NORTH-HOLLAND

Tangent Spaces of Rational Matrix Functions*

U. Helmke
Universität Würzburg
Institute für Mathematik
Würzburg, Germany

and

P. A. Fuhrmann†
Department of Mathematics
Ben-Gurion University of the Negev
Beer Sheva, Israel

Submitted by Shmuel Friedland

ABSTRACT

We consider the task of determining tangent spaces for classes of rational matrix valued functions. Our analysis is based on methods from control theory, and in particular the theory of polynomial models. Explicit descriptions of tangent spaces of rational transfer functions, stable rational transfer functions, rational inner functions, and symmetric rational transfer functions are obtained. Moreover, a new proof of Delchamps's decomposition formula for the tangent bundle of rational transfer functions is given. A Riemannian metric as well as a symplectic structure is defined. © 1998 Elsevier Science Inc.

1. INTRODUCTION

The current activity in research on analog computation and neural networks has led to a resurgence of interest in steepest descent gradient flow techniques as a tool to investigate constrained optimization tasks, which are

* Partially supported by GIF under grant I 184.
† Earl Katz Family Chair in Algebraic System Theory.

0024-3795/98/$19.00
PII S0024-3795(97)00252-8

difficult to approach by other methods. To develop gradient flow methods for minimizing smooth functions on a manifold requires explicit knowledge of tangent spaces as well as specification of a Riemannian metric. Thus it becomes important to be able to determine such basic objects from differential geometry. As an example of such a constrained optimization problem from control theory consider the task of minimizing the H^2 distance of a stable transfer function to the class of all stable rational transfer functions of a fixed degree. In this example the constraint set is the smooth manifold $\mathrm{Rat}^-(n, m, p)$ of stable, real rational $p \times m$ transfer functions of McMillan degree n. To determine the gradient flow of the smooth H^2 distance function one has to specify a Riemannian metric as well as needing an explicit description of the tangent spaces of $\mathrm{Rat}^-(n, m, p)$. Likewise one might consider the optimization task for smooth functions defined on various other classes of rational matrix functions. A case of special importance here is the class $\mathrm{Rat}(n, m, p)$ of arbitrary $p \times m$ real rational matrix valued functions of constant degree n. In fact, various problems in system identification, model reduction, and H^∞ control can be cast as minimization problems on manifolds of rational matrix valued functions, thus leading to the demand of computing tangent spaces for classes of linear systems.

Delchamps (1986) was the first who derived an explicit description of the tangent bundle of $\mathrm{Rat}(n, m, p)$. Using state space methods, he proved that the tangent bundle of $\mathrm{Rat}(n, m, p)$ has the direct sum decomposition

$$T\,\mathrm{Rat}(n, m, p) = \mathrm{Hom}(X, \mathbf{R}^p) \oplus \mathrm{Hom}(\mathbf{R}^m, X)$$

Here X denotes the n- dimensional vector bundle on $\mathrm{Rat}(n, m, p)$ whose fiber at a rational function $G \in \mathrm{Rat}(n, m, p)$ is the abstract state space of the system defined by G. Despite the apparent beauty of such a decomposition, it is hard to work with it. In fact, the tangent spaces of $\mathrm{Rat}(n, m, p)$ are themselves vector spaces of rational functions. This interpretation is lost in the above description. Thus the question arises how to interpret such a decomposition in terms of rational functions. Closely related to the work of Delchamps (1986) is the recent work by Alpay, Baratchart, and Gombani (1994). They consider the manifold $\mathrm{Rat}^-(n, m, p)$ of stable rational functions as a subspace of the Hardy space H^2. Explicit formulas for the tangent spaces as subspaces of H^2 are derived. Moreover a decomposition formula for the tangent bundle similar to that of Delchamps is proved. Here the Hilbert space structure of the Hardy space H^2 plays an essential role in simplifying the analysis considerably.

In this paper we develop a new approach to determine tangent spaces of classes of rational functions. Instead of trying to employ Hilbert space techniques as in Alpay, Baratchart, and Gombani (1994) [which would be possible using the factorization of Douglas, Shapiro, and Shields (1971), but is restricted to the case of stable rational functions], we heavily use methods from the theory of polynomial models. This leads us to interesting interpretations of the tangent spaces as rational model spaces for tensor products of matrix polynomials. Using state feedback and output injection transformations via solutions of the algebraic Riccati equation, an explicit form of the Delchamps decomposition is obtained. Thus we prove the existence of a canonical decomposition of vector spaces

$$T_G \operatorname{Rat}(n, m, p) = \operatorname{Hom}(\mathbf{R}^m[z]/\operatorname{Ker} H_G, \mathbf{R}^p) \oplus \operatorname{Hom}(\mathbf{R}^m, \operatorname{Im} H_G),$$

where H_G denotes the Hankel operator associated with the transfer function $G(z)$. We also describe explicit choices for Riemannian metrics, as well as proving that for $m = p$ the space $\operatorname{Rat}(n, m, p)$ is a simplectic manifold. The previous results of Delchamps (1986) and Alpay, Baratchart, and Gombani (1994) follow directly from our results, by minor modifications.

2. TENSORED POLYNOMIAL MODELS

Since our results on the representation of the tangent space of $\operatorname{Rat}(n, m, p)$ are best expressed in terms of a tensored form of polynomial and rational models, we develop in this section the basic results about tensored polynomials models. Some of these results are needed in the sequel, but we feel that they are of intrinsic interest and will be useful for later applications. For the standard polynomial model theory, see Fuhrmann (1976, 1983) and the further references therein.

Given the space $\mathbf{R}^{p\times m}$ of $p \times m$ real matrices, and square matrices A and B of size $p \times p$ and $m \times m$ respectively, we define $A \otimes B : \mathbf{R}^{p\times m} \to \mathbf{R}^{p\times m}$ by

$$(A \otimes B)(X) = AXB.$$

We extend this definition in a natural way to polynomial matrices.

To define polynomial models in this context we imitate the standard construction; see Fuhrmann (1976, 1983). Given nonsingular polynomial matrices D_1 and D_2, we define a map $\tau_{D_1 \otimes D_2} : \mathbf{R}^{p\times m}[z] \to \mathbf{R}^{p\times m}[z]$ by

$$\pi_{D_1 \otimes D_2} F = (D_1 \otimes D_2)\pi_-\big((D_1 \otimes D_2)^{-1} F\big) = D_1\big[\pi_-\big(D_1^{-1} F D_2^{-1}\big)\big] D_2.$$

Clearly, $\pi_{D_1 \otimes D_2}$ is a projection map, and it is easily computed that

$$\operatorname{Ker} \pi_{D_1 \otimes D_2} = (D_1 \otimes D_2)\mathbf{R}^{p \times m}[z] = D_1 \mathbf{R}^{p \times m}[z] D_2.$$

Obviously $D_1 \mathbf{R}^{p \times m}[z] D_2$ is an $\mathbf{R}[z]$ submodule of $\mathbf{R}^{p \times m}[z]$. We set

$$X_{D_1 \otimes D_2} = \operatorname{Im} \pi_{D_1 \otimes D_2}.$$

We then have the following easy characterization of elements of $X_{D_1 \otimes D_2}$. An element $F \in X_{D_1 \otimes D_2}$ if and only if $\pi_+ D_1^{-1} F D_2^{-1} = 0$, that is, if and only if $D_1^{-1} F D_2^{-1}$ is strictly proper. This is the analog of the set of all remainder polynomials modulo a scalar polynomial d. The *tensored polynomial model* $X_{D_1 \otimes D_2}$ has a natural $\mathbf{R}[z]$-module structure given by

$$S_{D_1 \otimes D_2} F = z \cdot F = \pi_{D_1 \otimes D_2} z F.$$

The map $S_{D_1 \otimes D_2}$, a tensored restricted shift operator, is the analog of (in the scalar case) the multiplication by z modulo the polynomial d map. This in turn is a functional version of the companion matrix; see Fuhrmann [1983, p. 175]. For the standard multivariable case, see Fuhrmann [1976, Equation 4.7].

As an aside we remark that a representation of the form $D_1 \mathbf{R}^{p \times m}[z] D_2$ is not a general representation of submodules. In fact, if we consider

$$\mathscr{M} = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{11} & a_{12} \end{pmatrix} \middle| a_{ij} \in \mathbf{R}[z] \text{ and } a_{12}(0) = 0 \right\},$$

then $\mathscr{M}$ is a submodule of $\mathbf{R}^{2 \times 2}[z]$, but does not have a representation as $\mathscr{M} = D_1 \mathbf{R}^{p \times m}[z] D_2$.

With the polynomial model $X_{D_1 \otimes D_2}$ we associate a closely related *tensored rational model* $X^{D_1 \otimes D_2}$ defined by

$$X^{D_1 \otimes D_2} = (D_1 \otimes D_2)^{-1} X_{D_1 \otimes D_2}.$$

Clearly, $H \in X^{D_1 \otimes D_2}$ if and only if $D_1 H D_2$ is a polynomial matrix. The $\mathbf{R}[z]$ module structure on $X^{D_1 \otimes D_2}$ is given by $z \cdot H = \pi_- z H$.

