Conflict-Free Coloring and its Applications

Shakhar Smorodinsky*

September 12, 2011

Abstract

Let $H = (V, E)$ be a hypergraph. A *conflict-free* coloring of H is an assignment of colors to V such that, in each hyperedge $e \in E$, there is at least one uniquely-colored vertex. This notion is an extension of the classical graph coloring. Such colorings arise in the context of frequency assignment to cellular antennae, in battery consumption aspects of sensor networks, in RFID protocols, and several other fields. Conflict-free coloring has been the focus of many recent research papers. In this paper, we survey this notion and its combinatorial and algorithmic aspects.

1 Introduction

1.1 Notations and Definitions

In order to introduce the main notion of this paper, we start with several basic definitions: Unless otherwise stated, the term log denotes the base 2 logarithm.

A *hypergraph* is a pair $(V, \mathcal{E})$ where V is a set and $\mathcal{E}$ is a collection of subsets of V. The elements of V are called *vertices* and the elements of $\mathcal{E}$ are called *hyperedges*. When all hyperedges in $\mathcal{E}$ contain exactly two elements of V then the pair $(V, \mathcal{E})$ is a *simple graph*. For a subset $V' \subset V$ refer to the hypergraph $H(V') = (V', \{S \cap V' | S \in \mathcal{E}\})$ as the *sub-hypergraph* induced by V'. A k-coloring, for some $k \in \mathbb{N}$, of (the vertices of) H is a function $\varphi : V \to \{1, \ldots, k\}$. Let $H = (V, \mathcal{E})$ be a hypergraph. A k-coloring φ of H is called *proper* or *non-monochromatic* if every hyperedge $e \in \mathcal{E}$ with $|e| \geq 2$ is non-monochromatic. That is, there exists at least two vertices $x, y \in e$ such that $\varphi(x) \neq \varphi(y)$. Let $\chi(H)$ denote the least integer k for which H admits a proper coloring with k colors.

In this paper, we focus on the following colorings which are more restrictive than proper coloring:

Definition 1.1 (Conflict-Free and Unique-Maximum Colorings). *Let $H = (V, \mathcal{E})$ be a hypergraph and let $C : V \to \{1, \ldots, k\}$ be some coloring of H. We say that C is a* conflict-free *coloring (*CF-coloring *for short) if every hyperedge $e \in \mathcal{E}$ contains at least one uniquely colored vertex. More formally, for every hyperedge $e \in \mathcal{E}$ there is a vertex $x \in e$ such that $\forall y \in e, y \neq x \Rightarrow C(y) \neq C(x)$. We say that C is a* unique-maximum *coloring (*UM-coloring *for short) if the maximum color in every hyperedge is unique. That is, for every hyperedge $e \in \mathcal{E}$, $\left|e \cap C^{-1}(\max_{v \in e} C(v))\right| = 1$.*

Let $\chi_{\mathrm{cf}}(H)$ (respectively, $\chi_{\mathrm{um}}(H)$) denote the least integer k for which H admits a CF-coloring (respectively, a UM-coloring) with k colors. Obviously, every UM-coloring of a hypergraph H is also a CF-coloring of H, and every CF-coloring of H is also a proper coloring of H. Hence, we have the followng inequalities:

$$\chi(H) \leq \chi_{\mathrm{cf}}(H) \leq \chi_{\mathrm{um}}(H)$$

Notice that for simple graphs, the three notions of coloring (non-monochromatic, CF and UM) coincide. Also, for 3-uniform hypergraphs (i.e., every hyperedge has cardinality 3), the two first notions (non-monochromatic and CF) coincide. However, already for 3-uniform hypergraphs there can be an arbitrarily

*Mathematics department, Ben-Gurion University, Beer Sheva, Israel; `http://www.math.bgu.ac.il/~shakhar/` ; `shakhar@math.bgu.ac.il`.

large gap between $\chi_{\rm cf}(H)$ and $\chi_{\rm um}(H)$. Consider, for example, two sets A and B each of cardinality $n > 1$. Let $H = (A \cup B, \mathcal{E})$ where $\mathcal{E}$ consists of all triples of elements e such that $e \cap A \neq \emptyset$ and $e \cap B \neq \emptyset$. In other words $\mathcal{E}$ consists of all triples containing two elements from one of the sets A or B and one element from the other set. It is easily seen that $\chi_{\rm cf}(H) = 2$ by simply coloring all elements of A with 1 and all elements of B with 2. It is also not hard to verify that $\chi_{\rm um}(H) \geq n$ (in fact $\chi_{\rm um}(H) = n + 1$). Indeed, let C be a UM-coloring of H. If all elements of A are colored with distinct colors we are done. Otherwise, there exist two elements u, v in A with the same color, say i. We claim that all elements of B are colored with colors greater than i. Assume to the contrary that there is an element $w \in B$ with color $C(w) = j \leq i$. However, in that case the hyperedge $\{u, v, w\}$ does not have the unique-maximum property. Hence all colors of B are distinct for otherwise if there are two vertices w_1, w_2 with the same color, again the hyperedge $\{w_1, w_2, u\}$ does not have the unique-maximum property.

Let us describe a simple yet an important example of a hypergraph H and analyze its chromatic number $\chi(H)$ and its CF-chromatic number $\chi_{\rm cf}(H)$. The vertices of the hypergraph consist of the first n integers $[n] = \{1, \ldots, n\}$. The hyperedge-set is the set of all (non-empty) subsets of $[n]$ consisting of consecutive elements of $[n]$, e.g., $\{2, 3, 4\}$, $\{2\}$, the set $[n]$, etc. We refer to such hypergraphs as *hypergraphs induced by points on the line with respect to intervals* or as the *discrete intervals hypergraph*. Trivially, we have $\chi(H) = 2$. We will prove the following proposition:

Proposition 1.2. $\chi_{\rm cf}(H) = \chi_{\rm um}(H) = \lfloor \log n \rfloor + 1$.

Proof. First we prove that $\chi_{\rm um}(H) \leq \lfloor \log n \rfloor + 1$. Assume without loss of generality that n is of the form $n = 2^k - 1$ for some integer k. If $n < 2^k - 1$ then we can add the vertices $n + 1, n + 2, \ldots, 2^k - 1$ and this can only increase the CF-chromatic number. In this case we will see that $\chi_{\rm um}(H) \leq k$ and that for $n \geq 2^k$ $\chi_{\rm cf}(H) \geq k + 1$. The proof is by induction on k. For $k = 1$ the claim holds trivially. Assume that the claim holds for some integer k and let $n = 2^{k+1} - 1$. Consider the median vertex 2^k and color it with a unique (maximum color), say $k + 1$, not to be used again. By the induction hypothesis, the set of elements to the right of 2^k, namely the set $\{2^k + 1, 2^k + 2, \ldots, 2^{k+1} - 1\}$ can be colored with k colors, say '1','2'...,'k', so that any of its subsets of consecutive elements has unique maximum color. The same holds for the set of elements to the left of 2^k. We will use the same set of k colors for the right set and the left set (and color the median with the unique color 'k+1'). It is easily verified that this coloring is indeed a UM-coloring for H. Thus we use a total of $k + 1$ colors and this completes the induction step.

Next, we need to show that for $n \geq 2^k$ we have $\chi_{\rm cf}(H) \geq k + 1$. Again, the proof is by induction on k. The base case $k = 0$ is trivial. For the induction step, let $k > 0$ and put $n = 2^k$. Let C be some CF-coloring of the underlying discrete intervals hypergraph. Consider the hyperedge $[n]$. There must be a uniquely colored vertex in $[n]$. Let x be this vertex. Either to the right of x or to its left we have at least 2^{k-1} vertices. That is, there is a hyperedge $S \subset [n]$ that does not contain x such that $|S| \geq 2^{k-1}$, so, by the induction hypothesis, any CF-coloring for S uses at least k colors. Thus, together with the color of x, C uses at least $k + 1$ colors in total. This completes the induction step. □

The notion of CF-coloring was first introduced and studied in [46] and [24]. This notion attracted many researchers and has been the focus of many research papers both in the computer science and mathematics communities. Recently, it has been studied also in the infinite settings of the so-called *almost disjoint set systems* by Hajnal et al. [26]. In this survey, we mostly consider hypergraphs that naturally arise in geometry. These come in two types:

- **Hypergraphs induced by regions:** Let $\mathcal{R}$ be a finite collection of regions (i.e., subsets) in $\mathbb{R}^d$, $d \geq 1$. For a point $p \in \mathbb{R}^d$, define $r(p) = \{R \in \mathcal{R} : p \in R\}$. The hypergraph $(\mathcal{R}, \{r(p)\}_{p \in \mathbb{R}^d})$, denoted $H(\mathcal{R})$, is called the *hypergraph* induced *by* $\mathcal{R}$. Since $\mathcal{R}$ is finite, so is the power set $2^{\mathcal{R}}$. This implies that the hypergraph $H(\mathcal{R})$ is finite as well.

- **Hypergraphs induced by points with respect to regions:** Let $P \subset \mathbb{R}^d$ and let $\mathcal{R}$ be a family of regions in $\mathbb{R}^d$. We refer to the hypergraph $H_{\mathcal{R}}(P) = (P, \{P \cap S | S \in \mathcal{R}\})$ as the *hypergraph*

induced by P with respect to $\mathcal{R}$. When $\mathcal{R}$ is clear from the context we sometimes refer to it as *the hypergraph induced by P*. In the literature, hypergraphs that are induced by points with respect to geometric regions of some specific kind are sometimes referred to as *range spaces*.

Definition 1.3 (Delaunay-Graph). *For a hypergraph $H = (V, \mathcal{E})$, denote by $G(H)$ the* Delaunay-graph *of H which is the graph $(V, \{S \in \mathcal{E} \mid |S| = 2\})$.*

In most of the coloring solutions presented in this paper we will see that, in fact, we get the stronger UM-coloring. It is also interesting to study hypergraphs for which $\chi_{\mathrm{cf}}(H) < \chi_{\mathrm{um}}(H)$. This line of research has been pursued in [14, 16]

1.2 Motivation

We start with several motivations for studying CF-colorings and UM-colorings.

1.2.1 Wireless Networks

Wireless communication is used in many different situations such as mobile telephony, radio and TV broadcasting, satellite communication, etc. In each of these situations a frequency assignment problem arises with application-specific characteristics. Researchers have developed different modeling approaches for each of the features of the problem, such as the handling of interference among radio signals, the availability of frequencies, and the optimization criterion.

The work of Even et al. [24] and of Smorodinsky [46] proposed to model frequency assignment to cellular antennas as CF-coloring. In this new model, one can use a very "small" number of distinct frequencies in total, to assign to a large number of antennas in a wireless network. Cellular networks are heterogeneous networks with two different types of nodes: *base-stations* (that act as servers) and *clients*. The base-stations are interconnected by an external fixed backbone network. Clients are connected only to base stations; connections between clients and base-stations are implemented by radio links. Fixed frequencies are assigned to base-stations to enable links to clients. Clients, on the other hand, continuously scan frequencies in search of a base-station with good reception. This scanning takes place automatically and enables smooth transitions between base-stations when a client is mobile. Consider a client that is within the reception range of two base stations. If these two base stations are assigned the same frequency, then mutual interference occurs, and the links between the client and each of these conflicting base stations are rendered too noisy to be used. A base station may serve a client provided that the reception is strong enough and interference from other base stations is weak enough. The fundamental problem of frequency assignment in cellular network is to assign frequencies to base stations so that every client is served by some base station. The goal is to minimize the number of assigned frequencies since the available spectrum is limited and costly.

The problem of frequency assignment was traditionally treated as a graph coloring problem, where the vertices of the graph are the given set of antennas and the edges are those pairs of antennas that overlap in their reception range. Thus, if we color the vertices of the graph such that no two vertices that are connected by an edge have the same color, we guarantee that there will be no conflicting base stations. However, this model is too restrictive. In this model, if a client lies within the reception range of say, k antennas, then every pair of these antennas are conflicting and therefore they must be assigned k distinct colors (i.e., frequencies). But note that if one of these antennas is assigned a color (say 1) that no other antenna is assigned (even if all other antennas are assigned the same color, say 2) then we use a total of two colors and this client can still be served. See Figure 1 for an illustration with three antennas.

A natural question thus arises: Suppose we are given a set of n antennas. The location of each antenna (base station) and its radius of transmission is fixed and is known (and is modeled as a disc in the plane). We seek the least number of colors that always suffice such that each of the discs is assigned one of the colors and such that every covered point p is also covered by some disc D whose assigned color is distinct from all the colors of the other discs that cover p. This is a special case of CF-coloring where the underlying hypergraph is induced by a finite family of discs in the plane.

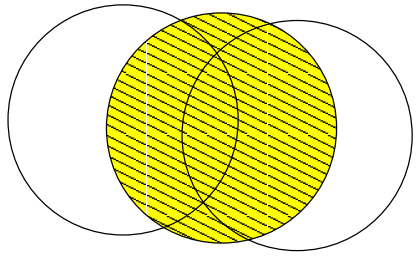

Figure 1: An example of three antennas presented as discs in the plane. In the classical model three distinct colors are needed where as in the new model two colors are enough as depicted here.

1.2.2 RFID networks

Radio frequency identification (RFID) is a technology where a reader device can sense the presence of a close by object by reading a tag device attached to the object. To improve coverage, multiple RFID readers can be deployed in the given region. RFID systems consist of readers and tags. A tag has an ID stored in its memory. The reader is able to read the IDs of the tags in the vicinity by using wireless protocol. In a typical RFID application, tags are attached to objects of interest, and the reader detects the presence of an object by using an available mapping of IDs to objects. We focus on passive tags i.e., tags that do not carry a battery. The power needed for passive tags to transmit their IDs to the reader is supplied by the reader itself. Assume that we are given a set D of readers where each reader is modeled by some disc in the plane. Let P be a set of tags (modeled as points) that lie in the union of the discs in D. Suppose that all readers in D use the same wireless frequency. For the sake of simplicity, suppose also that each reader is only allowed to be activated once. The goal is to schedule for each reader $d \in D$ a time slot $t(d)$ for which the reader d will be active. That is, at time $t(d)$ reader d would initiate a 'read' action. We further assume that a given tag $p \in P$ can be read by reader $d \in D$ at time t if $p \in d$ and d is initiating a 'read' action at time t (namely, $t = t(d)$) and no other reader d' for which $p \in d'$ is active at time t. We say that P is read by our schedule, if for every $p \in P$ there is at least one $d \in D$ and a time t such that p is read by d at time t. Obviously, we would like to minimize the total time slots used in the schedule. Thus our goal is to find a function $t : D \to \{1, \ldots, k\}$ which is conflict-free for the hypergraph $H(D)$. Since we want to minimize the total time slots used, again the question of what is the minimum number of colors that always suffice to CF-color any hypergraph induced by a finite set of n discs is of interest.

1.2.3 Vertex ranking

Let $G = (V, E)$ be a simple graph. An *ordered coloring* (also a *vertex ranking*) of G is a coloring of the vertices $\chi : V \to \{1, \ldots, k\}$ such that whenever two vertices u and v have the same color i then every simple path between u and v contains a vertex with color greater than i. Such a coloring has been studied before and has several applications. It was studied in the context of VLSI design [45] and in the context of parallel Cholesky factorization of matrices [36]. The vertex ranking problem is also interesting for the Operations Research community. It has applications in planning efficient assembly of products in manufacturing systems [30]. In general, it seems that the vertex ranking problem can model situations where inter-related tasks have to be accomplished fast in parallel, with some constrains (assembly from parts, parallel query optimization in databases, etc.). See also [31, 44]

The vertex ranking coloring is yet another special form of UM-coloring. Given a graph G, consider the hypergraph $H = (V, E')$ where a subset $V' \subseteq V$ is a hyperedge in E' if and only if V' is the set of vertices in some simple path of G. It is easily observed that an ordered coloring of G is equivalent to a UM-coloring of H.

1.3 A General Conflict-Free coloring Framework

Let P be a set of n points in $\mathbb{R}^2$ and let $\mathcal{D}$ be the set of all planar discs. In [24, 46] it was proved that $\chi_{\text{um}}(H_{\mathcal{D}}(P)) = O(\log n)$ and that this bound is asymptotically tight since for any $n \in \mathbb{N}$ there exist hypergraphs induced by sets of n points in the plane (w.r.t discs) which require $\Omega(\log n)$ in any CF-coloring. In fact, Pach and Tóth [42] proved a stronger lower-bound by showing that for any set P of n points it holds that $\chi_{\text{cf}}(H_{\mathcal{D}}(P)) = \Omega(\log n)$. The proofs of [24, 46] are algorithmic and rely on two crucial properties: The first property is that the Delaunay graph $G(H_{\mathcal{D}}(P))$ always contains a "large" independent set. The second is the following shrinkability property of discs: For every disc d containing a set of $i \geq 2$ points of P there is another disc d' such that $d' \cap P \subseteq d \cap P$ and $|d' \cap P| = 2$.

In [24, 46] it was also proved that, if D is a set of n discs in the plane, then $\chi_{\text{um}}(H(D)) = O(\log n)$. This bound was obtained by a reduction to a three-dimensional problem of UM-coloring a set of n points in $\mathbb{R}^3$ with respect to lower half-spaces. Later, Har-Peled and Smorodinsky [27] generalized this result to pseudo-discs using a probabilistic argument. Pach and Tardos [39] provided several non-trivial upper-bounds on the CF-chromatic number of arbitrary hypergraphs. In particular they showed that for every hypergraph H with m hyperedges

$$\chi_{\text{cf}}(H) \leq 1/2 + \sqrt{2m + 1/4}$$

Smorodinsky [47] introduced the following general framework for UM-coloring any hypergraph. This framework holds for arbitrary hypergraphs and the number of colors used is related to the chromatic number of the underlying hypergraph. Informally, the idea is to find a proper coloring with very 'few' colors and assign to all vertices of the largest color class the final color '1', discard all the colored elements and recursively continue on the remaining sub-hypergraph. See Algorithm 1 below.

