PROPOSITION 18. *If X is a convex set in a TAP $\mathcal{A}$ and $x \in X$, the following are equivalent:*

1. $X \setminus \{x\}$ is convex
2. There are no points $y, z \in X \setminus \{x\}$ such that x lies between y and z.

Proof. $(1) \Rightarrow (2)$ is clear, since if x were between y and z then $X \setminus \{x\}$ would not be convex.

Conversely, suppose $X \setminus \{x\}$ is not convex. Since X *is*, we have $x \in \text{conv}\,(X \setminus \{x\})$. Therefore, by Theorem 14, there are points $w, y, z \in X \setminus \{x\}$ such that $x \in$ conv (w, y, z). If x lies in the boundary of the pseudotriangle conv (w, y, z), then $x \in$ (say) conv (y, z), and we are done. Otherwise, x lies in the interior of the pseudotriangle, as in Figure 14.

Consider any pseudoline l through x. It meets the sides of the pseudotriangle in two points — either one of w, y, z plus another point v on one of the sides of the pseudotriangle, or else in two points u, v lying on different sides of the pseudotriangle, say u on $\overline{wy}$ and v on $\overline{wz}$. In each case, however, we get a contradiction to condition (2). □

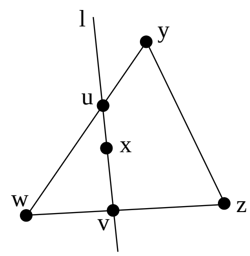


Figure 14: Minkowski's theorem on extreme points

If the situation in Proposition 18 holds, we say that x is an *extreme point* of the convex set X.

THEOREM 19. *A compact convex set X in a TAP $\mathcal{A}$ is the convex hull of its extreme points.*

Proof. Take a point $x \in X$. Choose a pseudoline l through x. l meets the boundary ∂X in two points, y and z, lying on opposite sides of x. Consider a supporting pseudoline to X at y; it exists by Theorem 15. The pseudosegment of ∂X containing y has extreme endpoints p and q (one or both of which may be y itself). Similarly for z: extreme endpoints r and s. Then $x \in \text{conv}\,(p,q,r,s)$. $\square$

As a final corollary to Carathéodory's theorem we get the so-called *anti-exchange principle* for convex sets in a TAP:

THEOREM 20. *If S is a compact convex set in a TAP $\mathcal{A}$ with $x,y \notin S$ but $y \in$ conv $(S \cup \{x\})$ and $x \in$ conv $(S \cup \{y\})$, then $y = x$.*

Proof. It follows from Theorem 14 plus the hypothesis $x,y \notin S$ that there are points $s_1,s_2,s_3,s_4 \in S$ such that $y \in$ conv (s_1,s_2,x) and $x \in$ conv (s_3,s_4,y). Let l be a pseudoline (strictly) separating x from S. It follows that l (strictly) separates y from S as well, since otherwise the entire set conv$(S \cup \{y\})$ would lie in the (closed) halfplane determined by l that does not contain x, so that x could not lie in conv $(S \cup \{y\})$.

As a consequence of Fact 1 above, we may assume that the seven pseudolines consisting of l plus the sides of the pseudotriangles s_1s_2x and s_3s_4y are straight, and that $\mathcal{A}$ is the Euclidean plane. Assuming $y \neq x$, this gives the situation depicted in Figure 15. Since x is in triangle s_3s_4y, the distance from x to l must be less than the distance from y to l. But the reverse is true for the corresponding reason, and this gives a contradiction. □

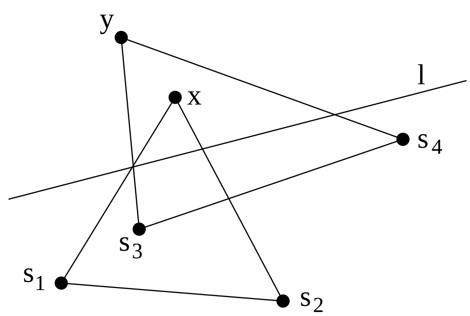


Figure 15: Anti-exchange

We remark that Theorem 20 can also be proven without recourse to the Euclidean plane, by simply considering the order of points along the directed pseudoline $\overrightarrow{xy}$.

4. Conclusion

The reader may wonder why we have extended the standard convexity theorems only in dimension 2. What about topological affine d-spaces? The reason is simply that there are none, besides the standard Euclidean ones. It has been known for well over a century (see [8], for example) that as soon as the dimension is 3 or more, the standard axioms of geometry imply Desargues's theorem; it follows from this that the space in question is isomorphic to the usual Euclidean d-space [2]. Thus it is only in dimension $d = 2$, where nonstretchable pseudoline arrangements proliferate, that non-Euclidean topological affine d-spaces can exist.

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