The importance of also considering rational model spaces is due to the fact that it allows us to do computations in a coordinate free way. We proceed to compute the dimension of a tensored polynomial model, that is, of $X_{D_1 \otimes D_2}$.

PROPOSITION 2.1. *We have*

$$\dim X_{D_1 \otimes D_2} = \dim X^{D_1 \otimes D_2} = m \deg\det D_1 + p \deg\det D_2 . \qquad (1)$$

Proof. We give two proofs. The first proof uses the Smith canonical form. Assume $D_1 \in \mathbf{R}^{p\times p}[z]$ and $D_2 \in \mathbf{R}^{m\times m}[z]$ are both nonsingular, and let $\Delta_1 = \operatorname{diag}(d_1^{(1)}, \dots, d_p^{(1)})$ and $\Delta_2 = \operatorname{diag}(d_1^{(2)}, \dots, d_m^{(2)})$ be their Smith forms. Let U_1, U_2, V_1, V_2 be unimodular matrices such that $U_1 D_1 = \Delta_1 V_1$ and $D_2 U_2 = V_2 \Delta_2$. The polynomial models $X_{D_1 \otimes D_2}$ and $X_{\Delta_1 \otimes \Delta_2}$ are isomorphic via $F \mapsto \pi_{\Delta_1 \otimes \Delta_2}(U_1 \otimes U_2)F$. Thus we have $\dim X_{D_1 \otimes D_2} = \dim X_{\Delta_1 \otimes \Delta_2}$. Now $F = (f_{ij}) \in X_{\Delta_1 \otimes \Delta_2}$ if and only if $f_{ij}/d_i^{(1)} d_j^{(2)}$ is strictly proper, or equivalently if and only if $\deg f_{ij} < \deg d_i^{(1)} + \deg d_j^{(2)}$. This implies

$$\dim X_{\Delta_1 \otimes \Delta_2} = \sum_{i=1}^{p} \sum_{j=1}^{m} \left(\deg d_i^{(1)} + \deg d_j^{(2)}\right)$$

$$= m \deg\det \Delta_1 = P \deg\det \Delta_2 .$$

We conclude from this that $\dim X_{D_1 \otimes D_2} = m \deg\det D_1 + p \deg\det D_2$.

There is another, independent way to prove the above dimension formula. We note that, given a (nonsingular) polynomial matrix D_1, there exists a unimodular polynomial matrix U_1 such that $U_1 U_1$ is in row proper form, with row degrees given by $\lambda_1 \geqslant \cdots \geqslant \lambda_p$, and the leading row coefficient matrix being the identity. Setting $\Delta_1(z) = \operatorname{diag}(z^{\lambda_1}, \dots, z^{\lambda_p})$, we have $U_1 D_1 = \Delta_1 \Gamma_1$ with Γ_1 biproper and normalized. Similarly, there exists a unimodular matrix U_2 such that $D_2 U_2 = \Gamma_2 \Delta_2$, with $\Delta_2(z) = \operatorname{diag}(z^{\kappa_1}, \dots, z^{\kappa_m})$, and Γ_2 biproper and normalized. Clearly, $\deg\det D_1 = \sum_{i=1}^{p} \lambda_i$ and $\deg\det D_2 = \sum_{j=1}^{m} \kappa_j$.

Now the map $F \mapsto U_1 F U_2$ provides an isomorphism of $X_{D_1 \otimes D_2}$ onto $X_{U_1 D_1 \otimes D_2 U_2}$ and in particular these spaces have the same dimensions. Now $F \in X_{U_1 D_1 \otimes D_2 U_2}$ if and only if

$$\pi_+ (U_1 D_1)^{-1} F (D_2 U_2)^{-1} = \pi_+ \Gamma_1^{-1} \Delta_1^{-1} F \Delta_2^{-1} \Gamma_2^{-1} = 0.$$

The last condition is equivalent to $\pi_+ \Delta_1^{-1} F \Delta_2^{-1} = 0$. Now $F = (F_{ij})$ satisfies this condition if and only if $\deg F_{ij} < \lambda_i + \kappa_j$, and hence

$$\dim X_{\Delta_1 \otimes \Delta_2} = \sum_{i=1}^{p} \sum_{j=1}^{m} (\lambda_i + \kappa_j) = m \deg\det \Delta_1 + p \deg\det \Delta_2 .$$

We conclude that

$$\dim X_{D_1 \otimes D_2} = m \deg\det D_1 + p \deg\det D_2 . \qquad \blacksquare$$

Of course, as an immediate corollary of the above result, we obtain the dimension formula for tensored rational models:

$$\dim X^{D_1 \otimes D_2} = m \deg\det D_1 + p \deg\det D_2 .$$

2.1. Submodules and Factorizations

As in the case of regular polynomial models, there is a close connection between factorizations and submodules. Assume $D_1 \otimes D_2 = (E_1 \otimes E_2)(F_1 \otimes F_2) = E_1 F_1 \otimes F_2 E_2$, i.e., $D_1 = E_1 F_1$ and $D_2 = F_2 E_2$. Then $(E_1 \otimes E_2) X_{F_1 \otimes F_2} \subset X_{D_1 \otimes D_2}$ is a submodule. However, again, not every submodule of $X_{D_1 \otimes D_2}$ has such a representation. Special cases are the following trivial factorizations.

$$D_1 \otimes D_2 = (D_1 \otimes I)(I \otimes D_2) = (I \otimes D_2)(D_1 \otimes I):$$

So $(D_1 \otimes I) X_{I \otimes D_2} = \{D_1 P \mid P \in \mathbf{R}^{p \times m}[z],\ \pi_+ P D_2^{-1} = 0\}$ and $(I \times D_2) X_{D_1 \otimes I} = \{R D_2 \mid R \in \mathbf{R}^{p \times m}[z],\ \pi_+ D_1^{-1} R = 0\}$ are both submodules of $X_{D_1 \otimes D_2}$, and so is their sum. Thus

$$(D_1 \otimes I) X_{I \otimes D_2} + (I \otimes D_2) X_{D_1 \otimes I} \subset X_{D_1 \otimes D_2}. \tag{2}$$

Using a dimensionality argument, equality in the previous inclusion would follow if and only if

$$(D_1 \otimes I) X_{I \otimes D_2} \cap (I \otimes D_2) X_{D_1 \otimes I} = 0. \tag{3}$$

PROPOSITION 2.2. *Let $D_1 \in \mathbf{R}^{p \times p}[z]$ and $D_2 \in \mathbf{R}^{m \times m}[z]$ be nonsingular polynomial matrices with determinants d_1 and d_2 respectively. Then*

$$X_{D_1 \otimes D_2} = (D_1 \otimes I) X_{I \otimes D_2} \oplus (I \otimes D_2) X_{D_1 \otimes I} \tag{4}$$

holds if and only if d_1 and d_2 are coprime.

Proof. We use the fact that, for any linear transformations A_1, A_2, there exists a nonzero linear transformation Z such that $ZA_1 = A_2 Z$ if and only if the characteristic polynomials of A_1, A_2 are coprime.

We have already noted the inclusion (2), so, using a dimensionality argument, it remains only to check the condition (3). Let $Q \in (D_1 \otimes I)X_{I\otimes D_2} \cap (I \otimes D_2)X_{D_1 \otimes I}$. Then $Q = RD_2 = D_1 P$ with $D_1^{-1}R$ and PD_2^{-1} strictly proper. The equality $RD_2 = D_1 P$ implies the existence of a homomorphism $Z: X_{D_2} \to X_{D_1}$, given by $Zf = \pi_{D_1} Rf$, i.e., we have $ZS_{D_2} = S_{D_1}Z$.

Now the characteristic polynomials d_1 and d_2 of S_{D_1}, S_{D_2}, are coprime if and only if $Z = 0$ is the only intertwining map. Thus necessarily $R = D_1 T$ holds for some polynomial matrix T and hence also $P = TD_2$. Therefore $T = D_1^{-1}R$ is strictly proper and a polynomial simultaneously, so necessarily $T = 0$, which in turn implies $Q = 0$.

Conversely, assume (4) holds, which implies (3). This in turn implies the nonexistence of a linear transformation intertwining S_{D_1} and S_{D_2}. It follows that the respective characteristic polynomials d_1 and d_2 are coprime. ■

2.2. *Quotient Modules*

Let G be a $p \times m$ strictly proper rational functions having the polynomial coprime factorizations

$$G = \overline{D}^{-1}\overline{N} = ND^{-1}. \tag{5}$$

We consider next the tensored polynomial model $X_{\overline{D}\otimes D}$. As a special case of Equation (2) we see that

$$\overline{D}X_{I\otimes D} + X_{\overline{D}\otimes I}D = (\overline{D} \otimes I)X_{I\otimes D} + (I \otimes D)X_{\overline{D}\otimes I} \subset X_{\overline{D}\otimes D}. \tag{6}$$

Our aim is to give a characterization of the quotient module $X_{\overline{D}\otimes D}/(\overline{D}X_{I\otimes D} + X_{\overline{D}\otimes I}D)$. To this end we need to introduce intertwining maps. Given linear transformations A and B, we say that a linear transformation Z *intertwines* A and B if $ZA = BZ$. The set of all linear transformations intertwining A and B is a linear space, which we denote by $\mathrm{Intw}(A, B)$.