Algorithm 1 UMcolor(H): *UM-coloring of a hypergraph $H = (V, \mathcal{E})$.*

1: $i \leftarrow 0$: *i denotes an unused color*
2: **while** $V \neq \emptyset$ **do**
3: **Increment:** $i \leftarrow i + 1$
4: **Auxiliary coloring:** find a proper coloring χ of the induced sub-hypergraph $H(V)$ with "few" colors
5: $V' \leftarrow$ **Largest color class of** χ
6: **Color:** $f(x) \leftarrow i$, $\forall x \in V'$
7: **Prune:** $V \leftarrow V \setminus V'$
8: **end while**

Theorem 1.4 ([47]). *Algorithm 1 outputs a valid UM-coloring of H.*

Proof. Formally, Algorithm 1 is not well defined as its output depends on the auxiliary coloring of step 4 of the algorithm. Nevertheless, we regard step 4 as given to us by some 'black' box and we treat this aspect of the algorithm later on. For a hyperedge $e \in \mathcal{E}$, let i be the maximal index (color) for which there is a vertex $v \in e$ colored with i. We claim that there is exactly one such vertex. Indeed, assume to the contrary that there is another such vertex $v' \in e$. Consider the ith iteration and let V' denote the set of vertices of V that are colored with color greater or equal to i. Namely, V' is the set of vertices that 'survived' all the prune steps up to iteration i and reached iteration i. Let χ denote the auxiliary proper coloring for the hypergraph $H(V')$ in iteration i. Since $e' = e \cap V'$ is a hyperedge of $H(V')$ and v and v' belong to the same color class of χ and $v, v' \in e'$ and since χ is a non-monochromatic coloring, there must exist a third vertex $v'' \in e'$ such that $\chi(v'') \neq \chi(v)$. This means that the final color of v'' is greater than i, a contradiction to the maximality of i in e. This completes the proof of the theorem. □

The number of colors used by Algorithm 1 is the number of iterations that are performed (i.e., the number of prune steps). This number depends on the 'black-box' auxiliary coloring provided in step 4

of the algorithm. If the auxiliary coloring χ uses a total of C_i colors on $|V_i|$ vertices, where V_i is the set of input vertices at iteration i, then by the pigeon-hole principle one of the colors is assigned to at least $\frac{|V_i|}{C_i}$ vertices so in the prune step of the same iteration at least $\frac{|V_i|}{C_i}$ vertices are discarded. Thus, after l iterations of the algorithm we are left with at most $|V| \cdot \Pi_{i=1}^{l}(1-\frac{1}{C_i})$ vertices. If this number is less than 1, then the number of colors used by the algorithm is at most l. If for example $C_i = 2$ for every iteration, then the algorithm discards at least $\frac{|V_i|}{2}$ vertices in each iteration so the number of vertices left after l iterations is at most $|V|\,(1-\frac{1}{2})^l$ so for $l = \lfloor \log n \rfloor + 1$ this number is less than 1. Thus the number of iterations is bounded by $\lfloor \log n \rfloor + 1$ where n is the number of vertices of the input hypergraph. In the next section we analyze the chromatic number $\chi(H)$ for several geometrically induced hypergraphs and use Algorithm 1 to obtain bounds on $\chi_{\mathrm{um}}(H)$.

We note that, as observed above, for a hypergraph H that admits a proper coloring with "few" colors hereditarily (that is, every induced sub-hypergraph admits a proper coloring with "few" colors), H also admits a UM-coloring with few colors. The following theorem summarizes this fact:

Theorem 1.5 ([47]). *Let $H = (V, \mathcal{E})$ be a hypergraph with n vertices, and let $k \in \mathbb{N}$ be a fixed integer, $k \geq 2$. If every induced sub-hypergraph $H' \subseteq H$ satisfies $\chi(H')) \leq k$, then $\chi_{\mathrm{um}}(H) \leq \log_{1+\frac{1}{k-1}} n = O(k \log n)$.*

Remark 1.6. We note that the parameter k in Theorem 1.5 can be replaced with a non-constant function $k = k(H')$. For example, if $k(H') = (n')^{\alpha}$ where $0 < \alpha \leq 1$ is a fixed real and n' is the number of vertices of H', an easy calculation shows that $\chi_{\mathrm{um}}(H) = O(n^{\alpha})$ where n is the number of vertices of H.

As we will see, for many of the hypergraphs that are mentioned in this survey, the two numbers $\chi(H), \chi_{\mathrm{um}}(H)$ are only a polylogarithmic (in $|V|$) factor apart. For the proof to work, the requirement that a hypergraph H admits a proper coloring with few colors hereditarily is necessary. One example is the 3-uniform hypergraph H with $2n$ vertices given above. We have $\chi(H) = 2$ and $\chi_{\mathrm{um}}(H) = n + 1$. Obviously H does not admit a proper 2-coloring hereditarily.

2 Conflict-Free Coloring of Geometric Hypergraphs

2.1 Discs and Pseudo-Discs in the Plane

2.1.1 Discs in $\mathbb{R}^2$

In [47] it was shown that the chromatic number of a hypergraph induced by a family of n discs in the plane is bounded by four. That is, for a finite family D of n discs in the plane we have:

Theorem 2.1 ([47]). $\chi(H(D)) \leq 4$

Combining Theorem 1.5 and Theorem 2.1 we obtain the following:

Theorem 2.2 ([47]). *Let $\mathcal{D}$ be a set of n discs in the plane. Then $\chi_{\mathrm{um}}(H(\mathcal{D})) \leq \log_{4/3} n$.*

Proof. We use Algorithm 1 and the auxiliary proper four coloring provided by Theorem 2.1 in each prune step. Thus in each step i we discard at least $|V_i|/4$ discs so the total number of iterations is bounded by $\log_{4/3} n$. □

Remark: The existence of a four coloring provided in Theorem 2.1 is algorithmic and uses the algorithm provided in the Four-Color Theorem [8, 9] which runs in linear time. It is easy to see that the total running time used by algorithm 1 for this case is therefore $O(n \log n)$. The bound in Theorem 2.2 holds also for the case of hypergraphs induced by points in the plane with respect to discs. This follows from the fact that such a hypergraph H satisfies $\chi(H) \leq 4$. Indeed, the Delaunay graph $G(H)$ is planar (and hence four colorable) and any disc containing at least 2 points also contains an edge of $G(H)$ [24].

Smorodinsky [47] proved that there exists an absolute constant C such that for any family $\mathcal{P}$ of pseudo-discs in the plane $\chi(H(\mathcal{P})) \leq C$. Hence, by Theorem 1.5 we have $\chi_{\mathrm{um}}(H(\mathcal{P})) = O(\log n)$. It is

not known what is the exact constant and it might be possible that it is still 4. By taking 4 pair-wise (openly-disjoint) touching discs, one can verify that it is impossible to find a proper coloring of the discs with less than 4 colors.

There are natural geometric hypergraphs which require n distinct colors even in any proper coloring. For example, one can place a set P of n points in general position in the plane (i.e., no three points lie on a common line) and consider those ranges that are defined by rectangles. In any proper coloring of P (w.r.t rectangles) every two such points need distinct colors since for any two points p, q there is a rectangle containing only p and q.

One might wonder what makes discs more special than other shapes? Below, we show that a key property that allows CF-coloring discs with a "small" number of colors unlike rectangles is the so called "low" *union-complexity* of discs.

Definition 2.3. *Let $\mathcal{R}$ be a family of n simple Jordan regions in the plane. The* union complexity *of $\mathcal{R}$ is the number of vertices (i.e., intersection of boundaries of pairs of regions in $\mathcal{R}$) that lie on the boundary $\partial \bigcup_{r \in \mathcal{R}} r$.*

As mentioned already, families of discs or pseudo-discs in the plane induce hypergraphs with chromatic number bounded by some absolute constant. The proof of [47] uses the fact that pseudo-discs have "linear union complexity" [32].

The following theorem bounds the chromatic number of a hypergraph induced by a finite family of regions $\mathcal{R}$ in the plane as a function of the union complexity of $\mathcal{R}$:

Theorem 2.4 ([47]). *Let $\mathcal{R}$ be a set of n simple Jordan regions and let $\mathcal{U} : \mathbb{N} \to \mathbb{N}$ be a function such that $U(m)$ is the maximum union complexity of any k regions in $\mathcal{R}$ over all $k \leq m$, for $1 \leq m \leq n$. We assume that $\frac{\mathcal{U}(m)}{m}$ is a non-decreasing function. Then, $\chi(H(\mathcal{R})) = O(\frac{\mathcal{U}(n)}{n})$. Furthermore, such a coloring can be computed in polynomial time under a proper and reasonable model of computation.*

As a corollary of Theorem 2.4, for any family $\mathcal{R}$ of n planar Jordan regions for which the union-complexity function $\mathcal{U}(n)$ is linear, we have that $\chi(H(\mathcal{R})) = O(1)$. Hence, combining Theorem 2.4 with Theorem 1.5 we have:

Theorem 2.5 ([47]). *Let $\mathcal{R}$ be a set of n simple Jordan regions and let $\mathcal{U} : \mathbb{N} \to \mathbb{N}$ be a function such that $U(m)$ is the maximum complexity of any k regions in $\mathcal{R}$ over all $k \leq m$, for $1 \leq m \leq n$. If $\mathcal{R}$ has linear union complexity in the sense that $\mathcal{U}(n) \leq Cn$ for some constant C, then $\chi_{\mathrm{um}}(H(\mathcal{R})) = O(\log n)$.*

2.2 Axis-Parallel rectangles

2.2.1 hypergraphs induced by axis-parallel rectangles

As mentioned already, a hypergraph induced by n rectangles in the plane might need n colors in any proper coloring. However, in the special case of axis-parallel rectangles, one can obtain non-trivial upper bounds. Notice that axis-parallel rectangles might have quadratic union complexity so using the above framework yields only the trivial upper bound of n. Nevertheless, in [47] it was shown that any hypergraph that is induced by a family of n axis-parallel rectangles, admits an $O(\log n)$ proper coloring. This bound is asymptotically tight as was shown recently by Pach and Tardos [40].

Theorem 2.6 ([47]). *Let $\mathcal{R}$ be a set of n axis-parallel rectangles in the plane. Then $\chi(H(\mathcal{R})) \leq 8 \log n$.*

Plugging this fact into Algorithm 1 yields:

Theorem 2.7 ([47]). *Let $\mathcal{R}$ be a set of n axis-parallel rectangles in the plane. Then $\chi_{\mathrm{um}}(H(\mathcal{R})) = O(\log^2 n)$.*

Remark: Notice that in particular there exists a family $\mathcal{R}$ of n axis-parallel rectangles for which $\chi_{\mathrm{cf}}(H(\mathcal{R})) = \Omega(\log n)$. Another example of a hypergraph H induced by n axis-parallel squares with $\chi(H) = 2$ and $\chi_{\mathrm{cf}}(H) = \Omega(\log n)$ is given in Figure 2. This hypergraph is, in fact, isomorphic to the discrete interval hypergraph with n vertices.

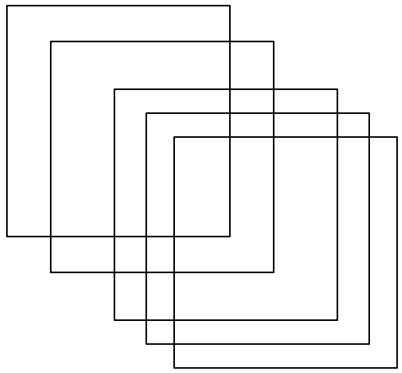

Figure 2: An example of n axis-parallel squares inducing the hypergraph H with $\chi(H) = 2$ and $\chi_{\mathrm{cf}}(H) = \Omega(\log n)$.

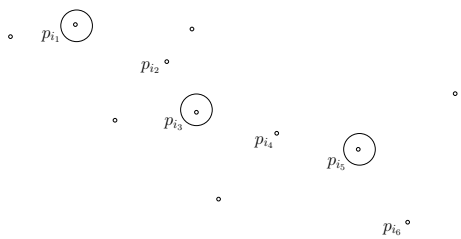

Figure 3: The circled points form an independent set in the Delaunay graph $D(P)$.

Problem 1. *Close the asymptotic gap between the best known upper bound $O(\log^2 n)$ and the lower bound $\Omega(\log n)$ on the CF-chromatic number of hypergraphs induced by n axis-parallel rectangles in the plane.*

2.2.2 Points with respect to axis-parallel rectangles

Let $\mathcal{R}$ be the family of all axis-parallel rectangles in the plane. For a finite set P in the plane, let $H(P)$ denote the hypergraph $H_{\mathcal{R}}(P)$. Let $D(P)$ denote the Delaunay graph of $H(P)$. It is easily seen that $\chi(D(P)) = \chi(H(P))$ since every axis-parallel rectangle containing at least two points, also contains an edge of $D(P)$.

The following problem seems to be rather elusive:

Problem 2. *Let $\mathcal{R}$ be the family of all axis-parallel rectangles in the plane. Let $d = d(n)$ be the least integer such that for any set P of n points in the plane $\chi(D(P)) \leq d(n)$. Provide sharp asymptotic bounds on $d(n)$.*

It was first observed in [27] that $d(n) = O(\sqrt{n})$ by a simple application of the classical Erdős-Szekeres theorem for a sequence of reals. This theorem states that in a sequence of $k^2 + 1$ reals there is always a monotone subsequence of length at least $k + 1$ (see, e.g., [51]).

One can show that for any set P of n points in the plane there is a subset $P' \subset P$ of size $\Omega(\sqrt{n})$ which is independent in the graph $D(P)$. To see this, sort the points $P = \{p_1, \ldots, p_n\}$ according to their x-coordinate. Write the sequence of y-coordinates of the points in P $y_1, \ldots, y_n$. By the Erdős-Szekeres theorem, there is a subsequence $y_{i_1}, \ldots, y_{i_k}$ with $k = \Omega(\sqrt{n})$ which is monotone. We refer to the corresponding subset of P as a *monotone chain*. Notice that by taking every other point in the monotone chain, the set $p_{i_1}, p_{i_3}, p_{i_5}, \ldots$ is a subset of size $k/2 = \Omega(\sqrt{n})$ which is independent in $D(P)$. See Figure 3 for an illustration. In order to complete the coloring it is enough to observe that one can iteratively partition P into $O(\sqrt{n})$ independent sets of $D(P)$.

The bounds on $d(n)$ were recently improved and the best known bounds are stated below:

Upper bound: [3] $d(n) = \tilde{O}(n^{0.382})$
Lower bound: [19] $d(n) = \Omega(\frac{\log n}{\log^2 \log n})$

We give a short sketch of the ideas presented in [3] in order to obtain the upper bound $d(n) = \tilde{O}(n^{0.382})$ where $\tilde{O}$ denotes the fact that a factor of polylog is hiding in the big-O notation. Our presentation of the ideas is slightly different from [22, 3] since our aim is to bound $d(n)$ which corresponds to coloring the Delaunay graph of n points rather than CF-coloring the points themselves. However, as mentioned above, such a bound implies also a similar bound on the CF-chromatic number of the underlying hypergraph. Assume that $d(n) \geq c \log n$ for some fixed constant c. We will show that $d(n) = O(n^{\alpha})$ for all $\alpha > \alpha_0 = \frac{3-\sqrt{5}}{2}$. The proof relies on the following key ingredient, first proved in [22]. For a point set P in the plane, let G_r be an $r \times r$ grid such that each row of G_r and each column of G_r contains at most $\lceil n/r \rceil$ points of P. Such a grid is easily seen to exists. A coloring of P is called a *quasi-coloring* with respect to G_r if every rectangle that is fully contained in a row of G_r or fully contained in a column of G_r is non-monochromatic. In other words, when coloring P, we do not care about rectangles that are not fully contained in a row or fully contained in a column (or contain only one point).

Lemma 2.8 ([22, 3]). *Let P be a set of n points in the plane. If $\Omega(\log n) = d(n) = O(n^{\alpha})$ then for every r, P admits a quasi -coloring with respect to G_r with $\tilde{O}((\frac{n}{r})^{2\alpha-\alpha^2})$ colors.*

The proof of the lemma uses a probabilistic argument. We first color each column in G_r independently with $d(n/r)$ colors. Then for each column we permute the colors randomly and then re-color all points in a given row that were assigned the same initial color. We omit the details of the proof and its probabilistic analysis.

Next, we choose an appropriate subset $P' \subset P$ which consists of $O(r)$ monotone chains and with the following key property: If a rectangle S contains points from at least two rows of G_r and at least two columns of G_r, then S also contains a point of P'. Note that a chain can be colored with 2 colors so altogether one can color P' with $O(r)$ colors, not to be used for $P \setminus P'$. Thus a rectangle that is not fully contained in a row or a column of G_r is served by the coloring. Hence, it is enough to quasi-color the points of $P \setminus P'$ with respect to G_r. By the above lemma, the total number of colors required for such a coloring is $\tilde{O}((\frac{n}{r})^{2\alpha-\alpha^2} + r)$. Choosing $r = n^{\frac{2\alpha-\alpha^2}{1+2\alpha-\alpha^2}}$ we obtain the bound $\tilde{O}(n^{\frac{2\alpha-\alpha^2}{1+2\alpha-\alpha^2}})$. Thus, taking α_0 to satisfy the equality

$$\alpha_0 = \frac{2\alpha_0 - {\alpha_0}^2}{1 + 2\alpha_0 - {\alpha_0}^2}$$

or $\alpha_0 = \frac{3-\sqrt{5}}{2}$, we have that for $\alpha > \alpha_0$ $d(n) = O(n^{\alpha})$ as asserted.

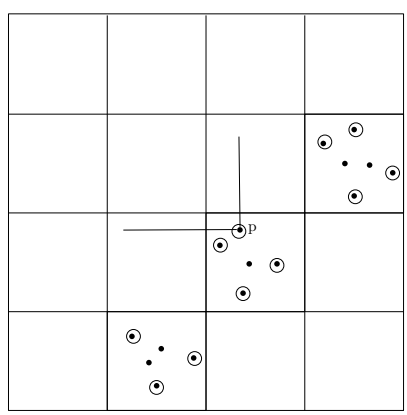

Figure 4: The grid G_r (for $r = 4$) and one of its positive diagonals. The circled points are taken to be in P' and the square points are in $P \setminus P'$. The point p is an extreme point of type 2 in that diagonal and is also an extreme point of type 1 in the negative diagonal that contains the grid cell of p.

To complete the proof, we need to construct the set P'. Consider the diagonals of the grid G_r. See Figure 4 for an illustration. In each positive diagonal we take the subset of (extreme) points of type 2 or 4, where a point p is said to be of type 2 (respectively, 4) if the 2'nd quadrant (respectively, the 4'th quadrant) with respect to p (i.e., the subset of all points above and to the left of p) does not contain any other point from the diagonal. Similarly, for diagonals with negative slope we take the points of type 1 and 3. If a point belongs to more than one type (in the two diagonals that contain the point) then we arbitrarily choose one of the colors it gets from one of the diagonals. It is easy to see that the set P' admits a proper coloring with $O(r)$ colors, as there are only $2r-1$ positive diagonals and $2r-1$ negative diagonals, and in each diagonal the extreme points of a fixed type form a monotone chain.

As mentioned, reducing the gap between the best known asymptotic upper and lower bounds mentioned above is a very interesting open problem.

2.3 Shallow Regions

As mentioned already, for every n there are sets D of n discs in the plane such that $\chi_{\text{cf}}(H(D)) = \Omega(\log n)$. For example, one can place n unit discs whose centers all lie on a line, say the x-axis, such that the distance between any two consecutive centers is less than $1/n$. It was shown in [24] that, for such a family D, $\chi_{\text{cf}}(H(D)) = \Omega(\log n)$ since $H(D)$ is isomorphic to the discrete interval hypergraph with n vertices. However, in this case there are points that are covered by many of the discs of D (in fact, by all of them). This leads to the following fascinating problem: What happens if we have a family of n discs D with the property that every point is covered by at most k discs of D, for some parameter k. It is not hard to see that in such a case, one can color D with $O(k)$ colors such that any two intersecting discs have distinct colors. However, we are interested only in CF-coloring of D. Let us call a family of regions, with the property that no point is covered by more than k of the regions, a *k-shallow* family.

Problem 3. *What is the minimum integer $f = f(k)$ such that for any finite family of k-shallow discs D, we have: $\chi_{\text{cf}}(H(D)) \leq f(k)$?*

As mentioned already, it is easy to see that $f(k) = O(k)$. However, it is conjectured that the true upper bound should be polylogarithmic in k.

In the further restricted case that any disc in D intersects at most k other discs, Alon and Smorodinsky [5] proved that $\chi_{\text{cf}}(H(D)) = O(\log^3 k)$ and this was recently improved by Smorodinsky [48] to $\chi_{\text{cf}}(H(D)) = O(\log^2 k)$. Both bounds also hold for families of pseudo-discs. We sketch the proof of the following theorem:

Theorem 2.9 ([48]). *Let D be a family of n discs in the plane such that any disc in D intersects at most k other discs in D. Then $\chi_{\text{cf}}(H(D)) = O(\log^2 k)$*

The proof of Theorem 2.9 is probabilistic and uses the Lovász Local Lemma [6]. We start with a few technical lemmas:

Denote by $E_{\leq \ell}(D)$ the subset of hyperedges of $H(D)$ of cardinality less than or equal to ℓ.

Lemma 2.10. *Let D be a finite set of n planar discs. Then $|E_{\leq k}(D)| = O(kn)$.*

Proof. This easily follows from the fact that discs have linear union-complexity [32] and the Clarkson-Shor probabilistic technique [20]. We omit the details of the proof. □

Lemma 2.11. *Let D be a set of n planar discs, and let $\ell > 1$ be an integer. Then the hypergraph $(D, E_{\leq \ell}(D))$ can be CF-colored with $O(\ell)$ colors.*

Remark: In fact, the proof of Lemma 2.11 which can be found in [7] provides a stronger coloring. The coloring has the property that every hyperedge in $E_{\leq \ell}(D)$ is colorful (i.e., all vertices have distinct colors). Such a coloring is referred to as ℓ-colorful coloring and is discussed in more details in Subsection 3.2

Lemma 2.12. *Let D be a set of discs such that every disc intersects at most k others. Then there is a constant C such that D can be colored with two colors (red and blue) and such that for every face $f \in \mathcal{A}(D)$ with depth at least $C \ln k$, there are at least $\frac{|d(f)|}{3}$ red discs containing f and at least $\frac{|d(f)|}{3}$ blue discs containing f, where $d(f)$ is the set of all discs containing the face f.*

Proof. Consider a random coloring of the discs in D, where each disc $d \in D$ is colored independently red or blue with probability $\frac{1}{2}$. For a face f of the arrangement $\mathcal{A}(D)$ with $|d(f)| \geq C \ln k$ (for some constant C to be determined later), let A_f denote the "bad" event that either less than $\frac{|d(f)|}{3}$ of the discs in $d(f)$ or more than $\frac{2|d(f)|}{3}$ of them are colored blue. By the Chernoff inequality (see, e.g., [6]) we have:

$$Pr[A_f] \leq 2e^{-\frac{|d(f)|}{72}} \leq 2e^{-\frac{C \ln k}{72}}$$

We claim that for every face f, the event A_f is mutually independent of all but at most $O(k^3)$ other events. Indeed A_f is independent of all events A_s for which $d(s) \cap d(f) = \emptyset$. By assumption, $|d(f)| \leq k+1$. Observe also that a disc that contains f, can contain at most $O(k^2)$ other faces, simply because the arrangement of k discs consists of at most $O(k^2)$ faces. Hence, the claim follows.

Let C be a constant such that:

$$e \cdot 2e^{-\frac{C \ln k}{72}} \cdot 2k^3 < 1$$

By the Lovász Local Lemma, (see, e.g., [6]) we have:

$$Pr[\bigwedge_{|d(f)| \geq C \ln k} \bar{A}_f] > 0$$

In particular, this means that there exists a coloring for which every face f with $|d(f)| \geq C \ln k$ has at least $\frac{|d(f)|}{3}$ red discs containing f and at least $\frac{|d(f)|}{3}$ blue discs containing it, as asserted. This completes the proof of the lemma. □

Proof of Theorem 2.9: Consider a coloring of D by two colors as in Lemma 2.12. Let B_1 denote the set of discs in D colored blue. We will color the discs of B_1 with $O(\ln k)$ colors such that $E_{\leq 2C \ln k}(B_1)$ is conflict-free, as guaranteed by Lemma 2.11, and recursively color the discs in $D \setminus B_1$ with colors disjoint from those used to color B_1. This is done, again, by splitting the discs in $D \setminus B_1$ into a set of red discs and a set B_2 of blue discs with the properties guaranteed by Lemma 2.12. We repeat this process until every face of the arrangement $\mathcal{A}(D')$ (of the set D' of all remaining discs) has depth at most $C \ln k$. At that time, we color D' with $O(\ln k)$ colors as described in Lemma 2.11. To see that this coloring scheme is a valid conflict-free coloring, consider a point $p \in \bigcup_{d \in D} d$. Let $d(p) \subset D$ denote the subset of all discs in D that contain p. Let i be the largest index for which $d(p) \cap B_i \neq \emptyset$. If i does not exist (namely, $d(p) \cap B_i = \emptyset \ \forall i$) then by Lemma 2.12 $|d(p)| \leq C \ln k$. However, this means that $d(p) \in E_{\leq C \ln k}(D)$ and thus $d(p)$ is conflict-free by the coloring of the last step. If $|d(p) \cap B_i| \leq 2C \ln k$ then $d(p)$ is conflict free since one of the colors in $d(p) \cap B_i$ is unique according to the coloring of $E_{\leq c \ln k}(B_i)$. Assume then, that $|d(p) \cap B_i| > 2C \ln k$. Let x denote the number of discs containing p at step i. By the property of the coloring of step i, we have that $x \geq 3C \ln k$. This means that after removing B_i, the face containing p is also contained in at least $C \ln k$ other discs. Hence, p must also belong to a disc of B_{i+1}, a contradiction to the maximality of i. To argue about the number of colors used by the above procedure, note that in each prune step, the depth of every face with depth $i \geq C \ln k$ is reduced with a factor of at least $\frac{1}{3}$. We started with a set of discs such that the maximal depth is $k+1$. After the first step, the maximal depth is $\frac{2}{3}k$ and for each step we used $O(\ln k)$ colors so, in total, we have that the maximum number of colors $f(k,r)$, needed for CF-coloring a family of discs with maximum depth r such that each disc intersects at most k others satisfies the recursion:

$$f(k,r) \leq O(\ln k) + f(k, \frac{2}{3}r).$$

This gives $f(k,r) = O(\ln k \log r)$. Since, in our case $r \leq k+1$, we obtain the asserted upper bound. This completes the proof of the theorem. $\square$

Remark: Theorem 2.9 works almost verbatim for any family of regions (not necessarily convex) with linear union complexity. Thus, for example, the result applies to families of homothetics or more generally to any family of pseudo-discs, since pseudo-discs have linear union complexity ([32]). We also note that, as in other cases mentioned so far, it is easily seen that the proof of the bound of Theorem 2.9 holds for UM-coloring.

The proof of Theorem 2.9 is non-constructive since it uses the Lovász Local Lemma. However, we can use the recently discovered algorithmic version of the Local Lemma of Moser and Tardos [38] to obtain a constructive proof of Theorem 2.9.

Problem 4. *As mentioned, the only lower bound that is known for this problem is $\Omega(\log k)$ which is obvious from taking the lower bound construction of [24] with k discs. It would be interesting to close the gap between this lower bound and the upper bound $O(\log^2 k)$.*

The following is a rather challenging open problem:

Problem 5. *Obtain a CF-coloring of discs with maximum depth $k+1$ (i.e., no point is covered by more than $k+1$ discs) with only polylogarithmic (in k) many colors. Obviously, the assumption of this subsection that a disc can intersect at most k others is much stronger and implies maximum depth $k+1$. However, the converse is not true. Assuming only bounded depth does not imply the former. In bounded depth, we still might have discs intersecting many (possibly all) other discs.*

3 Extensions of CF-Coloring

3.1 k-CF coloring

We generalize the notion of CF-coloring of a hypergraph to *k-CF-coloring*. Informally, we think of a hyperedge as being 'served' if there is a color that appears in the hyperedge (at least once and) at most k times, for some fix prescribed parameter k. For example, we will see that when the underlying hypergraph is induced by n points in $\mathbb{R}^3$ with respect to the family of all balls, there are n points for which any CF-coloring needs n colors but there exists a 2-CF-coloring with $O(\sqrt{n})$ colors (and a k-CF-coloring with $O(n^{1/k})$ colors for any fixed $k \geq 2$). We also show that any hypergraph $(V, \mathcal{E})$ with a finite VC-dimension c, can be k-CF-colored with $O(\log |P|)$ colors, for a reasonably large k. This relaxation of the model is applicable in the wireless scenario since the real interference between conflicting antennas (i.e., antennas that are assigned the same frequency and overlap in their coverage area) is a function of the number of such antennas. This suggests that if for any given point, there is some frequency that is assigned to at most a "small" number of antennas that cover this point, then this point can still be served using that frequency because the interference between a small number of antennas is low. This feature is captured by the following notion of k-CF-coloring.

Definition 3.1. **k-CF-coloring of a hypergraph***: Let $H = (V, \mathcal{E})$ be a hypergraph. A function $\chi : V \to \{1, \ldots, i\}$ is a k-*CF-coloring *of H if for every $S \in \mathcal{E}$ there exists a color j such that $1 \leq |\{v \in S | \chi(v) = j\}| \leq k$; that is, for every hyperedge $S \in \mathcal{E}$ there exists at least one color j such that j appears (at least once and) at most k times among the colors assigned to vertices of S.*

Let $\chi_{kCF}(H)$ denote the minimum number of colors needed for a k-CF-coloring of H.

Note that a 1-CF-coloring of a hypergraph H is simply a CF-coloring.

Here we modify Algorithm 1 to obtain a k-CF coloring of any hypergraph. We need yet another definition of the following relaxed version of a proper coloring:

Definition 3.2. *Let $H = (V, \mathcal{E})$ be a hypergraph. A coloring φ of H is called k-*weak *if every hyperedge $e \in \mathcal{E}$ with $|e| \geq k$ is non-monochromatic. That is, for every hyperedge $e \in \mathcal{E}$ with $|e| \geq k$ there exists at least two vertices $x, y \in e$ such that $\varphi(x) \neq \varphi(y)$.*

Notice that a k-weak coloring (for $k \geq 2$) of a hypergraph $H = (V, \mathcal{E})$ is simply a proper coloring for the hypergraph $(V, \mathcal{E}_{\geq k})$ where $\mathcal{E}_{\geq k}$ is the subset of hyperedges in $\mathcal{E}$ with cardinality at least k. This notion was used implicitly in [27, 47] and then was explicitly defined and studied in the Ph.D. of Keszegh [34, 33]. It is also related to the notion of cover-decomposability and polychromatic colorings (see, e.g., [25, 41, 43]).

We are ready to generalize Algorithm 1. See Algorithm 2 below.

Algorithm 2 k-CFcolor(H): *k-Conflict-Free-color a hypergraph $H = (V, \mathcal{E})$.*

$i \leftarrow 0$: *i denotes an unused color*
while $V \neq \emptyset$ **do**
  Increment: $i \leftarrow i + 1$
  Auxiliary coloring: find a weak $k+1$-coloring χ of $H(V)$ with "few" colors
  $V' \leftarrow$ **Largest color class of** χ
  Color: $f(x) \leftarrow i$, $\forall x \in V'$
  Prune: $V \leftarrow V \setminus V'$, $H \leftarrow H(V)$
end while

Theorem 3.3 ([27]). *Algorithm 2 outputs a valid k-CF-coloring of H.*

Proof. The proof is similar to the proof provided in Section 1.3 for the validity of Algorithm 1. In fact, again, the coloring provided by Algorithm 2 has the stronger property that for any hyperedge $S \in \mathcal{E}$ the maximal color appears at most k times. □

As a corollary similar to the one mentioned in Theorem 1.5, for a hypergraph H that admit a $k+1$-weak coloring with "few" colors hereditarily, H also admits a k-CF-coloring with few colors. The following theorem summarizes this fact:

Theorem 3.4 ([27]). *Let $H = (V, \mathcal{E})$ be a hypergraph with n vertices, and let $l, k \in \mathbb{N}$ be two fixed integers, $k \geq 2$. Assume that every induced sub-hypergraph $H' \subseteq H$ admits a $k+1$-weak coloring with at most l colors. Then H admits a k-CF-coloring with at most $\log_{1+\frac{1}{l-1}} n = O(l \log n)$ colors.*

Proof. The proof is similar to the proof of Theorem 1.5 □

3.1.1 CF-Coloring of Balls in Three Dimensions

Lemma 3.5. *Let $\mathcal{B}$ be the set of balls in three dimensions. There exists a hypergraph H induced by a finite set P of n points in $\mathbb{R}^3$ with respect to $\mathcal{B}$ such that $\chi_{1CF}(H) = n$. The same holds for the set $\mathcal{H}$ of halfspaces in $\mathbb{R}^d$, for $d > 3$.*

Proof. Take P to be a set of n points on the positive portion of the moment curve $\gamma = \{(t, t^2, t^3) | t \geq 0\}$ in $\mathbb{R}^3$. It is easy to verify that any pair of points $p, q \in P$ are connected in the Delaunay triangulation of P implying that there exists a ball whose intersection with P is $\{p, q\}$. Thus, all points must be colored using different colors.