Now with the coprime factorizations (5) we associate two linear transformations $S_D : X_D \to X_D$ and $S_{\overline{D}} : X_{\overline{D}} \to X_{\overline{D}}$, defined by $S_D f = \pi_D zf$ for $f \in X_D$ and analogously for $S_{\overline{D}}$. It is a basic result concerning polynomial models (see Fuhrmann, 1976) that $Z \in \mathrm{Intw}(S_D, S_{\overline{D}})$ if and only if there exist polynomial matrices A and $\overline{A}$ satisfying $\overline{A}D = \overline{D}A$ and Z is defined by $Zf\pi_{\overline{D}}\overline{A}f$. The polynomial matrices A and $\overline{A}$ are uniquely determined if we require $\overline{D}^{-1}\overline{A} = AD^{-1}$ be strictly proper. Moreover, Z is injective if and only if D, A are right coprime, and Z is surjective if and only if $\overline{D}$, $\overline{A}$ are left coprime. Since the factorizations (5) are assumed coprime, and the factorizations are equivalent to $\overline{N}D = \overline{D}N$, it follows that $S_{\overline{D}}$ and S_D are similar. In particular D and $\overline{D}$ have the same invariant factors.

We can state and prove now the principal result of this section.

THEOREM 2.1. *Let G be a $p \times m$ strictly proper rational functions having the polynomial coprime factorizations* (5). *Then there exists a linear isomorphism*

$$X_{\overline{D}\otimes D}/\left(\overline{D}X_{I\otimes D} + X_{\overline{D}\otimes I}D\right) \simeq \operatorname{Intw}(S_D, S_{\overline{D}}) \tag{7}$$

and

$$\dim \operatorname{Intw}(S_D, S_{\overline{D}}) = \delta_1 + 3\delta_2 + \cdots + (2p-1)\delta_p, \tag{8}$$

where $\delta_i = \deg d_i$, $i = 1, \dots, m$.

Proof. We assume $p \leqslant m$. The case $p > m$ can be treated in a completely analogous way. Let $\overline{d}_1, \dots, \overline{d}_p$ and $d_1, \dots, d_m$ be the monic invariant factors of $\overline{D}$ and D respectively, ordered so that $\overline{d}_i \mid \overline{d}_{-1}$ and $d_i \mid d_{i-1}$. Obviously, as $S_{\overline{D}}$ and S_D are similar, we have $\overline{d}_i = d_i$ for $i = 1, \dots, p$ and $d_{p+1} = \cdots = d_m = 1$. If n is the McMillan degree of G, it is clear that $n = \sum_{i=1}^m \delta_i$.

Assume now without loss of generality that $\overline{D} = \operatorname{diag}(d_1, \dots, d_p)$ and $D = \operatorname{diag}(d_1, \dots, d_m)$. Then

$$X_{\overline{D}\otimes D} = \left\{(p_{ij}) \middle| \deg p_{ij} < \delta_i + \delta_j\right\}$$

and, as we have seen already, $\dim X_{\overline{D}\otimes D} = n(p+m)$. Symbolically we can write $X_{\overline{D}\otimes D} = (X_{d_i d_j})$. Next, observe that

$$\overline{D}X_{I\otimes D} + X_{\overline{D}\otimes I}D = \left\{(d_i f_{ij} + g_{ij} d_j) \middle| \deg f_{ij} < \delta_j \text{ and } \deg g_{ij} < \delta_i\right\}.$$

Now

$$\left\{(d_i f_{ij} + g_{ij} d_j) \middle| \deg f_{ij} < \delta_j \text{ and } \deg g_{ij} < \delta_i\right\} = \begin{cases} d_j X_{d_i}, & j \geqslant i, \\ d_i X_{d_j}, & j \leqslant i. \end{cases}$$

So, symbolically,

$$\overline{D}X_{I\otimes D}+X_{\overline{D}\otimes I}D=\begin{pmatrix} d_1X_{d_1} & d_2X_{d_1} & \cdots & d_mX_{d_1}\\ d_2X_{d_1} & d_2X_{d_2} & \cdots & d_mX_{d_2}\\ \vdots & \vdots & & \vdots\\ d_pX_{d_1} & d_2X_{d_2} & \cdots & d_mX_{d_p}\end{pmatrix}.$$

This implies, for the case $p\leqslant m$,

$$\dim\left(\overline{D}X_{I\otimes D}+X_{\overline{D}\otimes I}D\right)$$

$$=(p+m-1)\delta_1+(p+m-3)\delta_2+\cdots+(m-p+1)\delta_p$$

and so

$$\dim\left[X_{\overline{D}\otimes D}/\left(\overline{D}X_{I\otimes D}+X_{\overline{D}\otimes I}D\right)\right]$$

$$=n(p+m)-\left[(p+m-1)\delta_1+(p+m-3)\delta_2+\cdots\right.$$

$$\left.+(m-p+1)\delta_p\right]$$

$$=\delta_1+3\delta_2+\cdots+(2p-1)\delta_p.$$

Next we compute $\operatorname{Intw}(S_D,S_{\overline{D}})$. Let A and B be matrices satisfying $AD=\overline{D}B$, with $\deg a_{ij}<\delta_i$ and $\deg b_{ij}<\delta_j$. Equivalently, $\deg a_{ij}d_j=d_i\deg b_{ij}$. So if $i\leqslant j$ we have $d_j\,|\,d_i$. Let, for $j\geqslant i$, $e_{ij}=d_i/d_j$. Thus $a_{ij}=e_{ij}b_{ij}$ and $\deg e_{ij}=\delta_i-\delta_j$. Thus we have

$$\operatorname{Intw}(S_D,S_{\overline{D}})=\begin{pmatrix} X_{d_1} & e_{12}X_{d_2} & \cdots & e_{1m}X_{d_1}\\ X_{d_1} & X_{d_2} & \cdots & e_{2m}X_{d_2}\\ \vdots & \vdots & & \vdots\\ X_{d_1} & X_{d_p} & \cdots & e_{pm}X_{d_p}\end{pmatrix}.$$

Note that the last $m - p$ columns are all zero. From this we conclude that

$$\dim \operatorname{Intw}(S_D, S_{\overline{D}}) = \delta_1 + 3\delta_2 + \cdots + (2p - 1)\delta_p .$$

Note that

$$\begin{aligned} &\delta_1 + 3\delta_2 + \cdots + (2p-1)\delta_p \\ &\quad + \left[(p+m-1)\delta_1 + (p+m-3)\delta_2 + \cdots + (m-p+1)\delta_p\right] \\ &\quad = \sum_{i=1}^{p} (p+m)\delta_i = (p+m)n. \end{aligned}$$

To conclude the proof, we define a map $\phi : X_{\overline{D}\otimes D} \to \operatorname{Intw}(S_D, S_{\overline{D}})$ by

$$\begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1m} \\ \vdots & \vdots & & \vdots \\ f_{p1} & f_{p2} & \cdots & f_{pm} \end{pmatrix} = \begin{pmatrix} \pi_{d_1} f_{11} & e_{12}\pi_{d_2} f_{12} & \cdots & e_{1m}\pi_{d_m} f_{1m} \\ \pi_{d_1} f_{21} & \pi_{d_2} f_{22} & \cdots & e_{2m}\pi_{d_m} f_{1m} \\ \vdots & \vdots & & \vdots \\ \pi_{d_p} f_{p1} & \pi_{d_p} f_{p2} & \cdots & \pi_{d_p} f_{pm} \end{pmatrix}.$$

It is easy to check that

$$\operatorname{Ker} \phi = \overline{D} X_{I \otimes D} + X_{\overline{D} \otimes I} D.$$

Also, from

$$\begin{aligned} e_{ij}\pi_{d_j} X_{d_i d_j} &= e_{ij} X_{d_j}, \qquad & j > i, \\ \pi_{d_i} X_{d_i d_j} &= X_{d_i}, \qquad & j \leqslant i, \end{aligned}$$

it follows that ϕ is surjective. From this, by considering the map induced by ϕ, the isomorphism (7) follows. ■

COROLLARY 2.1. *Let G be a $p \times m$ strictly proper rational function having the polynomial coprime factorizations* (5). *Let $\delta_1, \dots, \delta_p$ be the degrees of the invariant factors of D. Then for the commutant $C(S_D) = \{Z \mid ZS_D = S_D Z\}$ we have the dimension formula*

$$\dim C(S_D) = \delta_1 + 3\delta_2 + \cdots + (2p - 1)\delta_p .$$

2.3. Hankel Operators

Given two rational functions G_1 and G_2 of sizes $p_1 \times m_1$ and $p_2 \times m_2$ respectively, the Hankel operator $H_{G_1 \otimes G_2} : \mathbf{R}^{m_1 \times p_2}[z] \to z^{-1}\mathbf{R}^{p_1 \times m_2}[[z^{-1}]]$ is defined by

$$H_{G_1 \otimes G_2} F = \pi_-(G_1 \otimes G_2) F.$$

Assume now that $G_i = \overline{D}_i^{-1}\overline{N}_i = N_i D_i^{-1}$ are left and right coprime factorizations. Then it is trivial to verify that

$$\begin{aligned} &\operatorname{Ker} H_{G_1 \otimes G_2} \supset \big(D_1 \otimes \overline{D}_2\big)\mathbf{R}^{m_1 \times p_2}[z], \\ &\operatorname{Im} H_{G_1 \otimes G_2} \subset X^{\overline{D}_1 \otimes D_2}. \end{aligned} \tag{9}$$

However, for these inclusions to be replaced by equalities, extra conditions have to be imposed with are tantamount to the nonexistence of zero-pole cross cancellations.