The second claim follows by taking P to be n distinct points on the moment curve $\{(t, t^2, \ldots, t^d)\}$ in $\mathbb{R}^d$ (i.e, P is the set of vertices of a so-called *cyclic-polytope* $C(n, d)$. See, e.g., [49]). □

Theorem 3.6 ([27, 46]). *Let P be a set of n points in $\mathbb{R}^3$. Put $H = H_{\mathcal{B}}(P)$. Then $\chi_{kCF}(H) = O(n^{1/k})$, for any fixed constant $k \geq 1$.*

Proof. As is easily seen by Algorithm 2, it is enough to prove that H admits a $k+1$-weak coloring with $O(n^{1/k})$ colors. If so, then in every iteration we discard at least $\Omega(|P_i|^{1-\frac{1}{k}})$ elements so the total number of iterations (colors) used is $O(n^{1/k})$. The proof that H admits a $k+1$-weak coloring with $O(n^{1/k})$ colors uses the probabilistic method. We provide only a brief sketch of the proof. It is enough to consider all

balls containing exactly $k+1$ points since if a ball contains more than $k+1$ points then by perturbation and shrinking arguments it will also contain a subset of $k+1$ points that can be cut-off by a ball. So we may assume that in the underlying hypergraph $H = (P, \mathcal{E})$, all hyperedges have cardinality $k+1$ (such a hypergraph is also called a $k+1$-uniform hypergraph). So we want to color the set P with $O(n^{1/k})$ colors such that any hyperedge in $\mathcal{E}$ is non-monochromatic. By the Clarkson-Shor technique, it is easy to see that the number of hyperedges in $\mathcal{E}$ is $O(k^2 n^2)$. Thus the average degree of a vertex in H is $O(n)$ where the constant of proportionality depends on k. It is well known that such a hypergraph has chromatic number $O(n^{1/k})$. This is proved via the probabilistic method. The main ingredient is the Lovász Local Lemma (see, e.g., [6]). □

In a similar way we have:

Theorem 3.7 ([27, 46]). *Let $\mathcal{R}$ be a set of n balls in $\mathbb{R}^3$. Then $\chi_{kCF}(H(\mathcal{R})) = O(n^{1/k})$.*

3.1.2 VC-dimension and k-CF coloring

Definition 3.8. *Let $H = (V, \mathcal{E})$ be a hypergraph. The* Vapnik-Chervonenkis *dimension (or* VC-dimension*) of H, denoted by $VC(H)$, is the maximal cardinality of a subset $V' \subset V$ such that $\{V' \cap r | r \in \mathcal{E}\} = 2^{V'}$ (such a subset is said to be* shattered*). If there are arbitrarily large shattered subsets in V then $VC(H)$ is defined to be ∞. See [37] for discussion of VC-dimension and its applications.*

There are many hypergraphs with finite VC-dimension that arise naturally in combinatorial and computational geometry. One such example is the hypergraph $H = (\mathbb{R}^d, \mathcal{H}_d)$, where $\mathcal{H}_d$ is the family of all (open) halfspaces in $\mathbb{R}^d$. Any set of $d+1$ affinely independent points is shattered in this space, and, by Radon's theorem, no set of $d+2$ points is shattered. Therefore $VC(H) = d+1$.

Definition 3.9. *Let $(V, \mathcal{E})$ be a hypergraph with $|V| = n$ and let $0 < \epsilon \leq 1$. A subset $N \subset V$ is called an* ϵ-net *for $(V, \mathcal{E})$ if for every hyperedge $S \in \mathcal{E}$ with $|S| \geq \epsilon n$ we have $S \cap N \neq \emptyset$.*

Thus, an ϵ-net is a *hitting set* of all 'heavy' hyperedges, namely, those containing at least ϵn vertices.

An important consequence of the finiteness of the VC-dimension is the existence of small ϵ-nets, as shown by Haussler and Welzl in [28], where the notion of VC-dimension of a hypergraph was introduced to computational geometry.

Theorem 3.10 ([28]). *For any hypergraph $H = (V, \mathcal{E})$ with finite VC-dimension d and for any $\epsilon > 0$, there exists an ϵ-net $N \subset V$ of size $O(\frac{d}{\epsilon} \log \frac{d}{\epsilon})$.*

Remark: In fact, Theorem 3.10 is valid also in the case where H is equipped with an arbitrary probability measure μ. An ϵ-net in this case is a subset $N \subset V$ that meets all hyperedges with measure at least ϵ.

Since all hypergraphs mentioned so far have finite VC-dimension, and since some of them sometimes must be CF-colored with n colors, there is no direct relationship between a finite VC-dimension of a hypergraph and the existence of a CF-coloring of that hypergraph with a small number of colors. In this subsection we show that such a relationship does exist, if we are interested in k-CF-coloring with a reasonably large k.

We first introduce a variant of the general framework for k-CF-coloring of a hypergraph $H = (V, \mathcal{E})$. In this framework we modify lines 4 and 5 in Algorithm 2. In Algorithm 2 we first find a $k+1$-weak coloring of the underlying hypergraph (line 4) which is a partition of the vertices into sets such that each set has the following property: Every set in the partition cannot fully contain a hyperedge with cardinality at least $k+1$. Equivalently, every color class $V' \subset V$ has the property that every hyperedge containing at least $k+1$ vertices of V' also contain vertices of $V \setminus V'$. We modify that framework by directly finding a "large" such subset in the hypergraph.

Definition 3.11. *Let $H = (V, \mathcal{E})$ be a hypergraph. A subset $V' \subset V$ is* k-admissible *if for any hyperedge $S \in \mathcal{E}$ with $|S \cap V'| > k$ we have $S \cap (V \setminus V') \neq \emptyset$.*

Assume that we are given an algorithm $\mathbf{A}$ that computes, for any hypergraph $H = (V, \mathcal{E})$, a non-empty k-admissible set $V' = \mathbf{A}(H)$. We can now use algorithm $\mathbf{A}$ to k-CF-color the given hypergraph (i) Compute a $k+1$-admissible set $V' = \mathbf{A}(H)$, and assign to all the elements in V' the color 1. (ii) Color the remaining elements in $V \setminus V'$ recursively, where in the ith stage we assign the color i to the vertices in the resulting $k+1$-admissible set. We denote the resulting coloring by $C_A(H)$.

The proof of the following theorem is, yet, again, similar to that of Theorem 1.4.

Theorem 3.12 ([27, 46]). *Given a hypergraph $H = (V, \mathcal{E})$, the coloring $C_A(H)$ is a valid k-CF coloring of S.*

Lemma 3.13. *Let $H = (V, \mathcal{E})$ with $|V| = n$ be a hypergraph with VC-dimension d. For any $k \geq d$ there exists a k-admissible set $V' \subset V$ with respect to H of size $\Omega\big(n^{1-(d-1)/k}\big)$.*

Proof. Any coloring of V is valid as far as the small hyperedges of $\mathcal{E}$ are concerned; namely, those are the hyperedges that contain at most k vertices. Thus, let $\mathcal{E}'$ be the subset of hyperedges of $\mathcal{E}$ of size larger than k. By Sauer's Lemma (see, e.g., [6]) we have that $|\mathcal{E}'| \leq |\mathcal{E}| \leq n^d$.

Next, we randomly color V by black and white, where an element is being colored in black with probability p, where p would be specified shortly. Let I be the set of points of V colored in black. If a hyperedge $r \in \mathcal{E}'$ is colored only in black, we remove one of the vertices of r from I. Let I' be the resulting set. Clearly, I' is a k-admissible set for H.

Furthermore, by linearity of expectation, the expected size of I' is at least

$$pn - \sum_{r \in \mathcal{E}'} p^{|r|} \geq pn - \sum_{r \in \mathcal{E}'} p^{k+1} \geq pn - p^{k+1} n^d.$$

Setting $p = \big((k+1)n^{d-1}\big)^{-1/k}$, we have that the expected size of I' is at least $pn - p^{k+1}n^d = pn(1 - 1/(k+1)) = \Omega\big(n^{1-(d-1)/k}\big)$, as required. □

As was already seen, for geometric hypergraphs one might be able to get better bounds than the one guaranteed by Lemma 3.13.

Theorem 3.14 ([27, 46]). *Let $H = (V, \mathcal{E})$ with $|V| = n$ be a finite hypergraph with VC-dimension d. Then for $k \geq d \log n$ there exists a k-CFcoloring of H with $O(\log n)$ colors.*

Proof. By Lemma 3.13 the hypergraph H contains a k-admissible set of size at least $n/2$. Plugging this fact to the algorithm suggested by Theorem 3.12 completes the proof of the theorem. □

As remarked above, Theorem 3.14 applies to all hypergraphs mentioned in this paper. Note also, that Lemma 3.13 gives us a trade off between the number of colors and the threshold size of the coloring. As such, the bound of Theorem 3.14 is just one of a family of such bounds implied by Lemma 3.13.

3.2 k-Strong CF-Coloring

Here, we focus on the notion of *k-strong-conflict-free* (abbreviated, $kSCF$) which is yet another extension of the notion of CF-coloring of hypergraphs.

Definition 3.15 (k-strong conflict-free coloring:). *Let $H = (V, \mathcal{E})$ be a hypergraph and let $k \in \mathbb{N}$ be some fixed integer. A coloring of V is called* k-strong-conflict-free *for H ($kSCF$ for short) if for every hyperedge $e \in \mathcal{E}$ with $|e| \geq k$ there exists at least k vertices in e, whose colors are unique among the colors assigned to the vertices of e and for each hyperedge $e \in \mathcal{E}$ with $|e| < k$ all vertices in e get distinct colors. Let $f_H(k)$ denote the least integer l such that H admits a $kSCF$-coloring with l colors.*

Abellanas et al. [2] were the first to study kSCF-coloring[1]. They focused on the special case of hypergraphs induced by n points in $\mathbb{R}^2$ with respect to discs. They showed that in this case the hypergraph admits a kSCF-coloring with $O(\frac{\log n}{\log \frac{ck}{ck-1}})$ $(= O(k \log n))$ colors, for some absolute constant c.

The following notion was recently introduced and studied by Aloupis et al. [7] for the special case of hypergraphs induced by discs:

Definition 3.16 (k-colorful coloring). *Let $H = (V, \mathcal{E})$ be a hypergraph, and let φ be a coloring of H. A hyperedge $e \in \mathcal{E}$ is said to be* k-colorful *with respect to φ if there exist k vertices in e that are colored distinctively under φ. The coloring φ is called* k-colorful *if every hyperedge $e \in \mathcal{E}$ is* $\min\{|e|, k\}$*-colorful. Let $c_H(k)$ denote the least integer l such that H admits a k-colorful coloring with l colors.*

Aloupis et al. [7] introduced this notion explicitly and were motivated by a problem related to battery lifetime in sensor networks. This notion is also related to the notion of polychromatic colorings. In polychromatic colorings, the general question is to estimate the minimum number $f = f(k)$ such that one can k-color the hypergraph with the property that all hyperedges of cardinality at least $f(k)$ are colorful in the sense that they contain a representative color of each color class. (see, e.g., [25, 13, 43] for additional details on the motivation and related problems).

Remark: Every $kSCF$-coloring of a hypergraph H is a k-colorful coloring of H. However, the opposite claim is not necessarily true.

The following connection between k-colorful coloring and strong-conflict-free coloring of hypergraphs was proved by Horev et al. in [29]. If a hypergraph H admits a k-colorful coloring with a "small" number of colors (hereditarily) then it also admits a $(k-1)$SCF-coloring with a "small" number of colors. This connection is analogous to the connection between non-monochromatic coloring and CF-coloring as appear in Theorem 1.5 and the connection between $k+1$-weak coloring and k-CF-coloring as appear in Theorem 3.4. We start by introducing the general framework of [29] for kSCF-coloring a given hypergraph.

A Framework For Strong-Conflict-Free Coloring Let H be a hypergraph with n vertices and let k and l be some fixed integers such that H admits the hereditary property that every vertex-induced sub-hypergraph H' of H admits a k-colorful coloring with at most l colors. Then H admits a $(k-1)SCF$-coloring with $O(l \log n)$ colors. For the case when l is replaced with the function $kn(H')^{\alpha}$ we get a better bound without the $\log n$ factor. The proof is constructive. The following framework (denoted as Algorithm 3) produces a valid $(k-1)$SCF coloring for a hypergraph H.

Algorithm 3 (k-1)SCF-color(H): *$(k-1)$-Strong Conflict-Free-color a hypergraph $H = (V, \mathcal{E})$.*

```
i ← 1  i denotes an unused color
while V ≠ ∅ do
    Increment: i ← i + 1
    Auxiliary Coloring: find a k-colorful coloring φ of H(V) with "few" colors
    V' ← Largest color class of φ
    Color: χ(x) ← i , ∀x ∈ V'
    Prune: V ← V \ V'.
    Increment: i ← i + 1.
end while
Return χ.
```

Note that Algorithm 3 is a generalization of Algorithm 1. Indeed for $k = 2$ the two algorithms become identical since a 2-colorful coloring is equivalent to a proper coloring. Arguing about the number of colors used by the algorithm is identical to the arguments as in the coloring produced by Algorithm 1. The proof or correctness is slightly more subtle.

[1] They referred to such a coloring as k-conflict-free coloring.

For a hypergraph $H = (V, \mathcal{E})$, we write $n(H)$ to denote the number of vertices of H. As a corollary of the framework described in Algorithm 3 we obtain the following theorems:

Theorem 3.17 ([29]). *Let $H = (V, \mathcal{E})$ be a hypergraph with n vertices, and let $k, \ell \in \mathbb{N}$ be fixed integers, $k \geq 2$. If every induced sub-hypergraph $H' \subseteq H$ satisfies $c_{H'}(k) \leq \ell$, then $f_H(k-1) \leq \log_{1+\frac{1}{\ell-1}} n = O(l \log n)$.*

Theorem 3.18 ([29]). *Let $H = (V, \mathcal{E})$ be a hypergraph with n vertices, and let $k \geq 2$ be a fixed integer. let $0 < \alpha \leq 1$ be a fixed real. If every induced sub-hypergraph $H' \subseteq H$ satisfies $c_{H'}(k) = O(kn(H')^{\alpha})$, then $f_H(k-1) = O(kn^{\alpha})$.*

As a corollary of Theorem 3.17 and a result of Aloupis et al. [7] on k-colorful coloring of discs or points with respect to discs we obtain the following:

Theorem 3.19 ([29]). *If H is a hypergraph induced by n discs in the plane or a hypergraph induced by n points in the plane with respect to discs then $f_H(k) = O(k \log n)$.*

Proof. The proof follows by combining the fact that $c_H(k) = O(k)$ [7] with Theorem 3.17 □

Theorem 3.21 below provides an upper bound on the number of colors required by $kSCF$-coloring of geometrically induced hypergraphs as a function of the union-complexity of the regions that induce the hypergraphs.

Recall that, for a set $\mathcal{R}$ of n simple closed planar Jordan regions, $\mathcal{U}_{\mathcal{R}} : \mathbb{N} \to \mathbb{N}$ is the function defined in Theorem 2.4.

Theorem 3.20 ([29]). *Let $k \geq 2$, let $0 \leq \alpha \leq 1$, and let c be a fixed constant. Let $\mathcal{R}$ be a set of n simple closed Jordan regions such that $\mathcal{U}_{\mathcal{R}}(m) \leq cm^{1+\alpha}$, for $1 \leq m \leq n$, and let $H = H(\mathcal{R})$. Then $c_H(k) = O(kn^{\alpha})$.*

Combining Theorem 3.17 with Theorem 3.20 (for $\alpha = 0$) and Theorem 3.18 with Theorem 3.20 (for $0 < \alpha < 1$) yields the following result:

Theorem 3.21 ([29]). *Let $k \geq 2$, let $0 \leq \alpha \leq 1$, and let c be a constant. Let $\mathcal{R}$ be a set of n simple closed Jordan regions such that $\mathcal{U}_{\mathcal{R}}(m) = cm^{1+\alpha}$, for $1 \leq m \leq n$. Let $H = H(\mathcal{R})$. Then:*

$$f_H(k-1) = \begin{cases} O(k \log n), & \alpha = 0, \\ O(kn^{\alpha}), & 0 < \alpha \leq 1. \end{cases}$$

Axis-parallel rectangles: Consider $kSCF$-colorings of hypergraphs induced by axis-parallel rectangles in the plane. As mentioned before, axis-parallel rectangles might have quadratic union-complexity. For a hypergraph H induced by axis-parallel rectangles, Theorem 3.21 states that $f_H(k-1) = O(kn)$. This bound is meaningless, since the bound $f_H(k-1) \leq n$ is trivial. Nevertheless, the following theorem provides a better upper bound for this case:

Theorem 3.22 ([29]). *Let $k \geq 2$. Let $\mathcal{R}$ be a set of n axis-parallel rectangles, and let $H = H(\mathcal{R})$. Then $f_H(k-1) = O(k \log^2 n)$.*

In order to obtain Theorem 3.22 we need the following theorem:

Theorem 3.23 ([29]). *Let $H = H(\mathcal{R})$, be the hypergraph induced by a family $\mathcal{R}$ of n axis-parallel rectangles in the plane, and let $k \in \mathbb{N}$ be an integer, $k \geq 2$. For every induced sub-hypergraph $H' \subseteq H$ we have: $c_{H'}(k) \leq k \log n$.*

The proof of Theorem 3.22 is therefore an easy consequence of Theorem 3.23 combined with Theorem 3.17

Har-Peled and Smorodinsky [27] proved that any family $\mathcal{R}$ of n axis-parallel rectangles admit a CF-coloring with $O(\log^2 n)$ colors. Their proof uses the probabilistic method. They also provide a randomized algorithm for obtaining CF-coloring with at most $O(\log^2 n)$ colors. Later, Smorodinsky [47] provided a deterministic polynomial-time algorithm that produces a CF-coloring for n axis-parallel rectangles with $O(\log^2 n)$ colors. Theorem 3.22 thus generalizes the results of [27] and [47]. The upper bound provided in Theorem 3.21 for $\alpha = 0$ is optimal. Specifically, there exist matching lower bounds on the number of colors required by any $kSCF$-coloring of hypergraphs induced by (unit) discs in the plane.

Theorem 3.24 ([1]). *(i) There exist families $\mathcal{R}$ of n (unit) discs for which $f_{H(\mathcal{R})}(k) = \Omega(k \log n)$*
(ii) There exist families $\mathcal{R}$ of n axis-parallel squares for which $f_{H(\mathcal{R})}(k) = \Omega(k \log n)$.

Notice that for axis-parallel rectangles there is a logarithmic gap between the best known upper and lower bounds.