2.4. Duality

We discuss briefly the essentials of duality theory in the context of tensored polynomial models.

We can identify the dual space of $\mathbf{R}^{m \times p}[z]$ with the space $z^{-1}\mathbf{R}^{p \times m}[[z^{-1}]]$ by letting, for $H \in z^{-1}\mathbf{R}^{p \times m}[[z^{-1}]]$ and $P \in \mathbf{R}^{m \times p}[z]$,

$$[H, P] = (\operatorname{tr} HP)_{-1} = \operatorname{tr}(HP)_{-1}.$$

where $(X)_{-1}$ denotes the residue of a Laurent series of X, i.e., for $X(z) = \sum_{i=-\infty}^{n} x_i z^i$ we let $(X)_{-1} = x_{-1}$. The availability of this pairing allows us to prove the following.

PROPOSITION 2.3. *Let $D_1 \in \mathbf{R}^{p \times p}[z]$ and $D_2 \in \mathbf{R}^{m \times m}[z]$ be nonsingular. Then we have the identification*

$$X^{D_1 \otimes D_2} = \big(\mathbf{R}^{p \times m}[z] / D_1 \mathbf{R}^{p \times m}[z] D_2\big)^*, \tag{10}$$

where the dual space is defined with respect to the pairing defined for $H \in X^{D_1 \otimes D_2}$ and $P \in \mathbf{R}^{p \times m}[z]$ by

$$[H, P] = (\operatorname{tr} HP)_{-1} = \operatorname{tr}(HP)_{-1}. \tag{11}$$

Proof. We show that $X^{D_1 \otimes D_2}$ is the annihilator of the submodule $(D_2 \otimes D_1)\mathbf{R}^{m\times p}[z]$. Note that if $P \in (D_2 \otimes D_1)\mathbf{R}^{m\times p}[z]$ and $H \in X^{D_1 \otimes D_2}$, then $H = D_1^{-1}TD_2^{-1}$ and $P = D_2TD_1$ for some polynomial matrices S and T. Therefore

$$[H, P] = \left(\operatorname{tr} D_1^{-1}SD_2^{-1}D_2TD_1\right)_{-1}$$

$$= \left(\operatorname{tr} SD_2^{-1}D_2TD_1D_1^{-1}\right)_{-1} = (\operatorname{tr} ST)_{-1} = 0.$$

This shows that $X^{D_1 \otimes D_2} \subset ((D_2 \otimes D_1)\mathbf{R}^{m\times p}[z])^{\perp}$.

Conversely, let $H \in ((D_2 \otimes D_1)\mathbf{R}^{m\times p}[z])^{\perp}$. Then, for every polynomial matrix P, we have

$$0 = [H, D_2PD_1] = \operatorname{tr}(HD_2PD_1)_{-1} = \operatorname{tr}(D_1HD_2P)_{-1}.$$

This shows that D_1HD_2 is a polynomial matrix and hence that $H \in X^{D_1 \otimes D_2}$. The representation (10) follows from this. ■

2.5. *Direct Sum Decompositions*

The direct sum decomposition given in Proposition 2.2 is of importance in the proof and application of Theorem 3.1. In the next proposition we show, in the scalar case, how to actually compute this decomposition.

PROPOSITION 2.4. *Let d_1, d_2 be coprime polynomials. Then:*

1. *We have*

$$X_{d_1d_2} = X_{d_2d_1} = d_1X_{d_2} \oplus d_2X_{d_1}. \tag{12}$$

2. *Given $f \in X_{d_1d_2}$, $f = d_1f_1 + d_2f_2$, with $f_1 \in X_{d_2}$ and $f_2 \in X_{d_1}$, then*

$$\begin{aligned} f_1 &= \pi_{d_2}a_1f_2, \\ f_2 &= \pi_{d_1}a_2f, \end{aligned} \tag{13}$$

where a_1, a_2 are any solutions of the Bezout equation

$$a_1d_1 + a_2d_2 = 1. \tag{14}$$

3. *We have*

$$X^{d_1 d_2} = X^{d_1} \oplus X^{d_2}. \tag{15}$$

4. *If* $h \in X^{d_1 d_2}$ *and* $h = h_1 + h_2$ *with* $h_i \in X^{d_i}$, *then*

$$\begin{aligned} h_1 &= H_{a_1/d_2}(d_1 d_2) h, \\ h_2 &= H_{a_2/d_1}(d_1 d_2) h. \end{aligned} \tag{16}$$

Proof. Part 1 follows from the Chinese remainder theorem.
2: By coprimeness, there exist polynomials a_1, a_2 such that

$$a_1 d_1 + a_2 d_2 = 1.$$

Consequently we have the doubly comprime factorization

$$\begin{pmatrix} a_2 & a_1 \\ d_1 & -d_2 \end{pmatrix} \begin{pmatrix} d_2 & a_1 \\ d_1 & -a_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and hence also

$$\begin{pmatrix} d_2 & a_1 \\ d_1 & -a_2 \end{pmatrix} \begin{pmatrix} a_2 & a_1 \\ d_1 & -d_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using the equality $d_1 d_2 = d_2 d_1$, we define a map $Z : X_{d_2} \to X_{d_2}$ by

$$Zf = \pi_{d_2} a_1 f. \tag{17}$$

We claim that $Z^{-1} : X_{d_2} \to X_{d_2}$ is given by $Z^{-1} g = \pi_{d_2} a_1 g$. Indeed, we have

$$\begin{aligned} Z^{-1} Zf &= \pi_{d_2} a_1 \pi_{d_2} d_1 f \\ &= \pi_{d_2} a_1 d_1 f = \pi_{d_2} (a_1 d_1 + d_2 a_2) f \\ &= \pi_{d_2} f = f. \end{aligned}$$

Now, given $f \in X_{d_1 d_2}$, we would like to decompose it relative to the direct sum decomposition (12). Writing $f = d_1 f_1 + d_2 f_2$, with $f_1 \in X_{d_1}$ and $f_2 \in$

X_{d_2}, we compute

$$\pi_{d_2} f = \pi_{d_2}(d_1 f_1 + d_2 f_2) = \pi_{d_2} d_1 f_1 = Z f_1,$$

and hence

$$f_1 = Z^{-1} \pi_{d_2} f = \pi_{d_2} a_1 \pi_{d_2} f = \pi_{d_2} a_1 f.$$

That is, $f_1 = \pi_{d_2} a_1 f$. Similarly we get $f_2 = \pi_{d_1} a_2 f$.

Part 3 follows from (12) on multiplying by $(d_1 d_2)^{-1}$.

4: Assume now instead that we want to compute a decomposition relative to the related direct sum decomposition $X^{d_1 d_2} = X^{d_1} \oplus X^{d_2}$. For $h \in X^{d_1 d_2}$ we let $f = d_1 d_2 h$ and $h = h_1 + h_2$, with $h_i \in X_{d_i}$. Clearly $h_1 = d_2^{-1} \pi_{d_2} a_1 f = \pi_- d_2^{-1} a_1 f$, or

$$h_1 = H_{d_2^{-1} a_1} f. \tag{18}$$

Analogously, we have

$$h_2 = H_{d_1^{-1} a_2} f. \tag{19}$$

These equations exhibit the close relation of Hankel operators to projection operators. This completes the proof. ■

3. THE TANGENT SPACE OF Rat(n, m, p)

Let Rat(n, m, p) denote the set of all $p \times m$ strictly proper real rational transfer functions $G \in \mathbf{R}(z)^{p \times m}$ of McMillan degree $\delta(G) = n$. Every such transfer function Rat(n, m, p) has a minimal (that is, reachable and observable) realization (A, B, C):

$$G(z) = C(zI - A)^{-1} B,$$

where $A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times m}$, and $C \in \mathbf{R}^{p \times n}$. Moreover, by the state space isomorphism theorem, any two such minimal realizations (A_1, B_1, C_1) and (A_2, B_2, C_2) are related via a uniquely determined similarity transformation $T \in \mathrm{GL}(n)$ such that

$$(A_2, B_2, C_2) = \left(T A_1 T^{-1}, T B_1, C_1 T^{-1}\right).$$

It is a well-known fact, from the very beginning of the parameterization theory of linear systems, that the rational function space $\text{Rat}(n, m, p)$ is a smooth manifold of dimension $n(m + p)$; see e.g. Clark (1976). Thus it makes sense to consider the tangent spaces of $\text{Rat}(n, m, p)$. In the sequel we will always denote by $T_G \text{Rat}(n, m, p)$ the tangent space of $\text{Rat}(n, m, p)$ at an element G. The collection of all such tangent spaces with different base points G forms a smooth manifold, the tangent bundle $T \text{Rat}(n, m, p)$, which is in fact a smooth vector bundle on $\text{Rat}(n, m, p)$. We mention that in the more recent paper by Mann and Milgram (1991) a description of the normal bundle of $\text{Rat}(n, m, p)$ can be found. Here however we take a different, more algebraic approach to studying the tangent bundle. It may well be possible that our approach can be extended to compute the tangent bundle of certain smooth compactifications of $\text{Rat}(n, m, p)$ as well; see e.g. Ravi and Rosenthal (1994).

As $\text{Rat}(n, m, p)$ is a subset of the vector space $\mathbf{R}(z)$ of $p \times m$ matrices of real rational functions, it is natural to consider the tangent spaces $T_G \text{Rat}(n, m, p)$ as linear subspaces of $\mathbf{R}(z)^{p \times m}$. Thus each tangent vector can itself be regarded as a $p \times m$ matrix of rational functions. Note that at this stage we do not endow the vector space $\mathbf{R}(z)^{p \times m}$ with a manifold structure. In fact, it would be somewhat tedious to do the subsequent construction completely satisfactorily from a differential topology point of view. Thus we proceed in a rather formal way. To determine which linear subspaces of $\mathbf{R}(z)^{p \times m}$ correspond to the tangent spaces $T_G \text{Rat}(n, m, p)$ of $\text{Rat}(n, m, p)$ we proceed as follows. Let $\mathbf{R} \to \text{Rat}(n, m, p)$, $t \mapsto G_t$, denote a germ of a smooth curve in $\text{Rat}(n, m, p)$, passing through G at $t = 0$. By a simple canonical form argument there exist smooth curves $t \mapsto (N_t, D_t)$ of right comprime matrix polynomials of bounded order such that $G_t = N_t D_t^{-1}$ holds for all t sufficiently close to 0. Similarly for left coprime factorizations $\overline{D}_t^{-1} \overline{N}_t = G_t$. Thus every tangent vector of $\text{Rat}(n, m, p)$ can be expressed as the directional derivative of a smooth curve germ $t \mapsto G_t = N_t D_t^{-1}$ at $t = 0$, using left and right polynomial coprime factorizations.

THEOREM 3.1. *Let $G = \overline{D}^{-1}\overline{N} = ND^{-1}$ be polynomial left and right coprime factorizations of a strictly proper $p \times m$ rational transfer function G, and let $G(z) = C(zI - A)^{-1}B$ be a minimal realization. Then:*

1. *The tangent space $\mathcal{T}$ of* $\text{Rat}(n, m, p)$ *at G is equal to the tensored rational model $X^{\overline{D} \otimes D}$.*
2. *Let K be any stabilizing state feedback map for A, that is a map such that $A - BK$ is stable. Analogously let L be any output injection map such*

that $A - LC$ is antistable. Let Φ, Ψ be defined by

$$
\begin{aligned}
C(zI - A)^{-1} &= \bar{D}^{-1}\Psi(z),\\
(zI - A)^{-1}B &= \Phi(z)D^{-1},
\end{aligned}
\tag{20}
$$

and let

$$
\begin{aligned}
\Gamma(z) &= (I + K\Phi D^{-1}) = D_2 D^{-1} = (D + K\Phi)D^{-1},\\
\bar{\Gamma}(z) &= (I + \bar{D}^{-1}\Psi L) = \bar{D}^{-1}D_1 = \bar{D}^{-1}(\bar{D} + \Psi L).
\end{aligned}
$$

Then we have the following direct sum decomposition of the tangent space:

$$
X^{\bar{D}\otimes D} = \bar{\Gamma}\{S\Phi D^{-1} | S \in \mathbf{R}^{p\times n}\} \oplus \{\bar{D}^{-1}\Psi T | T \in \mathbf{R}^{n\times m}\}\Gamma. \tag{21}
$$

3. *We have the isomorphisms*

$$
\mathscr{T} \simeq \mathrm{Hom}(\mathbf{R}^m[z]/D\mathbf{R}^m[z], \mathbf{R}^p) \oplus \mathrm{Hom}(\mathbf{R}^m, X^{\bar{D}}) \tag{22}
$$

and

$$
\mathscr{T} \simeq \mathrm{Hom}(\mathbf{R}^m[z]/\mathrm{Ker}\, H_G, \mathbf{R}^p) \oplus \mathrm{Hom}(\mathbf{R}^m, \mathrm{Im}\, H_G). \tag{23}
$$

Proof. 1: Let $t \mapsto G_t \in \mathrm{Rat}(n, m, p)$ be the germ of a smooth curve in $\mathrm{Rat}(n, m, p)$, passing through G at $t = 0$, such that the velocity vector at $t = 0$ represents a given tangent vector in $T_G\,\mathrm{Rat}(n, m, p)$. Choose smooth curves $t \mapsto (N_t, D_t)$ and $t \mapsto (\bar{N}_t, \bar{D}_t)$ of polynomial left and right coprime factorizations of G_t. Let

$$
G = \frac{d}{dt}(G_t)\bigg|_{t=0}
$$

denote the velocity vector of G_t at $t = 0$, and similarly for $N_t, D_t, \bar{N}_t, \bar{D}_t$. Then, by the product rule,

$$
\begin{aligned}
\dot{G} &= \dot{N}D^{-1} - ND^{-1}\dot{D}D^{-1}\\
&= \dot{N}D^{-1} - \bar{D}^{-1}\bar{N}\dot{D}D^{-1}\\
&= \bar{D}^{-1}(\bar{D}\dot{N} - \bar{N}\dot{D})D^{-1}.
\end{aligned}
$$

Since $\overline{D}\dot{N} - \overline{N}\dot{D}$ is a $p \times m$ polynomial matrix, we have the inclusion

$$T_G \operatorname{Rat}(n, m, p) \subset X^{\overline{D} \otimes D}.$$

Both vectors spaces have the same dimension $n(m + p)$. Thus the result follows.

2: Let $G(z) = C(zI - A)^{-1}B$ be a minimal realization. We have left coprime and right coprime factorizations

$$\begin{aligned} C(zI - A)^{-1} &= \overline{D}^{-1}\Psi(z), \\ (zI - A)^{-1}B &= \Phi(z)D^{-1}. \end{aligned} \tag{24}$$

These factorizations are unique to a common left or right unimodular factor. By a result of Hautus and Heymann (1978) and Wimmer (1979), the columns of Ψ form a basis of the polynomial model $X_{\overline{D}}$. In the same way the columns of $\overline{D}^{-1}\Psi$ form a basis of the rational model $X^{\overline{D}}$. Similarly, the rows of Φ form a basis of X_D, and the rows of ΦD^{-1} form a basis of X^D.

Note that if $\overline{D}^{-1}\overline{N} = ND^{-1}$ are coprime factorizations of G, then $N(z) = C\Phi(z)$ and $\overline{N}(z) = \Psi(z)B$. Let us choose now any stabilizing state feedback K that is a map such that $A - BK$ is stable. Analogously, we choose an output injection map L such that $A - LC$ is antistable. Since, with $D_1 = \overline{D} + \Psi L$ and $D_2 = D + K\Phi$, we have

$$C(zI - A + LC)^{-1} = (\overline{D} + \Psi L)^{-1}\Psi = D_1^{-1}\Psi,$$

$$(zI - A + BK)^{-1}B = \Phi(D + K\Phi)^{-1} = \Phi D_2^{-1},$$

it follows that

$$\Gamma(z) = I + K\Phi D^{-1} = D_2 D^{-1},$$

$$\overline{\Gamma}(z) = I + \overline{D}^{-1}\Psi L = \overline{D}^{-1}D_1.$$

Using the last equalities and the fact that Γ and $\overline{\Gamma}$ are biproper, it is clear that $X_{\overline{D} \otimes D}$ and $X_{D_1 \otimes D_2}$ are equal as sets.