Theorems 3.17 and 3.18 asserts that in order to attain upper bounds on $f_H(k)$, for a hypergraph H, one may concentrate on attaining a bound on $c_H(k)$. Given a k-colorful coloring of H, Algorithm 3 obtains a strong-conflict-free coloring of H in a constructive manner. Here computational efficiency is not of main interest. However, it can be seen that for certain families of geometrically induced hypergraphs, Algorithm 3 is efficient. In particular, for hypergraphs induced by discs or axis-parallel rectangles, Algorithm 3 has a low degree polynomial running time. Colorful-colorings of such hypergraphs can be computed once the arrangement of the discs is computed together with the depth of every face.

3.3 List Colorings

In view of the motivation for CF-coloring in the context of wireless antennae, it is natural to assume that each antenna is restricted to use some subset of the spectrum of frequencies and that different antennae might have different such subsets associated with them (depending, for example, on the physical location of the antenna). Thus, it makes sense to study the following more restrictive notion of coloring:

Let $H = (V, \mathcal{E})$ with $V = \{v_1, \ldots, v_n\}$ be a hypergraph and let $\mathcal{L} = \{L_1, \ldots, L_n\}$ be a family of subsets of the integers. We say that H admits a proper coloring from $\mathcal{L}$ (respectively, a CF-coloring from $\mathcal{L}$, a UM-coloring from $\mathcal{L}$) if there exists a proper coloring (respectively a CF-coloring, a UM-coloring) $C\colon V \to \mathbb{N}$ such that $C(v_i) \in L_i$ for $i = 1, \ldots, n$.

Definition 3.25. *We say that a hypergraph $H = (V, \mathcal{E})$ is* k-choosable *(respectively,* k-CF-choosable, k-UM-choosable*) if for every family $\mathcal{L} = \{L_1, \ldots, L_n\}$ such that $|L_i| \geq k$ for $i = 1, \ldots, n$, H admits a proper-coloring (respectively a CF-coloring, a UM-coloring) from $\mathcal{L}$.*

We are interested in the minimum number k for which a given hypergraph is k-choosable (respectively, k-CF-choosable, k-UM-choosable). We refer to this number as the choice-number (respectively the *CF-choice-number, UM-choice-number*) of H and denote it by $ch(H)$ (respectively $ch_{cf}(H), ch_{\mathrm{um}}(H)$). Obviously, if the choice-number (respectively, the CF-choice-number, UM-choice-number) of H is k then it can be properly colored (respectively CF-colored, UM-colored) with at most k colors, as one can proper color (respectively, CF-color, UM-color) H from $\mathcal{L} = \{L_1, \ldots, L_n\}$ where for every i we have $L_i = \{1, \ldots, k\}$. Thus,

$$ch(H) \geq \chi(H)$$
$$ch_{cf}(H) \geq \chi_{\mathrm{cf}}(H).$$
$$ch_{\mathrm{um}}(H) \geq \chi_{\mathrm{um}}(H)$$

Hence, any lower bound on the number of colors required by a proper coloring of H (respectively, a CF-coloring, a UM-coloring of H) is also a lower bound on the choice number (respectively, the CF-choice-number, the UM-choice-number) of H.

The study of choice numbers in the special case of graphs was initiated by Vizing [50] and by Erdős Rubin and Taylor [23]. The study of the CF-choice number and the UM-choice number of hypergraphs was initiated very recently by Cheilaris, Smorodinsky and Sulovský [15].

Let us return to the discrete interval hypergraph H_n with n vertices, which was described in the introduction. As was shown already, we have $\chi_{\mathrm{cf}}(H_n) = \chi_{\mathrm{um}}(H_n) = \lfloor \log_2 n \rfloor + 1$. In particular we have the lower bound $ch_{cf}(H_n) \geq \lfloor \log_2 n \rfloor + 1$. Hence, the following upper-bound is tight:

Proposition 3.26. *For $n \geq 1$, $ch_{cf}(H_n) \leq \lfloor \log_2 n \rfloor + 1$.*

Proof. Assume, without loss of generality, that $n = 2^{k+1} - 1$. We will show that H_n is $k+1$ CF-choosable. The proof is by induction on k. Let $\mathcal{L} = \{L_i\}_{i \in [n]}$, such that $|L_i| = k+1$, for every i. Consider the median vertex $p = 2^k$. Choose a color $x \in L_p$ and assign it to p. Remove x from all other lists (for lists containing x), i.e., consider $\mathcal{L}' = \{L'_i\}_{i \in [n] \setminus p}$ where $L'_i = L_i \setminus \{x\}$. Note that all lists in $\mathcal{L}'$ have size at least k. The induction hypothesis is that we can CF-color any set of points of size $2^k - 1$ from lists of size k. Indeed, the number of vertices smaller (respectively, larger) than p is exactly $2^k - 1$. Thus, we CF-color vertices smaller than p and independently vertices larger than p, both using colors from the lists of $\mathcal{L}'$. Intervals that contain the median vertex p also have the conflict-free property, because color x is used only in p. This completes the induction step and hence the proof of the proposition. □

Note that, even in the discrete interval hypergraph, it is a more difficult problem to obtain any non-trivial upper bound on the UM-choice number. A divide and conquer approach, along the lines of the proof of Proposition 3.26 is doomed to fail. In such an approach, some vertex close to the median must be found, a color must be assigned to it from its list, and this color must be deleted from all other lists. However, vertices close to the median might have only "low" colors in their lists. Thus, while we are guaranteed that a vertex close to the median is uniquely colored for intervals containing it, such a unique color is not necessarily the maximal color for such intervals.

Instead, Cheilaris et al. used a different approach. This approach provides a general framework for UM-coloring hypergraphs from lists. Moreover, when applied to many geometric hypergraphs, it provides asymptotically tight bounds for the UM-choice number.

Below, we give an informal description of that approach, which is then summarized in Algorithm 4. It is similar in spirit to Algorithm 1.

Start by sorting the colors in the union of all lists in increasing order. Let c denote the minimum color. Let $V^c \subseteq V$ denote the subset of vertices containing c in their lists. Note that V^c might contain very few vertices, in fact, it might be that $|V^c| = 1$. We simultaneously color a suitable subset $U \subseteq V^c$ of vertices in V^c with c. We make sure that U is independent in the sub-hypergraph $H(V^c)$. The exact way in which we choose U is crucial to the performance of the algorithm and is discussed below. Next, for the uncolored vertices in $V^c \setminus U$, we remove the color c from their lists. This is repeated for every color in the union $\bigcup_{v \in V} L_v$ in increasing order of the colors. The algorithm stops when all vertices are colored. Notice that such an algorithm might run into a problem, when all colors in the list of some vertex are removed before this vertex is colored. Later, we show that if we choose the subset $U \subseteq V^c$ in a clever way and the lists are sufficiently large, then we avoid such a problem.

As mentioned, Algorithm 4 might cause some lists to run out of colors before coloring all vertices. However, if this does not happen, it is proved that the algorithm produces a UM-coloring.

Lemma 3.27. *[15] Provided that the lists associated with the vertices do not run out of colors during the execution of Algorithm 4, then the algorithm produces a UM-coloring from $\mathcal{L}$.*

Proof. The proof is similar to the validity proof of Algorithm 1 and we omit the details. □

The key ingredient, which will determine the necessary size of the lists of $\mathcal{L}$, is the particular choice of the independent set in the above algorithm. We assume that the hypergraph $H = (V, \mathcal{E})$ is hereditarily k-colorable for some fixed positive integer k. Recall that, as shown before, this is the case in many geometric

Algorithm 4 UMColorGeneric(H, $\mathcal{L}$): Unique-maximum color hypergraph $H = (V, \mathcal{E})$ from lists of family $\mathcal{L}$

```
while V ≠ ∅ do
    c ← min ⋃_{v∈V} L_v {c is the minimum color in the union of the lists}
    V^c ← {v ∈ V | c ∈ L_v} {V^c is the subset of remaining vertices containing c in their lists}
    U ← a "good" independent subset of the induced sub-hypergraph H(V^c)
    for x ∈ U do {for every vertex in the independent set,}
        f(x) ← c {color it with color c}
    end for
    for v ∈ V^c \ U do {for every uncolored vertex in V^c,}
        L_v ← L_v \ {c} {remove c from its list}
    end for
    V ← V \ U {remove the colored vertices}
end while
Return f.
```

hypergraphs. We must also put some condition on the size of the lists in the family $\mathcal{L} = \{L_v\}_{v \in V}$. With some hindsight, we require

$$\sum_{v \in V} \lambda^{-|L_v|} < 1,$$

where $\lambda := \frac{k}{k-1}$.

Theorem 3.28. *[15] Let $H = (V, \mathcal{E})$ be a hypergraph which is hereditarily k-colorable and set $\lambda := \frac{k}{k-1}$. Let $\mathcal{L} = \{L_v\}_{v \in V}$, such that $\sum_{v \in V} \lambda^{-|L_v|} < 1$. Then, H admits a UM-coloring from $\mathcal{L}$.*

Notice, that in particular for a hypergraph H which is hereditarily k-colorable we have:

$$ch_{\text{um}}(H) \leq \log_\lambda n + 1 = O(k \log n)$$

Thus, Theorem 3.28 subsumes all the theorems (derived from Algorithm 1) that are mentioned in Section 2.

Proof. The proof of Theorem 3.28 is constructive and uses a potential method: This method gives priority to coloring vertices that have fewer remaining colors in their lists, when choosing the independent sets. Towards that goal, we define a potential function on subsets of uncolored vertices and we choose the independent set with the highest potential (the potential quantifies how dangerous it is that some vertex in the set will run out of colors in its list).

For an uncolored vertex $v \in V$, let $r_t(v)$ denote the number of colors remaining in the list of v in the beginning of iteration t of the algorithm. Obviously, the value of $r_t(v)$ depends on the particular run of the algorithm. For a subset of uncolored vertices $X \subseteq V$ in the beginning of iteration t, let $P_t(X) := \sum_{v \in X} \lambda^{-r_t(v)}$. We define the potential in the beginning of iteration t to be $P_t := P_t(V_t)$, where V_t denotes the subset of all uncolored vertices in the beginning of iteration t. Notice that the value of the potential in the beginning of the algorithm (i.e., in the first iteration) is $P_1 = \sum_{v \in V} \lambda^{-|L_v|} < 1$.

Our goal is to show that, with the right choice of the independent set in each iteration, we can make sure that for any iteration t and every vertex $v \in V_t$ the inequality $r_t(v) > 0$ holds. In order to achieve this, we will show that, with the right choice of the subset of vertices colored in each iteration, the potential function P_t is non-increasing in t. This will imply that for any iteration t and every uncolored vertex $v \in V_t$ we have:

$$\lambda^{-r_t(v)} \leq P_t \leq P_1 < 1$$

and hence $r_t(v) > 0$, as required.

Assume that the potential function is non-increasing up to iteration t. Let P_t be the value of the potential function in the beginning of iteration t and let c be the color associated with iteration t. Recall that V_t denotes the set of uncolored vertices that are considered in iteration t, and $V^c \subseteq V_t$ denotes the subset of uncolored vertices that contain the color c in their lists. Put $P' = P_t(V_t \setminus V^c)$ and $P'' = P_t(V^c)$. Note that $P_t = P' + P''$. Let us describe how we find the independent set of vertices to be colored at iteration t. First, we find an auxiliary proper coloring of the hypergraph $H[V^c]$ with k colors (here we use the hereditary k-colorability property of the hypergraph). Consider the color class U which has the largest potential $P_t(U)$. Since the vertices in V^c are partitioned into at most k independent subsets $U_1, \ldots, U_k$ and $P'' = \sum_{i=1}^{k} P_t(U_i)$, then by the pigeon-hole principle there is an index j for which $P_t(U_j) \geq P''/k$. We choose $U = U_j$ as the independent set to be colored at iteration t. Notice that, in this case, the value $r_{t+1}(v) = r_t(v) - 1$ for every vertex $v \in V^c \setminus U$, and all vertices in U are colored. For vertices in $V_t \setminus V^c$, there is no change in the size of their lists. Thus, the value P_{t+1} of the potential function at the end of iteration t (and in the beginning of iteration $t+1$) is $P_{t+1} \leq P' + \lambda(1 - \frac{1}{k})P''$. Since $\lambda = \frac{k}{k-1}$, we have that $P_{t+1} \leq P' + P'' = P_t$, as required. □

3.3.1 A relation between chromatic and choice number in general hypergraphs

Using a probabilistic argument, Cheilaris et al. [15] proved the following general theorem for arbitrary hypergraphs and arbitrary colorings with the so-called refinement property:

Definition 3.29. *We call C' a* refinement *of a coloring C if $C(x) \neq C(y)$ implies $C'(x) \neq C'(y)$. A class $\mathcal{C}$ of colorings is said to have* the refinement property *if every refinement of a coloring in the class is also in the class.*

The class of conflict-free colorings and the class of proper colorings are examples of classes which have the refinement property. On the other hand, the class of unique-maximum colorings does not have this property.

For a class $\mathcal{C}$ of colorings, one can naturally extend the notions of chromatic number $\chi_{\mathcal{C}}$ and choice number $ch_{\mathcal{C}}$ to $\mathcal{C}$.

Theorem 3.30 ([15]). *For every class of colorings $\mathcal{C}$ that has the refinement property and every hypergraph H with n vertices, $ch_{\mathcal{C}}(H) \leq \chi_{\mathcal{C}}(H) \cdot \ln n + 1$.*

Proof. If $k = \chi_{\mathcal{C}}(H)$, then there exists a $\mathcal{C}$-coloring C of H with colors $\{1, \ldots, k\}$, which induces a partition of V into k classes: $V_1 \cup V_2 \cup \cdots \cup V_k$. Consider a family $\mathcal{L} = \{L_v\}_{v \in V}$, such that for every v, $|L_v| = k^* > k \cdot \ln n$. We wish to find a family $\mathcal{L}' = \{L'_v\}_{v \in V}$ with the following properties:

1. For every $v \in V$, $L'_v \subseteq L_v$.
2. For every $v \in V$, $L'_v \neq \emptyset$.
3. For every $i \neq j$, if $v \in V_i$ and $u \in V_j$, then $L'_v \cap L'_u = \emptyset$.

Obviously, if such a family $\mathcal{L}'$ exists, then there exists a $\mathcal{C}$-coloring from $\mathcal{L}'$: For each $v \in V$, pick a color $x \in L'_v$ and assign it to v.

We create the family $\mathcal{L}'$ randomly as follows: For each element in $\cup\mathcal{L}$, assign it uniformly at random to one of the k classes of the partition $V_1 \cup \cdots \cup V_k$. For every vertex $v \in V$, say with $v \in V_i$, we create L'_v, by keeping only elements of L_v that were assigned through the above random process to v's class, V_i.

The family $\mathcal{L}'$ obviously has properties 1 and 3. We will prove that with positive probability it also has property 2.

For a fixed v, the probability that $L'_v = \emptyset$ is at most

$$\left(1 - \frac{1}{k}\right)^{k^*} \leq e^{-k^*/k} < e^{-\ln n} = \frac{1}{n}$$

and therefore, using the union bound, the probability that for at least one vertex v, $L'_v = \emptyset$, is at most

$$n\left(1-\frac{1}{k}\right)^{k^*} < 1.$$

Thus, there is at least one family $\mathcal{L}'$ where property 2 also holds, as claimed. □

Corollary 3.31. *For every hypergraph H,*

$$ch_{\mathrm{cf}}(H) \leq \chi_{\mathrm{cf}}(H) \cdot \ln n + 1.$$

Corollary 3.32. *For every hypergraph H,*

$$ch(H) \leq \chi(H) \cdot \ln n + 1.$$

The argument in the proof of Theorem 3.30 is a generalization of an argument first given in [23], proving that any bipartite graph with n vertices is $O(\log n)$-choosable (see also [4]).

4 Non-Geometric Hypergraphs

Pach and Tardos [39] investigated the CF-chromatic number of arbitrary hypergraphs and proved that the inequality:

$$\chi_{\mathrm{cf}}(H) \leq 1/2 + \sqrt{2m + 1/4}$$

holds for every hypergraph H with m edges, and that this bound is tight. Cheilaris et al. [15] strengthened this bound in two ways by proving that:

$$ch_{\mathrm{um}}(H) \leq 1/2 + \sqrt{2m + 1/4}$$

If, in addition, every hyperedge contains at least $2t-1$ vertices (for $t \geq 3$) then Pach and Tardos showed that:

$$\chi_{\mathrm{cf}}(H) = O(m^{\frac{1}{t}} \log m)$$

Using the Lovász Local Lemma, they show that the same result holds for hypergraphs, in which the size of every edge is at least $2t-1$ and every edge intersects at most m other edges.

Hypergraphs induced by neighborhoods in graphs A particular interest arises when dealing with hypergraphs induced by neighborhoods of vertices of a given graph. Given a graph $G = (V, E)$ and a vertex $v \in V$, denote by $N_G(v) = N(v)$ the set of all neighbors of v in G together with v and refer to it as the *neighborhood of* v. Call the set $\dot{N}(G) = N_G(v) \setminus \{v\}$ the *pointed neighborhood of* v. The hypergraph H associated with the neighborhoods of G has its vertex set $V(H) = V$ and its edge set $E(H) = \{N_G(v) | v \in V\}$ and the hypergraph $\dot{H}$ associated with the pointed neighborhoods of G has $V(\dot{H}) = V$ and $E(\dot{H}) = \{\dot{N}_G(v) | v \in V\}$. The *conflict-free chromatic parameter* $\kappa_{CF}(G)$ is defined simply as $\chi_{\mathrm{cf}}(H)$ and the *pointed* version of this parameter $\dot{\kappa}_{CF}(G)$ is defined analogously as $\chi_{\mathrm{cf}}(\dot{H})$.