Having chosen K, L as above, it is clear that D_1 and D_2 are antistable and stable respectively, and hence their determinants are coprime polynomials. Thus, by Proposition 2.2, we get the submodule direct sum decomposition

$$X_{D_1 \otimes D_2} = (D_1 \otimes I) X_{I \otimes D_2} \oplus (I \otimes D_2) X_{D_1 \otimes I}. \tag{25}$$

This provides at the same time a direct sum decomposition of $X_{\overline{D} \otimes D}$ in terms of subspaces. More specifically, we have

$$X_{D_1 \otimes D_2} = \{D_1 S \Phi | S \in \mathbf{R}^{p \times n}\} \oplus \{\Psi T D_2 | T \in \mathbf{R}^{n \times m}\}.$$

Now $X^{\overline{D} \otimes D} = (\overline{D} \otimes D)^{-1} X_{\overline{D} \otimes D} = (\overline{D} \otimes D)^{-1} X_{D_1 \otimes D_2}$. So

$$\begin{aligned} X^{\overline{D} \otimes D} &= \{\overline{D}^{-1} D_1 S \Phi D^{-1} | S \in \mathbf{R}^{p \times n}\} \oplus \{\overline{D}^{-1} \Psi T D_2 D^{-1} | T \in \mathbf{R}^{n \times m}\} \\ &= \overline{\Gamma}\{S \Phi D^{-1} | S \in \mathbf{R}^{p \times n}\} \oplus \{\overline{D}^{-1} \Psi T | T \in \mathbf{R}^{n \times m}\} \Gamma. \end{aligned} \tag{26}$$

3: Next, we analyze the spaces $\{S\Phi D^{-1} \mid S \in \mathbf{R}^{p \times n}\}$ and $\{\overline{D}^{-1}\Psi T \mid T \in \mathbf{R}^{n \times m}\}$. We identify the space $\{\overline{D}^{-1}\Psi T \mid T \in \mathbf{R}^{n \times m}\}$ with $\mathrm{Hom}(\mathbf{R}^m, X^{\overline{D}})$, via the linear map which maps $\overline{D}^{-1}\Psi T$ into the linear transformation $\lambda_{\overline{D}^{-1}\Psi T} \in \mathrm{Hom}(\mathbf{R}^m, X^{\overline{D}})$ defined by

$$\lambda_{\overline{D}^{-1}\Psi T}\xi = \overline{D}^{-1}\Psi T \xi \in X^{\overline{D}}.$$

To prove that this defines an isomorphism of vector spaces we show first that the map λ is injective. Assume $\lambda_{\overline{D}^{-1}\Psi T} = 0$. Thus, for each ξ, we have $\overline{D}^{-1}\Psi T\xi = 0$. This implies $T = 0$ by the nonsingularity of $\overline{D}$ and the injectivity of Ψ. The surjectivity of λ then follows by a dimensionality argument, as

$$\dim \mathrm{Hom}(\mathbf{R}^m, X^{\overline{D}}) = \dim\{\overline{D}^{-1}\Psi T | T \in \mathbf{R}^{n \times m}\} = mn.$$

Next we identify $\{S\Phi D^{-1} \mid S \in \mathbf{R}^{p \times n}\}$ with $\mathrm{Hom}(F^m[z]/DF^m[z], \mathbf{R}^p)$ via the linear map $\mu : \{S\Phi D^{-1} \mid S \in \mathbf{R}^{p \times n}\} \to \mathrm{Hom}(F^m[z]/DF^m[z], \mathbf{R}^p)$ given by $\mu(S\Phi D^{-1}) = \mu_{S\Phi D^{-1}}$ and

$$\mu_{S\Phi D^{-1}} f = (S\Phi D^{-1} f)_{-1} \in \mathbf{R}^p.$$

Since for each $S \in \mathbf{R}^{p \times n}$ the submodule $D\mathbf{R}^m[z]$ is contained in the kernel of $\mu_{S\Phi D^{-1}}$, we can look at the induced map on $\mathbf{R}^m[z]/D\mathbf{R}^m[z]$. Thus μ can be considered to be a linear map from $\{S\Phi D^{-1} \mid S \in \mathbf{R}^{p\times n}\}$ to $\mathrm{Hom}(F^m[z]/DF^m[z], \mathbf{R}^p)$. Again, using a dimensionality argument and

$$\dim\{S\Phi D^{-1} | S \in \mathbf{R}^{p\times n}\} = \dim \mathrm{Hom}(F^m[z]/DF^m[z], \mathbf{R}^p) = pn,$$

it suffices to show that μ is injective. Assume $\mu_{S\Phi D^{-1}} = 0$, that is, $S\Phi D^{-1}f = 0$ for all $f \in \mathbf{R}[z]$. This implies that $S\Phi D^{-1} = T$ is a polynomial matrix. Now $S\Phi D^{-1}$ is polynomial and strictly proper at the same time, so necessarily $S = 0$ and hence also $S\Phi D^{-1} = 0$. ■

REMARK 3.1. It is an open problem to find explicit ways to compute the decomposition (20) of the tangent space. In the scalar case such a constructive approach is available using Proposition 2.4.

Next we specialize our choice of the state feedback and output injection maps, K and L, in such a way that they are both canonical and depend smoothly on the transfer function G. In this we follow the ideas of Delchamps (1986).

With any minimal realization $G(z) = C(zI - A)^{-1}B$, there are associated two dual Riccati equations, namely

$$A^*X + XA - XBB^*X + C^*C = 0,$$

$$AY + YA^* - YC^*CY + BB^* = 0.$$

The first of these equations has a unique maximal, positive definite solution X_+ that is stabilizing, namely, for which $A - BB^*X_+$ is stable. Similarly, let Y_- be the unique antistabilizing solution of the second equation, that is, $A + Y_-C^*C$ is antistable. It is well known (see Delchamps, 1984) that both solutions depend smoothly on G. Note that X_+ and Y_- are related as

$$Y_- = -X_+^{-1}.$$

Thus one has to solve only one Riccati equation.

The biproper functions Γ and $\overline{\Gamma}$ have now the following realizations

$$\Gamma = \left(\begin{array}{c|c} A & -Y_-C^* \\ \hline C & I \end{array}\right), \qquad \overline{\Gamma} = \left(\begin{array}{c|c} A & B \\ \hline B^*X_+ & I \end{array}\right).$$

Clearly both Γ and $\overline{\Gamma}$ are independent of the realization and depend solely on G.

From Theorem 3.1 we deduce the following decomposition of the tangent bundle, due to Delchamps (1986).

COROLLARY 3.1 (Delchamps). *There exists an n-dimensional vector bundle X on* Rat(n, m, p) *such that the tangent bundle* T Rat(n, m, p) *is isomorphic to the Whitney sum of bundles*

$$T \operatorname{Rat}(n, m, p) \simeq \operatorname{Hom}(X, \mathbf{R}^p) \oplus \operatorname{Hom}(\mathbf{R}^m, X). \tag{27}$$

Proof. Let X denote the n-dimensional vector bundle X on Rat(n, m, p) whose fiber X_G at $G \in$ Rat(n, m, p) is the image space of the Hankel operator H_G. Then $\mathbf{R}^m[z]/\operatorname{Ker} H_G$ is canonically isomorphic to $\operatorname{Im} H_G$, and therefore we have an isomorphism of vector spaces

$$\operatorname{Hom}(\mathbf{R}^m[z]/\operatorname{Ker} H_G, \mathbf{R}^p) \oplus \operatorname{Hom}(\mathbf{R}^m, \operatorname{Im} H_G)$$

$$\simeq \operatorname{Hom}(X_G, \mathbf{R}^p) \oplus \operatorname{Hom}(\mathbf{R}^m, X_G).$$

These isomorphisms actually define isomorphisms of smooth vector bundles. Thus it remains to show that T Rat(n, m, p) is, as a vector bundle on Rat(n, m, p), smoothly isomorphic to the Whitney sum of vector bundles

$$\operatorname{Hom}(\mathbf{R}^m[z]/\operatorname{Ker} H_G, \mathbf{R}^p) \oplus \operatorname{Hom}(\mathbf{R}^m, \operatorname{Im} H_G).$$

The main new point we have to show is that the isomorphism (23) described in Theorem 3.1 depends smoothly on $G \in$ Rat(n, m, p). For this we have to choose the state feedback and output injection matrices K and L in a smooth way. An appropriate way to do so is via solutions of algebraic Riccati equations. Explicitly, let X_+ and Y_- denote the unique positive and negative definite solutions of the algebraic Riccati equations

$$A^* X_+ + X_+ A - X_+ BB^* X_+ + C^* C = 0,$$

$$AY_- + Y_- A^* + BB^* - Y_- C^* C Y_- = 0.$$

By multiplying the first Riccati equation on the left and on the right by $-X_+^{-1}$, a solution Y_- of the second Riccati equation is obtained. Thus, by uniqueness,

$$Y_- = -X_+^{-1}.$$

By a result of Delchamps (1984), these solutions depend smoothly on (A, B, C). Moreover, it is easily verified that under state space coordinate transformations, these solutions change as

$$X_+(TAT^{-1}, TB, CT^{-1}) = \tilde{T}^{-1} X_+(A, B, C) T^{-1},$$

$$Y_-(TAT^{-1}, TB, CT^{-1}) = \tilde{T} Y_-(A, B, C) \tilde{T}.$$

Furthermore, $A - B\tilde{B}X_+$ and $A - Y_-\tilde{C}C$ are stable and antistable respectively. Thus $K := \tilde{B}X_+$ and $L := Y_-\tilde{C}$ are appropriate state feedback and output injection matrices, which depend smoothly on (A, B, C). Now, with these choices of K and L, consider the rational matrix valued functions Γ and $\overline{\Gamma}$ introduced in the proof of Theorem 3.1. Then

$$\Gamma(z) = I + K\Phi D^{-1} = I + \tilde{B}X_+(zI - A)^{-1} B,$$

$$\overline{\Gamma}(z) = I + \overline{D}^{-1}\Psi L = I + C(zI - A)^{-1} Y_- \tilde{C}.$$

From the above, Γ and $\overline{\Gamma}$ are rational functions of McMillan degree n which vary smoothly with (A, B, C). Since Γ and $\overline{\Gamma}$ do not depend on the choice of realization (A, B, C) of G, they are actually smooth functions of $G \in \mathrm{Rat}(n, m, p)$. This shows that the decomposition of vector spaces (27) is actually smoothly varying with G. Thus it defines a smooth bundle isomorphism. This completes the proof. ■

4. THE TANGENT SPACE OF STABLE RATIONAL FUNCTIONS

In the following we specialize the computation of the tangent space to the manifold of stable rational transfer functions. Of course the previously obtained result, namely Theorem 3.1, holds in this case too. However we can reinterpret our result in terms of Douglas-Shapiro-Shields (DSS) factorizations and thus make contact with the results of Alpay, Baratchart, and Gombani (1994).