We start with an example taken from [39] in order to provide some basic insights into the relation between these two parameters. Let K'_s be the graph obtained from the complete graph K_s on s vertices by subdividing each edge with a new vertex. Each pair of the s original vertices form the pointed neighborhood of one of the new vertices, so all original vertices must receive different colors in any conflict-free coloring of the corresponding hypergraph $\dot{H}$. Thus, we have $\dot{\kappa}_{CF}(K'_s) \geq s$ and it is easy to see that equality holds here. On the other hand, K'_s is bipartite and any proper coloring of a graph is also a conflict-free coloring of the hypergraph formed by the neighborhoods of its vertices. This shows that $\kappa_{CF}(K'_s) = 2$, for any $s \geq 2$. The example illustrates that the pointed conflict-free chromatic parameter of a graph cannot be bounded from above by any function of its non-pointed variant. For many other graphs, the non-pointed parameter can be larger than the pointed parameter. For instance, let G denote

the graph obtained from the complete graph K_4 by subdividing a single edge with a vertex. It is easy to check that $\kappa_{CF}(G) = 3$, while $\dot{\kappa}_{CF}(G) = 2$. However, it is not difficult to verify that

$$\kappa_{CF}(G) \leq 2\dot{\kappa}_{CF}(G)$$

for any graph G. This inequality holds, because in a conflict-free coloring of the pointed neighborhoods, each neighborhood $N(x)$ also has a vertex whose color is not repeated in $N(x)$, unless x has degree one in the subgraph spanned by one of the color classes. One can fix this by carefully splitting each color class into two. The following theorems were proved in [39]:

Theorem 4.1 ([39]). *The conflict-free chromatic parameter of any graph G with n vertices satisfies $\kappa_{CF}(G) = O(\log^2 n)$. The corresponding coloring can be found by a deterministic polynomial time algorithm.*

Theorem 4.2 ([39]). *There exist graphs of n vertices with conflict-free chromatic parameter $\Omega(\log n)$.*

Problem 6. *Close the gap between the last two bounds.*

For graphs with maximum degree Δ, a slightly better upper-bound is known:

Theorem 4.3 ([39]). *The conflict-free chromatic parameter of any graph G with maximum degree Δ satisfies $\kappa_{CF}(G) = O(\log^{2+\epsilon} \Delta)$ for any $\epsilon > 0$. The corresponding coloring can be found by a deterministic polynomial time algorithm.*

Hypergraphs induced by simple paths in graphs As mentioned in the introduction, a particular interest is in hypergraphs induced by simple paths in a given graph: Recall the that given a graph G, we consider the hypergraph $H = (V, E')$ where a subset $V' \subset V$ is a hyperedge in E' if and only if V' is the set of vertices in some simple path of G. As mentioned before, the parameter $\chi_{\mathrm{um}}(H)$ is known as the vertex ranking number of G and was studied in other context in the literature (see, e.g., [31, 44]). An interesting question arises when trying to understand the relation between the two parameters $\chi_{\mathrm{cf}}(H)$ and $\chi_{\mathrm{um}}(H)$. This line of research was pursued in [14] and [16]. Cheilaris and Tóth proved the following:

Theorem 4.4 ([16]). *(i) Let G be a simple graph and let H be the hypergraph induced by paths in G as above: Then $\chi_{\mathrm{um}}(H) \leq 2^{\chi_{\mathrm{cf}}(H)} - 1$.*

(ii) There is is a sequence of such hypergraphs $\{H_i\}_{i=1}^{\infty}$ induced by paths such that

$$\lim_{n\to\infty} \frac{\chi_{\mathrm{um}}(H_n)}{\chi_{\mathrm{cf}}(H_n)} = 2.$$

Narrowing the gaps between the two parameters for such hypergraphs is an interesting open problem:

Problem 7. *Let $f(k)$ denote the function of the least integer such that for every hypergraph H induced by path in a graph G we have that $\chi_{\mathrm{um}}(H) \leq f(\chi_{\mathrm{cf}}(H))$. Find the asymptotic behavior of f.*

5 Algorithms

Until now we were mainly concerned with the combinatorial problem of obtaining bounds on the CF-chromatic number of various hypergraphs. We now turn our attention to the computational aspect of the corresponding optimization problem. Even et al. [24] proved that given a finite set D of discs in the plane, it is NP-hard to compute an optimal CF-coloring for $H(D)$; namely, a CF-coloring of $H(D)$ using a minimum number of colors. This hardness result holds even if all discs have the same radius. However, as mentioned in the introduction, any set D of n discs admits a CF-coloring that uses $O(\log n)$ colors and such a coloring can be found in deterministic polynomial time (in fact in $O(n \log n)$ time). This trivially implies that such an algorithm serves as an $O(\log n)$ approximation algorithm for the corresponding optimization problem.

5.1 Approximation Algorithms

Given a finite set D of discs in the plane, the *size ratio* of D denoted by $\rho = \rho(D)$ is the ratio between the maximum and minimum radii of discs in D. For simplicity, we may assume that the smallest radius is 1. For each $i \geq 1$, let D^i denote the subset of discs in D whose radius is in the range $[2^{i-1}, 2^i)$. Let $\phi_{2^i}(D^i)$ denote the maximum number of centers of discs in D^i that are contained in a $2^i \times 2^i$ square. Refer to $\phi_{2^i}(D^i)$ as the *local density* of D^i (with respect to $2^i \times 2^i$ square). For a set of points X in $\mathbb{R}^2$ let $D_r(X)$ denote the set of $|X|$ discs with radius r centered at the points of X. The following algorithmic results were provided in [24].

Theorem 5.1 ([24]). *1. Given a finite set D of discs with size-ratio ρ, there exists a polynomial-time algorithm that compute a CF-coloring of D using $O\left(\min\{(\log \rho) \cdot \max_i\{\log \phi_{2^i}(D^i)\}, \log |D|\}\right)$ colors.*

2. Given a finite set of centers $X \subset \mathbb{R}^2$, there exists a polynomial-time algorithm that computes a UM-coloring χ of the hypergraph induced X with respect to all discs using $O(\log |X|)$ colors. This is equivalent to the following: If we color $D_r(X)$ by assigning each disc $d \in D_r(X)$ the color of its center then this is a valid UM-coloring of the hypergraph $H(D_r(X))$ for every radius r.

The tightness of Theorem 5.1 follows from the fact that for any integer n, there exists a set D of n unit discs with $\phi_1(D) = n$ for which $\Omega(\log n)$ colors are necessary in every CF-coloring of D.

In the first part of Theorem 5.1 the discs are not necessarily congruent. That is, the size-ratio ρ may be bigger than 1. In the second part of Theorem 5.1, the discs are congruent (i.e., the size-ratio equals 1). However, the common radius is not determined in advance. Namely, the order of quantifiers in the second part of the theorem is as follows: Given the locations of the disk centers, the algorithm computes a coloring of the centers (of the discs) such that this coloring is conflict-free *for every* radius r.

Building on Theorem 5.1, Even et al. [24] also obtain two bi-criteria CF-coloring algorithms for discs having the same (unit) radius. In both cases the algorithm uses only few colors. In the first case this comes at a cost of not serving a small area that is covered by the discs (i.e., an area close to the boundary of the union of the discs). In the second case, all the area covered by the discs is served, but the discs are assumed to have a slightly larger radius. A formal statement of these bi-criteria results is as follows:

Theorem 5.2 ([24]). *For every $0 < \varepsilon < 1$ and every finite set of centers $X \subset \mathbb{R}^2$, there exist polynomial-time algorithms that compute colorings as follows:*

1. *A coloring χ of $D_1(X)$ using $O\left(\log \frac{1}{\varepsilon}\right)$ colors for which the following holds: The area of the set of points in $\bigcup D_1(X)$ that are not served with respect to χ is at most an ε-fraction of the total area of $D_1(X)$.*

2. *A coloring of $D_{1+\varepsilon}(X)$ that uses $O\left(\log \frac{1}{\varepsilon}\right)$ colors such that every point in $\bigcup \mathcal{D}_1(X)$ is served.*

In other words, in the first case, the portion of the total area that is not served is an exponentially small fraction as a function of the number of colors. In the second case, the increase in the radius of the discs is exponentially small as a function of the number of colors.

The following problem seems like a non-trivial challenge.

Problem 8. *Is there a constant factor approximation algorithm for finding an optimal CF-coloring for a finite set of discs in the plane?*

Remark: In the special case that all discs are congruent (i.e., have the same radius) Lev-Tov and Peleg [35] have recently provided a constant-factor approximation algorithm.

5.1.1 An $O(1)$-Approximation for CF-Coloring of Rectangles and Regular Hexagons

Recall that Theorem 2.7 states that every set of n axis-parallel rectangles can be CF-colored with $O(\log^2 n)$ colors and such a coloring can be found in polynomial time.

Let $\mathcal{R}$ denote a set of axis-parallel rectangles. Given a rectangle $R \in \mathcal{R}$, let $w(R)$ ($h(R)$, respectively) denote the width (height, respectively) of R. The *size-ratio* of $\mathcal{R}$ is defined by $\max\left\{\frac{w(R_1)}{w(R_2)}, \frac{h(R_1)}{h(R_2)}\right\}_{R_1,R_2\in\mathcal{R}}$.

The size ratio of a collection of regular hexagons is simply the ratio of the longest side length and the shortest side length.

Theorem 5.3 ([24]). *Let $\mathcal{R}$ denote either a set of axis-parallel rectangles or a set of homothets of a regular hexagons. Let ρ denote the size-ratio of $\mathcal{R}$ and let $\chi_{\text{opt}}(\mathcal{R})$ denote an optimal CF-coloring of $\mathcal{R}$.*

1. *If $\mathcal{R}$ is a set of rectangles, then there exists a polynomial-time algorithm that computes a CF-coloring χ of $\mathcal{R}$ such that $|\chi(\mathcal{R})| = O((\log\rho + 1)^2 \cdot |\chi_{\text{opt}}(\mathcal{R})|)$.*
2. *If $\mathcal{R}$ is a set of hexagons, then there exists a polynomial-time algorithm that computes a CF-coloring χ of $\mathcal{R}$ such that $|\chi(\mathcal{R})| = O((\log\rho + 1) \cdot |\chi_{\text{opt}}(\mathcal{R})|)$.*

For a constant size-ratio ρ, Theorem 5.3 implies a constant approximation algorithm.

5.2 Online CF-Coloring

Recall the motivation to study CF-coloring in the context of cellular antanae. To capture a dynamic scenario where antennae can be added to the network, Chen et al. [17] introduced an online version of the CF coloring problem. As we shall soon see, the online version of the problem is considerably harder, even in the one-dimensional case, where the static version (i.e., CF-coloring the discrete intervals hypergraph) is trivial and fully understood.

5.2.1 Points with respect to intervals

Let us start with the simplest possible example where things become highly non-trivial in an online setting. We start with the dynamic extension of the discrete interval hypergraph case. Thats is, we deal with coloring of points on the line, with respect to interval ranges. We maintain a finite set $P \subset \mathbb{R}$. Initially, P is empty, and an adversary repeatedly insert points into P, one point at a time. We denote by $P(t)$ the set P after the tth point has been inserted. Each time a new point p is inserted, we need to assign a color $c(p)$ to it, which is a positive integer. Once the color has been assigned to p, it cannot be changed in the future. The coloring should remain a valid CF-coloring at all times. That is, as in the static case, for any interval I that contains points of $P(t)$, there is a color that appears exactly once in I.

We begin by examining a natural, simple, and obvious coloring algorithm (referred to as the UniMax greedy algorithm) which might be inefficient in the worst case. Chen et al. [17] presented an efficient 2-stage variant of the UniMax greedy algorithm and showed that the maximum number of colors that it uses is $\Theta(\log^2 n)$.

As in the case in most CF-coloring of hypergraphs that were tackled so far, we wish to maintain the unique maximum invariant. At any given step t the coloring of $P(t)$ is a UM-coloring.

The following simple-minded algorithm for coloring an inserted point p into the current set $P(t)$ is used. We say that the newly inserted point p *sees* a point x if all the colors of the points between p and x (exclusive) are smaller than $c(x)$. In this case we also say that p sees the color $c(x)$. Then p gets the smallest color that it does not see. (Note that a color can be seen from p either to the left or to the right, but not in both directions; see below.) Refer to this algorithm as the *Unique Maximum Greedy* algorithm, or the UniMax greedy algorithm, for short.

Below is an illustration of the coloring rule of the UniMax greedy algorithm. The left column gives the colors (integers in the range $1, 2, \ldots, 6$) assigned to the points in the current set P and the location

of the next point to be inserted (indicated by a period). The right column gives the colors "seen" by the new point. The colors seen to the left precede the $\cdot$, and those seen to the right succeed the $\cdot$.

$1\cdot$	$[1\cdot]$
$1 \cdot 2$	$[1 \cdot 2]$
$1 \cdot 32$	$[1 \cdot 3]$
$12 \cdot 32$	$[2 \cdot 3]$
$121 \cdot 32$	$[21 \cdot 3]$
$121 \cdot 432$	$[21 \cdot 4]$
$121 \cdot 3432$	$[21 \cdot 34]$
$1215 \cdot 3432$	$[5 \cdot 34]$
$1215 \cdot 13432$	$[5 \cdot 134]$
$12152 \cdot 13432$	$[52 \cdot 134]$
$121526 \cdot 13432$	$[6 \cdot 134]$

Correctness. The correctness of the algorithm is established by induction on the insertion order. First, note that no color can be seen twice from p: This is obvious for two points that lie both to the left or both to the right of p. If p sees the same color at a point u to its left and at a point v to its right, then the interval $[u, v]$, before p is inserted, does not have a unique maximum color; thus this case is impossible, too. Next, if p is assigned color c, any interval that contains p still has a unique maximum color: This follows by induction when the maximum color is greater than c. If the maximum color is c, then it cannot be shared by another point u in the interval, because then p would have seen the nearest such point and thus would not be assigned color c. It is also easy to see that the algorithm assigns to each newly inserted point the smallest possible color that maintains the invariant of a unique maximum color in each interval. This makes the algorithm *greedy* with respect to the unique maximum condition.

Special insertion orders. Denote by $C(P(t))$ the sequence of colors assigned to the points of $P(t)$, in left-to-right order along the line.

The *complete binary tree sequence* S_k of order k is defined recursively as $S_1 = (1)$ and $S_k = S_{k-1}\|(k)\|S_{k-1}$, for $k > 1$, where $\|$ denotes concatenation. Clearly, $|S_k| = 2^k - 1$.

For each pair of integers $a < b$, denote by $C_0(a, b)$ the following special sequence. Let k be the integer satisfying $2^{k-1} \leq b < 2^k$. Then $C_0(a, b)$ is the subsequence of S_k from the ath place to the bth place (inclusive). For example, $C_0(5, 12)$ is the subsequence $(1, 2, 1, 4, 1, 2, 1, 3)$ of $(1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1)$.

Lemma 5.4. (a) *If each point is inserted into P to the right of all preceding points, then $C(P(t)) = C_0(1, t)$.*

(b) *If each point is inserted into P to the left of all preceding points, then $C(P(t)) = C_0(2^k - t,\, 2^k - 1)$, where k satisfies $2^{k-1} \leq t < 2^k$.*

Proof. The proof is easy and is left as an exercise to the reader. □

Unfortunately, the UniMax greedy algorithm might be very inefficient as was shown in [17]:

Theorem 5.5 ([17]). *The UniMax greedy algorithm may require $\Omega(\sqrt{n})$ colors in the worst case for a set of n points.*

Problem 9. *Obtain an upper bound for the maximum number of colors that the algorithm uses for n inserted points. It is conjectured that the bound is close to the $\Omega(\sqrt{n})$ lower bound. At the moment, there is no known sub-linear upper bound.*

Related algorithms

The First-Fit algorithm—another greedy strategy. The UniMax greedy algorithm is greedy for maintaining the unique maximum invariant. Perhaps it is more natural to consider a greedy approach in which we want only to enforce the standard CF property. That is, we want to assign to each newly inserted

point the *smallest* color for which the CF property continues to hold. There are cases where this *First-Fit* greedy algorithm uses fewer colors than the UniMax greedy algorithm: Consider an insertion of five points in the order (1 3 2 4 5). The UniMax greedy algorithm produces the color sequence (1 3 2 1 4), whereas the First-Fit algorithm produces the coloring (1 3 2 1 2). Unfortunately, Bar-Noy et al. [11] have shown that there are sequences with $2i+3$ elements that force the algorithm to use $i+3$ colors, and this bound is tight.

CF coloring for unit intervals. Consider the special case where we want the CF property to hold only for *unit intervals*. In this case, $O(\log n)$ colors suffice: Partition the line into the unit intervals $J_i = [i, i+1)$ for $i \in \mathbb{Z}$. Color the intervals J_i with even i as white, and those with odd i as black. Note that any unit interval meets only one white and one black interval. We color the points in each J_i independently, using the same set of "light colors" for each white interval and the same set of "dark colors" for each black interval. For each J_i, we color the points that it contains using the UniMax greedy algorithm, except that new points inserted into J_i between two previously inserted points get a special color, color 0. It is easily checked that the resulting coloring is CF with respect to unit intervals. Since we effectively insert points into any J_i only to the left or to the right of the previously inserted points, Lemma 5.4(c) implies that the algorithm uses only $O(\log n)$ (light and dark) colors. We remark that this algorithm satisfies the unique maximum color property for unit-length intervals.

We note that, in contrast to the static case (which can always be solved with $O(1)$ colors), $\Omega(\log n)$ colors may be needed in the worst case. Indeed, consider a left-to-right insertion of n points into a sufficiently small interval. Each contiguous subsequence σ of the points will be a suffix of the whole sequence at the time the rightmost element of σ is inserted. Since such a suffix can be cut off the current set by a unit interval, it must have a unique color. Hence, at the end of insertion, *every* subsequence must have a unique color, which implies (see [24, 46]) that $\Omega(\log n)$ colors are needed.

An efficient online deterministic algorithm for points with respect to intervals We describe an efficient online algorithm for coloring points with respect to intervals that was obtained in [17]. This is done by modifying the UniMax greedy algorithm into a deterministic 2-stage coloring scheme. It is then shown that it uses only $O(\log^2 n)$ colors. The algorithm is referred to as the *leveled UniMax greedy algorithm.*

Let x be the point which we currently insert. We assign a color to x in two steps. First we assign x to a *level*, denoted by $\ell(x)$. Once x is assigned to level $\ell(x)$ we give it an actual color among the set of colors dedicated to $\ell(x)$. We maintain the invariant that each color is used by at most one level. Formally, the colors that we use are pairs $(\ell(x), c(x)) \in \mathbb{Z}^2$, where $\ell(x)$ is the level of x and $c(x)$ is its integer color within that level.

Modifying the definition from the UniMax greedy algorithm, we say that point x *sees* point y (or that point y is *visible* to x) if and only if for every point z between x and y, $\ell(z) < \ell(y)$. When x is inserted, we set $\ell(x)$ to be the smallest level ℓ such that either to the left of x or to the right of x (or in both directions) there is no point y visible to x at level ℓ.

To give x a color, we now consider only the points of level $\ell(x)$ that x can see. That is, we discard every point y such that $\ell(y) \neq \ell(x)$, and every point y such that $\ell(y) = \ell(x)$ and there is a point z between x and y such that $\ell(z) > \ell(y)$. We apply the UniMax greedy algorithm so as to color x with respect to the sequence P_x of the remaining points, using the colors of level $\ell(x)$ only. That is, we give x the color $(\ell(x), c(x))$, where $c(x)$ is the smallest color that ensures that the coloring of P_x maintains the unique maximum color condition. This completes the description of the algorithm. See Figure 5 for an illustration.

We begin the analysis of the algorithm by making a few observations on its performance.

(a) Suppose that a point x is inserted and is assigned to level $i > 1$. Since x was not assigned to any level $j < i$, it must see a point ℓ_j at level j that lies to its left, and another such point r_j that lies to its right. Let $E_j(x)$ denote the interval $[\ell_j, r_j]$. Note that, by definition, these intervals are *nested*, that is, $E_j(x) \subset E_k(x)$ for $j < k < i$. See Figure 5.

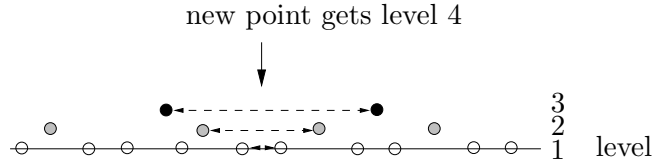

Figure 5: Illustrating the 2-stage deterministic algorithm. An insertion order that realizes the depicted assignment of levels to points is to first insert all level-1 points from left to right, then insert the level-2 points from left to right, and then the level-3 points.

(b) We define a *run* at level i to be a maximal sequence of points $x_1 < x_2 < \cdots < x_k$ at level i, such that all points between x_1 and x_k that are distinct from $x_2, x_3, \ldots, x_{k-1}$ are assigned to levels smaller than i. Whenever a new point x is assigned to level i and is inserted into a run of that level, it is always inserted either to the left or to the right of all points in the run. Moreover, the actual color that x gets is determined solely from the colors of the points already in the run. See Figure 5.

(c) The runs keep evolving as new points are inserted. A run may either grow when a new point of the same level is inserted at its left or right end (note that other points at smaller levels may separate the new point from the former end of the run) or split into two runs when a point of a higher level is inserted somewhere between its ends.

(d) As in observation (a), the points at level i define *intervals*, called *i-intervals*. Any such interval E is a contiguous subsequence $[x, y]$ of P, so that x and y are both at level i and all the points between x and y have smaller levels. E is formed when the second of its endpoints, say x, is inserted. We say that x *closes* the interval E and refer to it as a *closing point*. Note that, by construction, x cannot close another interval.

(e) Continuing observation (a), when x is inserted, it *destroys* the intervals $E_j(x)$, for $j < i$, into which it is inserted, and only these intervals. That is, each of these intervals now contains a point with a level greater than that of its endpoints, so it is no longer a valid interval. We charge x to the set of the closing endpoints of all these intervals. Clearly, none of these points will ever be charged again by another insertion (since it is the closing endpoint of only one interval, which is now destroyed). We maintain a forest F, whose nodes are all the points of P. The leaves of F are all the points at level 1. When a new point x is inserted, we make it a new root of F, and the parent of all the closing points that it charges. Since these points have smaller levels than x, and since none of these points becomes a child of another parent, it follows that F is indeed a forest.

Note that the nonclosing points can only be roots of trees of F. Note also that a node at level i has exactly $i - 1$ children, exactly one at each level $j < i$. Hence, each tree of F is a *binomial tree* (see [21]); if its root has level i, then it has 2^i nodes.

This implies that if m is the maximal level assigned after n points have been inserted, then we must have $2^m \leq n$, or $m \leq \log n$. That is, the algorithm uses at most $\log n$ levels.

We next prove that the algorithm uses only $O(\log n)$ colors at each level. We recall the way runs evolve: They grow by adding points at their right or left ends, and split into prefix and suffix subruns, when a point with a larger level is inserted in their middle.

Lemma 5.6. *At any time during the insertion process, the colors assigned to the points in a run form a sequence of the form $C_0(a, b)$. Moreover, when the jth smallest color of level i is given to a point x, the run to which x is appended has at least $2^{j-2} + 1$ elements (including x).*

Proof. The proof proceeds by induction through the sequence of insertion steps and is based on the following observation. Let σ be a contiguous subsequence of the complete binary tree sequence S_{k-1}, and let x be a point added, say, to the left of σ. If we assign to x color $c(x)$, using the UniMax greedy algorithm, then $(c(x))\|\sigma$ is a contiguous subsequence of either S_{k-1} or S_k. The latter happens only if σ contains $S_{k-2}\|(k-1)$ as a prefix. Symmetric properties hold when x is inserted to the right of σ. We omit the straightforward proof of this observation. □

As a consequence we have.

Theorem 5.7 ([17]). (a) *The algorithm uses at most* $(2 + \log n) \log n$ *colors.*

(b) *At any time, the coloring is a valid CF-coloring.*

(c) *In the worst case the algorithm may be forced to use* $\Omega(\log^2 n)$ *colors after* n *points are inserted.*

Proof. (a) We have already argued that the number of levels is at most $\log n$. Within a level i, the kth smallest color is assigned when a run contains at least 2^{k-2} points. Hence $2^{k-2} \leq n$, or $k \leq 2 + \log n$, and (a) follows.

To show (b), consider an arbitrary interval I. Let ℓ be the highest level of a point in I. Let $\sigma = (y_1, y_2, \ldots, y_j)$ be the sequence of the points in I of level ℓ. Since ℓ is the highest level in I, σ is a contiguous subsequence of some run, and, by Lemma 5.6, the sequence of the colors of its points is also of the form $C_0(a', b')$. Hence, there is a point $y_i \in \sigma$ which is uniquely colored among $y_1, y_2, \ldots, y_j$ by a color of level ℓ.

To show (c), we construct a sequence P so as to force its coloring to proceed level by level. We first insert 2^{k-1} points from left to right, thereby making them all be assigned to level 1 and colored with k different colors of that level. Let P_1 denote the set of these points. We next insert a second batch of 2^{k-2} points from left to right. The first point is inserted between the first and second points of P_1, the second point between the third and fourth points of P_1, and so on, where the jth new point is inserted between the $(2j-1)$th and $(2j)$th points of P_1. By construction, all points in the second batch are assigned to level 2, and they are colored with $k-1$ different colors of that level. Let P_2 denote the set of all points inserted so far. P_2 is the concatenation of 2^{k-2} triples, where the levels in each triple are $(1, 2, 1)$. We now insert a third batch of 2^{k-3} points from left to right. The first point is inserted between the first and second triples of P_2, the second point between the third and fourth triples of P_2, and so on, where the jth new point is inserted between the $(2j-1)$th and $(2j)$th triples of P_2. By construction, all points in the third batch are assigned to level 3, and they are colored with $k-2$ different colors of that level.

The construction is continued in this manner. Just before inserting the ith batch of 2^{k-i} points, we have a set P_{i-1} of $2^{k-1} + \cdots + 2^{k-i+1}$ points, which is the concatenation of 2^{k-i+1} tuples, where the sequences of levels in each of these tuples are all identical and equal to the "complete binary tree sequence" $C_0(1, 2^{i-1} - 1)$, as defined above (whose elements now encode levels rather than colors). The points of the ith batch are inserted from left to right, where the jth point is inserted between the $(2j-1)$th and $(2j)$th tuples of P_{i-1}. By construction, all points in the ith batch are assigned to level i and are colored with $k-i+1$ different colors of that level. Proceeding in this manner, we end the construction by inserting the $(k-1)$th batch, which consists of a single point that is assigned to level k. Altogether we have inserted $n = 2^k - 1$ points and forced the algorithm to use $k + (k-1) + \cdots + 1 = k(k+1)/2 = \Omega(\log^2 n)$ different colors. □

Given that the only known lower bound for this online CF-coloring problem is $\Omega(\log n)$ which holds also in the static problem, its a major open problem to close the gap with the $O(\log^2 n)$ upper bound provided by the algorithm above.

Problem 10. *Find a deterministic online CF-coloring for coloring points with respect to intervals which uses* $o(\log^2 n)$ *colors in the worst case or improve the* $\Omega(\log n)$ *lower bound.*

Other Online models For the case of online CF-coloring points with respect to intervals, other models of a weaker adversary were studied in [12]. For example, a natural assumption is that the adversary reveals, for a newly inserted point, its final position among the set of all points in the end of the online input. This is referred to as the *online absolute positions model.* In this model an online CF-coloring algorithm that uses at most $O(\log n)$ colors is presented in [12].

5.2.2 points with respect to halfplanes or unit discs

In [17] it was shown that the two-dimensional variant of online CF-coloring a given sequence of inserted points with respect to arbitrary discs is hopeless as there exists sequences of n points for which every CF-coloring requires n distinct colors. However if we require a CF-coloring with respect to congruent discs or with respect to half-planes, there is some hope. Even though no efficient deterministic online algorithms are known for such cases, some efficient randomized algorithms that uses expected $O(\log n)$ colors are provided in [18, 10] under the assumption that the adversary is oblivious to the random bits used by the algorithm.

Chen Kaplan and Sharir [18] introduced an $O(log^3 n)$ deterministic algorithm for online CF-coloring any n nearly-equal axis-parallel rectangles in the plane.

5.2.3 Degenerate hypergraphs

Next, we describe the general framework of [10] for online CF-coloring any hypergraph. This framework is used to obtain efficient randomized online algorithms for hypergraphs provided that a special parameter referred to as the *degeneracy* of the underlying hypergraph is small. This notion extends the notion of a degenerate graph to that of a hypergraph:

Definition 5.8. *Let $k > 0$ be a fixed integer and let $H = (V, E)$ be a hypergraph on the n vertices v_1, $\ldots$, v_n. For a permutation $\pi\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ define the n partial sums, indexed by $t = 1, \ldots, n$,*

$$S_t^\pi = \sum_{j=1}^{t} d(v_{\pi(j)}),$$

where

$$d(v_{\pi(j)}) = \left|\left\{i < j \mid \{v_{\pi(i)}, v_{\pi(j)}\} \in G(H(\{v_{\pi(1)}, ..., v_{\pi(j)}\}))\right\}\right|,$$

that is, $d(v_{\pi(j)})$ is the number of neighbors of $v_{\pi(j)}$ in the Delaunay graph of the hypergraph induced by $\{v_{\pi(1)}, ..., v_{\pi(j)}\}$. Assume that for all permutations π and for every $t \in \{1, \ldots, n\}$ we have

$$S_t^\pi \leq kt. \tag{1}$$

*Then, we say that H is k-*degenerate.

Let $H = (V, E)$ be any hypergraph. We define a framework that colors the vertices of V in an online fashion, i.e., when the vertices of V are revealed by an adversary one at a time. At each time step t, the algorithm must assign a color to the newly revealed vertex v_t. This color cannot be changed in future times $t' > t$. The coloring has to be conflict-free for all the induced hypergraphs $H(V_t)$ with $t = 1, \ldots, n$, where $V_t \subseteq V$ is the set of vertices revealed by time t.

For a fixed positive integer h, let $A = \{a_1, \ldots, a_h\}$ be a set of h *auxiliary* colors. This auxiliary colors set should not be confused with the set of *main* colors used for the conflict-free coloring: $\{1, 2, \ldots\}$. Let $f\colon \mathbb{N}^+ \to A$ be some fixed function. In the following, we define the framework that depends on the choice of the function f and the parameter h.

A table (to be updated online) is maintained with row entries indexed by the variable i with range in $\mathbb{N}^+$. Each row entry i at time t is associated with a subset $V_t^i \subseteq V_t$ in addition to an auxiliary proper non-monochromatic coloring of $H(V_t^i)$ with at most h colors. The subsets V_t^i are nested. Namely, $V_t^{i+1} \subset V_t^i$ for every i. Informally, we think of a newly inserted vertex as trying to reach its final entry by some decision process. It starts with entry 1 and continue "climbing" to higher levels as long as it does not succeed to get its final color. We say that $f(i)$ is the auxiliary color that *represents* entry i in the table. At the beginning all entries of the table are empty. Suppose all entries of the table are updated until time $t - 1$ and let v_t be the vertex revealed by the adversary at time t. The framework first checks if an auxiliary color can be assigned to v_t such that the auxiliary coloring of V_{t-1}^1 together with the color of

v_t is a proper non-monochromatic coloring of $H(V_{t-1}^1 \cup \{v_t\})$. Any (proper non-monochromatic) coloring procedure can be used by the framework. For example a first-fit greedy method in which all colors in the order $a_1, \ldots, a_h$ are checked until one is found. If such a color cannot be found for v_t, then entry 1 is left with no changes and the process continues to the next entry. If however, such a color can be assigned, then v_t is added to the set V_{t-1}^1. Let c denote such an auxiliary color assigned to v_t. If this color is the same as $f(1)$ (the auxiliary color that represents entry 1), then the final color in the online conflict-free coloring of v_t is 1 and the updating process for the t-th vertex stops. Otherwise, if an auxiliary color cannot be found or if the assigned auxiliary color is not the same as $f(1)$, then the updating process continues to the next entry. The updating process stops at the first entry i for which v_t is both added to V_t^i and the auxiliary color assigned to v_t is the same as $f(i)$. Then, the main color of v_t in the final conflict-free coloring is set to i. See Figure 6 for an illustration.

$f : N \to \{a, b, c\}$	v_1	v_3	v_4	v_2	
f(1)=a	a	c	a	b	
f(2)=b		b		a	
f(3)=a				a	
•					
•					
•					
•					

Figure 6: An example of the updating process of the table for the hypergraph induced by points with respect to intervals. 3 auxiliary colors denoted $\{a, b, c\}$ are used. In each line i the auxiliary coloring is given. It serves as a proper coloring for the hypergraphs $H(V_t^i)$ induced by the subset V_t^i of all points revealed up to time t that reached line i. The first point v_1 is inserted to the left. The second point v_2 to the right and the third point v_3 in the middle, etc. For instance, at the first entry (i.e., line) of the table, the auxiliary color of v_2 is b. In the second line it is a and in the third line it is a. Since $f(3) = a$, the final color of v_2 is 3. Similarly, the final color of v_1 is 1, of v_3 is 2, and of v_4 is 1.

It is possible that v_t never gets a final color. In this case we say that the framework does not halt. However, termination can be guaranteed by imposing some restrictions on the auxiliary coloring method and the choice of the function f. For example, if first-fit is used for the auxiliary colorings at any entry and if f is the constant function $f(i) = a_1$, for all i, then the framework is guaranteed to halt for any time t. Later, a randomized online algorithm based on this framework is derived under the oblivious adversary model. This algorithm always halts, or to be more precise halts with probability 1, and moreover it halts after a "small" number of entries with high probability. We prove that the above framework produces a valid conflict-free coloring in case it halts.

Lemma 5.9. *If the above framework halts for any vertex v_t then it produces a valid online conflict-free coloring of H.*

Proof. Let $H(V_t)$ be the hypergraph induced by the vertices already revealed at time t. Let S be a hyperedge in this hypergraph and let j be the maximum integer for which there is a vertex v of S colored with j. We claim that exactly one such vertex in S exists. Assume to the contrary that there is another

vertex v' in S colored with j. This means that at time t both vertices v and v' were present at entry j of the table (i.e., $v, v' \in V_t^j$) and that they both got an auxiliary color (in the auxiliary coloring of the set V_t^j) which equals $f(j)$. However, since the auxiliary coloring is a proper non-monochromatic coloring of the induced hypergraph at entry j, $S \cap V_t^j$ is not monochromatic so there must exist a third vertex $v'' \in S \cap V_t^j$ that was present at entry j and was assigned an auxiliary color different from $f(j)$. Thus, v'' got its final color in an entry greater than j, a contradiction to the maximality of j in the hyperedge S. This completes the proof of the lemma. □

The above algorithmic framework can also describe some well-known deterministic algorithms. For example, if first-fit is used for auxiliary colorings and f is the constant function, $f(i) = a_1$, for all i, then, for the hypergraph induced by points on a line with respect to intervals, the algorithm derived from the framework becomes identical to the UniMax greedy algorithm described above.