Recall [see Fuhrmann (1981), where a more general result is proved] that given $G \in \mathrm{RH}^\infty_+$—that is, G is a stable, proper rational $p \times m$ matrix valued function, where stability means analyticity and boundedness in the open left half plane—there exist representations

$$G = \overline{Q}^{-1}\overline{R} = RQ^{-1} \tag{28}$$

where $S, \overline{S}, Q, \overline{Q}$ are all in appropriate RH_-^∞ spaces with $Q, \overline{Q}$ inner functions, that is, satisfying $Q^*Q = I_m$ and $\overline{Q}^*\overline{Q} = I_p$. Moreover, these factorizations, to which we refer as DSS factorizations, can be taken to be coprime in the sense that there exist RH_-^∞ solutions to the Bezout equations

$$AH + BQ = I_m,$$

$$\overline{H}\overline{A} + \overline{Q}\overline{B} = I_p.$$

THEOREM 4.1. *Let $G \in RH_+^\infty$, i.e., it is a $p \times m$ real, stable, proper rational matrix function. Let $G = \overline{Q}^{-1}\overline{R} = RQ^{-1}$ be left and right coprime DSS factorizations of G. Then the tangent space of* $\text{Rat}^-(n, m, p)$ *of stable, real rational, strictly proper $p \times m$ transfer functions at G is given by*

$$\mathcal{T}_G = \overline{Q}^{-1}H(\overline{Q}, Q)Q^{-1}, \tag{29}$$

where $H(\overline{Q}, Q) = \{M \in RH_-^\infty | \overline{Q}^{-1}MQ^{-1} \in RH_+^\infty\}$.

Proof. Let $G = \overline{D}^{-1}\overline{N} = ND^{-1}$ be polynomial left and right coprime factorizations. Clearly $D, \overline{D}$ are stable, that is they have all their zeros in the open left half plane. Let us choose antistable polynomial matrices E and $\overline{E}$ so that DE^{-1} and $\overline{E}^{-1}\overline{D}$ are both inner functions in the open left half plane. For $Q = DE^{-1}$ to be inner E must be an antistable solution to the spectral factorization problem $E^*E = D^*D$. Similarly, $\overline{E}$ must be an antistable solution to the spectral factorization problem $\overline{E}\,\overline{E}^* = \overline{D}\,\overline{D}^*$. The inner functions $Q, \overline{Q}$ can be obtained also by use of state space methods. In fact, if $G(z) = C(zI - A)^{-1}B$ is a minimal realization, then

$$Q = \left(\begin{array}{c|c} A + BB^*X_+ & B \\ \hline B^*X_+ & I \end{array}\right)$$

where X_+ is the positive definite solution to the homogeneous Riccati equation $A^*X + XA + XBB^*X = 0$. In a completely analogous way we have

$$\overline{Q} = \left(\begin{array}{c|c} A + Y_+C^*C & Y_+C^* \\ \hline C & I \end{array}\right),$$

where Y_+ is the positive definite solution to the homogeneous Riccati equation $AY + YA^* + YC^*CY = 0$.

Now $\mathrm{Rat}^-(n, m, p)$ is an open subset of $\mathrm{Rat}(n, m, p)$, hence a submanifold. By Theorem 3.1 the tangent space at G is given by $\mathcal{T}_G = X^{\overline{D}\otimes D}$. So if $H \in X^{\overline{D}\otimes D}$ we can write

$$\begin{aligned} H &= \overline{D}^{-1}PD^{-1} = (\overline{D}^{-1}\overline{E})(\overline{E}^{-1}PE^{-1})(ED^{-1}) \\ &= (\overline{E}^{-1}\overline{D})^{-1}(\overline{E}^{-1}PE^{-1})(DE^{-1})^{-1} \\ &= \overline{Q}^{-1}MQ^{-1}. \end{aligned}$$

Here Q and $\overline{Q}$ are, by construction, inner functions in H^∞, whereas M is clearly real, proper rational, and stable. This shows that $X^{\overline{D}\otimes D} \subset \overline{Q}^{-1}H(\overline{Q}, Q)Q^{-1}$. Conversely, let $H \in \overline{Q}^{-1}H(\overline{Q}, Q)Q^{-1}$. Then $H = \overline{Q}^{-1}MQ^{-1}$, where $M \in H(\overline{Q}, Q)$. Since $H = (\overline{D}^{-1}\overline{E})M(ED^{-1})$ is analytic in the right half plane, so is $\overline{E}ME$. But M is analytic in the open left half plane, which shows that necessarily $P = \overline{E}ME$ is a polynomial matrix. This implies that $H = \overline{Q}^{-1}MQ^{-1} = \overline{D}^{-1}PD^{-1}$ and $\overline{Q}^{-1}H(\overline{Q}, Q)Q^{-1} \subset X^{\overline{D}\otimes D}$. Thus equality follows. ■

5. THE TANGENT SPACE OF RATIONAL INNER FUNCTIONS

We denote by $\mathrm{In}(n, m)$ the set of all $m \times m$ rational inner functions of McMillan degree n. Inner functions are generally of interest only up to a right constant unitary factor. We use this freedom to normalize the rational inner function G by requiring $G(\infty) = I$. Since inner functions are proper and not strictly proper, we cannot view them as embedded in $\mathrm{Rat}^-(n, m, m)$. However, as regarding the tangent space, the extra additive constant does not play a role. Since an $m \times m$ normalized inner function G with minimal realization (A, B, C, I) is completely determined by the similarity orbit of the reachable pair (A, B), it is clear from the work of Hazewinkel and Kalman (1976) and Helmke (1983) that $\mathrm{In}(n, m)$ can be considered as a smooth manifold of dimension nm. We will proceed with a concrete characterization of the tangent space. The result we obtain is very close to that of Alpay, Baratchart, and Gombani (1994).

In the proof of the principal result we shall make use of the following proposition, which may be of independent interest.

PROPOSITION 5.1. *Let G be an $m \times m$ normalized inner function G with minimal realization*

$$G = \left(\begin{array}{c|c} A & B \\ \hline C & I \end{array}\right).$$

Assume

$$G = ND^{-1} \tag{30}$$

is a polynomial right coprime factorization of G. Then:

1. *The expression*

$$G = N^{-*}D^{*} \tag{31}$$

is a left coprime factorization.

2. *The reachability and observability indices of this realization coincide.*

Proof. 1: Since G is inner, it satisfies $G = G^{-*}$. This implies

$$G = ND^{-1} = N^{-*}D^{*}, \tag{32}$$

and the last factorization is obviously left coprime.

2: we give two different proofs. The first is functional, whereas the second is state space oriented.

We assume G has the polynomial coprime factorization (32). Since $G(\infty) = I$, the column indices of D and N coincide. The column indices of D are the controllability indices of any minimal realization of G. On the other hand, using the left coprime factorization in (32), the row indices of N^*, which are the column indices of N, are the observability indices of any minimal realization of G.