An online randomized conflict-free coloring algorithm We devise a randomized online conflict-free coloring algorithm in the oblivious adversary model. In this model, the adversary has to commit to a permutation according to the order of which the vertices of the hypergraph are revealed to the algorithm. Namely, the adversary does not have access to the random bits that are used by the algorithm. The algorithm always produces a valid coloring and the number of colors used is related to the degeneracy of the underlying hypergraph in a manner described in the following theorem.

Theorem 5.10 ([10]). *Let $H = (V, E)$ be a k-degenerate hypergraph on n vertices. Then, there exists a randomized online conflict-free coloring algorithm for H which uses at most $O(\log_{1+\frac{1}{4k+1}} n) = O(k \log n)$ colors with high probability against an oblivious adversary.*

The algorithm is based on the framework presented above. In order to define the algorithm, we need to state what is (a) the set of auxiliary colors of each entry, (b) the function f, and (c) the algorithm we use for the auxiliary coloring at each entry. We use the set of auxiliary colors $A = \{a_1, \ldots, a_{2k+1}\}$. For each entry i, the representing color $f(i)$ is chosen uniformly at random from A. We use a first-fit algorithm for the auxiliary coloring.

Our assumption on the hypergraph H (being k-degenerate) implies that at least half of the vertices up to time t that *reached* entry i (but not necessarily added to entry i), denoted by $X_{\geq i}^t$, have been actually given some auxiliary color at entry i (that is, $\left|V_t^i\right| \geq \frac{1}{2}\left|X_{\geq i}^t\right|$). This is due to the fact that at least half of those vertices v_t have at most $2k$ neighbors in the Delaunay graph of the hypergraph induced by $X_{\geq i}^{t-1}$ (since the sum of these quantities is at most $k\left|X_{\geq i}^t\right|$ and since $V_t^i \subseteq X_{\geq i}^t$). Therefore, since we have $2k+1$ colors available, there is always an available color to assign to such a vertex. The following lemma shows that if we use one of these available colors then the updated coloring is indeed a proper non-monochromatic coloring of the corresponding induced hypergraph as well.

Lemma 5.11. *Let $H = (V, E)$ be a k-degenerate hypergraph and let V_t^j be the subset of V at time t and at level j as produced by the above algorithm. Then, for any j and t if v_t is assigned a color distinct from all its neighbors in the Delaunay graph $G(H(V_t^j))$ then this color together with the colors assigned to the vertices V_{t-1}^j is also a proper non-monochromatic coloring of the hypergraph $H(V_t^j)$.*

Proof. Follows from Lemma 5.9 □

We also prove that for every vertex v_t, the algorithm always halts, or more precisely halts with probability 1.

Proposition 5.12. *For every vertex v_t, the algorithm halts with probability 1.*

Proof.

$$\begin{aligned}
&\Pr[\text{algorithm does not halt for } v_t] = \\
&\Pr[\text{algorithm does not assign a main color to } v_t \text{ in any entry}] \leq \\
&\Pr[\text{algorithm does not assign a main color to } v_t \text{ in any empty entry}] = \\
&\Pr[\bigcap_{i:\text{ empty entry}} (\text{algorithm does not assign a main color to } v_t \text{ in entry } i)] = \\
&\prod_{i:\text{ empty entry}} \Pr[\text{algorithm does not assign a main color to } v_t \text{ in entry } i] = \\
&\prod_{i:\text{ empty entry}} (1-h^{-1}) = \lim_{j\to\infty}(1-h^{-1})^j = 0
\end{aligned}$$

and therefore $\Pr[\text{algorithm halts for } v_t] = 1$. □

We proceed to the analysis of the number of colors used by the algorithm, proving theorem 5.10.

Lemma 5.13. *Let $H = (V, E)$ be a hypergraph and let C be a coloring produced by the above algorithm on an online input $V = \{v_t\}$ for $t = 1, \ldots, n$. Let X_i (respectively $X_{\geq i}$) denote the random variable counting the number of points of V that were assigned a final color at entry i (respectively a final color at some entry $\geq i$). Let $\mathbf{E}_i = \mathbf{E}[X_i]$ and $\mathbf{E}_{\geq i} = \mathbf{E}[X_{\geq i}]$ (note that $X_{\geq i+1} = X_{\geq i} - X_i$). Then:*

$$\mathbf{E}_{\geq i} \leq \left(\frac{4k+1}{4k+2}\right)^{i-1} n.$$

Proof. By induction on i. The case $i = 1$ is trivial. Assume that the statement holds for i. To complete the induction step, we need to prove that $\mathbf{E}_{\geq i+1} \leq (\frac{4k+1}{4k+2})^i n$. By the conditional expectation formula, we have for any two random variables X, Y that $\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X \mid Y]]$. Thus,

$$\mathbf{E}_{\geq i+1} = \mathbf{E}[\mathbf{E}[X_{\geq i+1} \mid X_{\geq i}]] = \mathbf{E}[\mathbf{E}[X_{\geq i} - X_i \mid X_{\geq i}]] = \mathbf{E}[X_{\geq i} - \mathbf{E}[X_i \mid X_{\geq i}]].$$

It is easily seen that $\mathbf{E}[X_i \mid X_{\geq i}] \geq \frac{1}{2}\frac{X_{\geq i}}{2k+1}$ since at least half of the vertices of $X_{\geq i}$ got an auxiliary color by the above algorithm. Moreover each of those elements that got an auxiliary color had probability $\frac{1}{2k+1}$ to get the final color i. This is the only place where we need to assume that the adversary is oblivious and does not have access to the random bits. Thus,

$$\mathbf{E}[X_{\geq i} - \mathbf{E}[X_i \mid X_{\geq i}]] \leq \mathbf{E}[X_{\geq i} - \frac{1}{2(2k+1)}X_{\geq i}] = \frac{4k+1}{4k+2}\mathbf{E}[X_{\geq i}] \leq \left(\frac{4k+1}{4k+2}\right)^i n,$$

by linearity of expectation and by the induction hypotheses. This completes the proof of the lemma. □

Lemma 5.14. *The expected number of colors used by the above algorithm is at most* $\log_{\frac{4k+2}{4k+1}} n + 1$.

Proof. Let I_i be the indicator random variable for the following event: some points are colored with a main color in entry i. We are interested in the number of colors used, that is $Y := \sum_{i=1}^{\infty} I_i$. Let $b(k, n) = \log_{\frac{4k+2}{4k+1}} n$. Then,

$$\mathbf{E}[Y] = \mathbf{E}[\sum_{1\leq i} I_i] \leq \mathbf{E}[\sum_{1\leq i\leq b(k,n)} I_i] + \mathbf{E}[X_{\geq b(k,n)+1}] \leq b(k, n) + 1,$$

by Markov's inequality and lemma 5.13. □

We notice that:

$$b(k,n) = \frac{\ln n}{\ln \frac{4k+2}{4k+1}} \leq (4k+2)\ln n = O(k \log n).$$

We also have the following concentration result:

$$\Pr[\text{more than } c \cdot b(k,n) \text{ colors are used}] = \Pr[X_{\geq c \cdot b(k,n)+1} \geq 1] \leq \mathbf{E}_{\geq c \cdot b(k,n)+1} \leq \frac{1}{n^{c-1}},$$

by Markov's inequality and by lemma 5.13.

This completes the performance analysis of the algorithm.

Remark In the above description of the algorithm, all the random bits are chosen in advance (by deciding the values of the function f in advance). However, one can be more efficient and calculate the entry $f(i)$ only at the first time we need to update entry i, for any i. Since at each entry we need to use $O(\log k)$ random bits and we showed that the number of entries used is $O(k \log n)$ with high probability then the total number of random bits used by the algorithm is $O(k \log k \log n)$ with high probability.

Acknowledgments.

I would like to thank the two anonymous referees for providing very valuable comments and suggestions.

References

[1] M. A. Abam, M. de Berg, and S.-H. Poon. Fault-tolerant conflict-free coloring. In *Proceedings of the 20th Annual Canadian Conference on Computational Geometry (CCCG), Montreal, Canada, August 13-15*, 2008.

[2] M. Abellanas, P. Bose, J. Garcia, F. Hurtado, M. Nicolas, and P. A. Ramos. On properties of higher order delaunay graphs with applications. In *EWCG*, 2005.

[3] D. Ajwani, K. Elbassioni, S. Govindarajan, and S. Ray. Conflict-free coloring for rectangle ranges using $\tilde{O}(n^{.381+\epsilon})$ colors. In *SPAA '07: Proc. 19th ACM Symp. on Parallelism in Algorithms and Architectures*, pages 181–187, 2007.

[4] N. Alon. Choice numbers of graphs: a probabilistic approach. 1:107–114, 1992.

[5] N. Alon and S. Smorodinsky. Conflict-free colorings of shallow discs. *Int. J. Comput. Geometry Appl.*, 18(6):599–604, 2008.

[6] N. Alon and J. H. Spencer. *The Probabilistic Method*. Wiely Inter-Science, 2000.

[7] G. Aloupis, J. Cardinal, S. Collette, S. Langerman, and S. Smorodinsky. Coloring geometric range spaces. *Discrete & Computational Geometry*, 41(2):348–362, 2009.

[8] K. Appel and W. Haken. Every planar map is 4-colorable - 1: Discharging. *Illinois Journal of Mathematics*, (21):421–490, 1977.

[9] K. Appel and W. Haken. Every planar map is 4-colorable - 2: Reducibility. *Illinois Journal of Mathematics*, (21):491–567, 1977.

[10] A. Bar-Noy, P. Cheilaris, S. Olonetsky, and S. Smorodinsky. Online conflict-free colouring for hypergraphs. *Combinatorics, Probability & Computing*, 19(4):493–516, 2010.

[11] A. Bar-Noy, P. Cheilaris, and S. Smorodinsky. Conflict-free coloring for intervals: from offline to online. In *Proc. ACM Symposium on Parallelism in Algorithms and Architectures*, pages 128–137, New York, NY, USA, 2006. ACM Press.

[12] A. Bar-Noy, P. Cheilaris, and S. Smorodinsky. Deterministic conflict-free coloring for intervals: From offline to online. *ACM Transactions on Algorithms*, 4(4), 2008.

[13] A. L. Buchsbaum, A. Efrat, S. Jain, S. Venkatasubramanian, and K. Yi. Restricted strip covering and the sensor cover problem. In *Proc. Annu. ACM-SIAM Symposium on Discrete Algorithms*, pages 1056–1063, 2007.

[14] P. Cheilaris, B. Keszegh, and D. Pálvölgyi. Unique-maximum and conflict-free colorings for hypergraphs and tree graphs. *arXiv:1002.4210v1*, 2010.

[15] P. Cheilaris, S. Smorodinsky, and M. Sulovský. The potential to improve the choice: list conflict-free coloring for geometric hypergraphs. In *Proc. 27th Annu. ACM Sympos. Comput. Geom.*, 2011.

[16] P. Cheilaris and G. Tóth. Graph unique-maximum and conflict-free colorings. In *Proc. 7th International Conference on Algorithms and Complexity (CIAC)*, pages 143–154, 2010.

[17] K. Chen, A. Fiat, M. Levy, J. Matoušek, E. Mossel, J. Pach, M. Sharir, S. Smorodinsky, U. Wagner, and E. Welzl. Online conflict-free coloring for intervals. *Siam. J. Comput.*, 36:545–554, 2006.

[18] K. Chen, H. Kaplan, and M. Sharir. Online conflict-free coloring for halfplanes, congruent disks, and axis-parallel rectangles. *ACM Transactions on Algorithms*, 5(2), 2009.

[19] X. Chen, J. Pach, M. Szegedy, and G. Tardos. Delaunay graphs of point sets in the plane with respect to axis-parallel rectangles. In *Proc. Annu. ACM-SIAM Symposium on Discrete Algorithms*, pages 94–101, 2008.

[20] K. L. Clarkson and P. W. Shor. Application of random sampling in computational geometry, ii. *Discrete & Computational Geometry*, 4:387–421, 1989.

[21] T. H. Cormen, C. E. Leiserson, and R. L. Rivest. *Introduction to Algorithms*. MIT Press, Cambridge, MA, 1990.

[22] K. Elbassioni and N. Mustafa. Conflict-free colorings of rectangle ranges. In *STACS '06: Proc. 23rd International Symposium on Theoretical Aspects of Computer Science*, pages 254–263, 2006.

[23] P. Erdős, A. L. Rubin, and H. Taylor. Choosability in graphs. In *Proc. West Coast Conf. on Combinatorics, Graph Theory and Computing, Congressus Numerantium XXVI*, pages 125–157, 1979.

[24] G. Even, Z. Lotker, D. Ron, and S. Smorodinsky. Conflict-free colorings of simple geometric regions with applications to frequency assignment in cellular networks. *Siam. J. Comput.*, 33:94–136, 2003.

[25] M. Gibson and K. R. Varadarajan. Decomposing coverings and the planar sensor cover problem. In *50th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2009, October 25-27, 2009, Atlanta, Georgia, USA*, pages 159–168, 2009.

[26] A. Hajnal, I. Juhász, L. Soukup, and Z. Szentmiklóssy. Conflict free colorings of (strongly) almost disjoint set-systems. *Acta Mathematica Hungarica*, 2011, to appear.

[27] S. Har-Peled and S. Smorodinsky. Conflict-free coloring of points and simple regions in the plane. *Discrete & Computational Geometry*, 34(1):47–70, 2005.

[28] D. Haussler and E. Welzl. Epsilon-nets and simplex range queries. *Discrete & Computational Geometry*, 2:127–151, 1987.

[29] E. Horev, R. Krakovski, and S. Smorodinsky. Conflict-free coloring made stronger. In *Proc. 12th Scandinavian Symposium and Workshops on Algorithm Theory*, 2010.

[30] A. V. Iyer, H. D. Ratliff, and G. Vijayan. Optimal node ranking of trees. *Information Processing Letters*, 28(5):225–229, 1988.

[31] M. Katchalski, W. McCuaig, and S. M. Seager. Ordered colourings. *Discrete Mathematics*, 142(1-3):141–154, 1995.

[32] K. Kedem, R. Livne, J. Pach, and M. Sharir. On the union of Jordan regions and collision-free translational motion amidst polygonal obstacles. *Discrete & Computational Geometry*, 1:59–71, 1986.

[33] B. Keszegh. Weak conflict-free colorings of point sets and simple regions. In *Proceedings of the 19th Annual Canadian Conference on Computational Geometry, CCCG 2007, August 20-22, 2007, Carleton University, Ottawa, Canada*, pages 97–100, 2007.

[34] B. Keszegh. *Combinatorial and computational problems about points in the plane.* PhD thesis, Central European University, Budapest, Department of Mathematics and its Applications, 2009.

[35] N. Lev-Tov and D. Peleg. Conflict-free coloring of unit disks. *Discrete Applied Mathematics*, 157(7):1521–1532, 2009.

[36] J. W. H. Liu. Computational models and task scheduling for parallel sparse cholesky factorization. *Parallel Computing*, 3(4):327–342, 1986.

[37] J. Matoušek. *Lectures on Discrete Geometry.* Springer-Verlag New York, Inc., Secaucus, NJ, USA, 2002.

[38] R. A. Moser and G. Tardos. A constructive proof of the general lovász local lemma. *J. ACM*, 57(2), 2010.

[39] J. Pach and G. Tardos. Conflict-free colourings of graphs and hypergraphs. *Comb. Probab. Comput.*, 18(5):819–834, 2009.

[40] J. Pach and G. Tardos. Coloring axis-parallel rectangles. *Journal of Combinatorial Theory, Series A*, in press, 2009.

[41] J. Pach, G. Tardos, and G. Tóth. Indecomposable coverings. In *The China-Japan Joint Conference on Discrete Geometry, Combinatorics, and Graph Theory (CJCDGCGT 2005), Lecture Notes in Computer Sceince*, pages 135–148, 2007.

[42] J. Pach and G. Tóth. Conflict free colorings. *Discrete & Computational Geometry, The Goodman-Pollack Festschrift*, pages 665–671, 2003.

[43] J. Pach and G. Tóth. Decomposition of multiple coverings into many parts. *Comput. Geom.: Theory and Applications*, 42(2):127–133, 2009.

[44] A. A. Schäffer. Optimal node ranking of trees in linear time. *Information Processing Letters*, 33(2):91–96, 1989.

[45] A. Sen, H. Deng, and S. Guha. On a graph partition problem with application to vlsi layout. *Information Processing Letters*, 43(2):87–94, 1992.

[46] S. Smorodinsky. *Combinatorial Problems in Computational Geometry.* PhD thesis, School of Computer Science, Tel-Aviv University, 2003.

[47] S. Smorodinsky. On the chromatic number of some geometric hypergraphs. *Siam. J. Discrete Mathematics*, 21:676–687, 2007.

[48] S. Smorodinsky. Improved conflict-free colorings of shallow discs, manuscript. 2009.

[49] R. P. Stanley. *Combinatorics and Commutative Algebra, Second Edition.* Birkhuser, Boston, Inc., 1996.

[50] V. Vizing. Coloring the vertices of a graph in prescribed colors (in russian). *Diskret. Analiz.*, Metody Diskret. Anal. v. Teorii Kodov i Shem 101(29):3–10, 1976.

[51] D. B. West. *Introduction to Graph Theory.* Prentice Hall, 2ed edition, 2001.