Here we use the state space characterization of inner functions. Since G is inner we must have $C = -B^*X$, where X is the positive definite solution of the Riccati equation $A^*X + XA + XBB^*X = 0$. Since $G^* = G^{-1}$, we have

$$\left(\begin{array}{c|c} -A^* & XB \\ \hline B^* & I \end{array}\right) = \left(\begin{array}{c|c} A - BB^*X & B \\ \hline B^*X & I \end{array}\right).$$

From the isomorphism of these realizations we conclude that the observability indices of (A^*, B^*) and $(A + BB^*X, B^*X)$, and hence also of (A, B^*, X), coincide. So (A, B) and (A^*, XB) have the same reachability indices. But the reachability indices of (A^*, XB) are the observability indices of $(A, -B^*X)$, and the proof is complete. ■

We introduce another piece of notation. Given nonsingular $m \times m$ polynomial matrices D_1 and D_2, we denote by $X^{\text{asym}}_{D_1 \otimes D_2}$ the set of all $G \in X_{D_1 \in D_2}$ which are skew symmetric, i.e., satisfy $\tilde{G} = -G$. We set

$$X^{D_1 \otimes D_2}_{\text{asym}} = (D_1 \otimes D_2)^{-1} X^{\text{asym}}_{D_1 \otimes D_2} = D_1^{-1} X^{\text{asym}}_{D_1 \otimes D_2} D_2^{-1}.$$

THEOREM 5.1. *Let $G \in \mathrm{In}(n, m)$. Then*

1. *We have*

$$\dim X^{D_1 \otimes D_2}_{\text{asym}} = \text{nm}. \tag{33}$$

2. *The tangent space $\mathscr{T}$ of* $\mathrm{In}(n, m)$ *at $G \in \mathrm{In}(n, m)$ is $X^{D_1 \otimes D_2}_{\text{asym}}$.*

Proof. 1: Without loss of generality we assume

$$G = \left(\begin{array}{c|c} A & B \\ \hline -B^*X & I \end{array} \right)$$

where X is the positive definite solution of the homogeneous Riccati equation $A^*X + XA + XBB^*X = 0$. By Proposition 5.1, the reachability and observability of any minimal realization of G coincide. Denote the indices by $\kappa_1 \geqslant \cdots \geqslant \kappa_m$, and set $\Delta = \mathrm{diag}(z^{\kappa_1}, \dots, z^{\kappa_m})$. By an argument similar to one used in the proof of Proposition 2.1, it suffices to compute $\dim X^{\Delta_1 \otimes \Delta_2}_{\text{asym}}$.

An element in $X^{\Delta_1 \otimes \Delta_2}_{\text{asym}}$ is completely determined by its upper triangular part. If $G = (g_{ij})$ the constraints are

$$\deg g_{ij} < \kappa_i + \kappa_j,$$
$$g^*_{ij} = -g_{ji}.$$

This shows that the diagonal elements g_{ii} are odd polynomials of degree $< 2\kappa_i$. The dimension of such a space is clearly κ_i. Thus we have

$$\begin{aligned}
\dim X^{\Delta_1 \otimes \Delta_2}_{\text{asym}} = {} & \kappa_1 + (\kappa_1 + \kappa_2) + \cdots + (\kappa_1 + \kappa_m) \\
& + \kappa_2 + (\kappa_2 + \kappa_3) + \cdots + (\kappa_2 + \kappa_m) \\
& + \cdots \\
& + \kappa_{m-1} + (\kappa_{m-1} + \kappa_m) \\
& + \kappa_m \\
= {} & m(\kappa_1 + \cdots + \kappa_m) = nm.
\end{aligned}$$

2: As in the proof of Theorem 3.1, using $G = ND^{-1}$, we have

$$\begin{aligned}\dot{G} &= \dot{N}D^{-1} - ND^{-1}\dot{D}D^{-1}\\ &= \dot{N}D^{-1} - N^{-*}D^{*}\dot{D}D^{-1}\\ &= N^{-*}(N^{*}\dot{N} - D^{*}\dot{D})D^{-1}.\end{aligned}$$

On the other hand, starting from $G = N^{-*}D^{*}$ and using $(\dot{A})^{*} = \dot{A}^{*}$, we compute

$$\begin{aligned}\dot{G} &= N^{-*}\dot{D}^{*} - N^{-*}\dot{N}^{*}N^{-*}D^{*}\\ &= N^{-*}\dot{D}^{*} - N^{-*}\dot{N}^{*}ND^{-1}\\ &= N^{-*}(\dot{D}^{*}D - \dot{N}^{*}N)D^{-1}.\end{aligned}$$

Setting $S = N^{*}\dot{N} - D^{*}\dot{D}$, and comparing the two representations, we conclude that $S^{*} = -S$, that is S is a skew symmetric polynomial matrix in $X_{N^{*}\otimes D}$, or $S \in X^{\text{asym}}_{N^{*}\otimes D}$. Clearly $\dot{G} \in X^{N^{*}\otimes D}_{\text{asym}}$, and we have obtained the inclusion

$$T_G\,\mathrm{In}(n,m) \subset X^{N^{*}\otimes D}_{\text{asym}}.$$

As both spaces have dimension nm, we must have equality. ∎

6. THE TANGENT SPACE OF SYMMETRIC RATIONAL FUNCTIONS

We denote by $\mathrm{Rat}_{\text{sym}}(n, m)$ the space of all symmetric, $m \times m$ strictly proper real matrix functions of McMillan degree n. $\mathrm{Rat}_{\text{sym}}(n, m)$ is a manifold of dimension $n(m + 1)$. We proceed to compute its tangent space.

Given a nonsingular polynomial matrix D, we define the symmetric tensored polynomial model $X^{\text{sym}}_{\tilde{D}\otimes D}$ by

$$X^{\text{sym}}_{\tilde{D}\otimes D} = \left\{P \in X_{\tilde{D}\otimes D} \,\middle|\, \tilde{P} = P\right\}. \tag{34}$$

Here $\tilde{A}$ denotes the transpose of A. The symmetric rotational model $X^{\tilde{D}\otimes D}_{\text{sym}}$ is similarly defined by

$$X^{\tilde{D}\otimes D}_{\text{sym}} = \left\{H \in X^{\tilde{D}\otimes D} \,\middle|\, \tilde{H} = H\right\}. \tag{35}$$

THEOREM 6.1. *Let* $G \in \mathrm{Rat}_{\mathrm{sym}}(n, m)$, *and let*

$$G = ND^{-1} = \tilde{D}^{-1}\tilde{N}$$

be coprime factorizations of G. Then:

1. *We have*

$$\dim X_{\mathrm{sym}}^{\tilde{D} \otimes D} = n(m+1). \tag{36}$$

2. *The tangent space* $\mathcal{T}$ *of* $\mathrm{Rat}_{\mathrm{sym}}(n, m)$ *at G is equal to* $X_{\mathrm{sym}}^{\tilde{D} \otimes D}$.

Proof. 1: The spaces $X_{\mathrm{sym}}^{\tilde{D} \otimes D}$ and $X_{\tilde{D} \otimes D}^{\mathrm{sym}}$ have the same dimension. Thus it suffices to compute the dimension of the second space. We assume without loss of generality that D is column proper with column indices $\kappa_1 \geqslant \cdots \geqslant \kappa_m$. The space $X_{\tilde{D} \otimes D}^{\mathrm{sym}}$ is completely determined by the upper triangular part of its elements. Counting dimensions by rows, we get

$$\begin{aligned} \dim X_{\tilde{D} \otimes D}^{\mathrm{sym}} = {} & \left[(\kappa_1 + \kappa_1) + \cdots + (\kappa_1 + \kappa_m)\right] \\ & + \left[(\kappa_2 + \kappa_2) + \cdots = (\kappa_2 + \kappa_m)\right] \\ & + \left[(\kappa_m + \kappa_m)\right] \\ = {} & (m+1)(\kappa_1 + \cdots + \kappa_m) = n(m+1). \end{aligned}$$

2: As in the proof of Theorem 3.1, using $ND^{-1} = \tilde{D}^{-1}\tilde{N}$, we have

$$\begin{aligned} \dot{G} &= \dot{N}D^{-1} - ND^{-1}\dot{D}D^{-1} \\ &= \dot{N}D^{-1} - \tilde{D}^{-1}\tilde{N}\dot{D}D^{-1} \\ &= \tilde{D}^{-1}\left(\tilde{D}\dot{N} - \tilde{N}\dot{D}\right)D^{-1}. \end{aligned}$$

In an analogous way, we obtain $\dot{G} = \tilde{D}^{-1}(\dot{\tilde{N}}D - \dot{\tilde{D}}N)D^{-1}$, and clearly, as $\dot{\tilde{N}} = \tilde{\dot{N}}$ and $\dot{\tilde{D}} = \tilde{\dot{D}}$, it follows that $\tilde{D}\dot{N} - \tilde{N}\dot{D}$ is a symmetric polynomial matrix. Thus we get

$$T_G \, \mathrm{Rat}_{\mathrm{sym}}(n, m) \subset X_{\mathrm{sym}}^{\tilde{D} \otimes D}.$$

We know that $\dim T_G \mathrm{Rat}_{\mathrm{sym}}(n, m) = n(m+1)$; there the equality

$$T_G \ \mathrm{Rat}_{\mathrm{sym}}(n, m) = X_{\mathrm{sym}}^{\tilde{D} \otimes D}$$

follows. ■

7. RIEMANNIAN METRICS AND SYMPLECTIC STRUCTURES

A *Riemannian metric* on a smooth manifold M is a family of inner products $\langle \ , \ \rangle_x$, $x \in M$, defined on each tangent space $T_x M$ such that $\langle \ , \ \rangle_x$ varies smoothly with $x \in M$. Thus a Riemannian metric is a smooth section, $s : M \to \mathrm{Bil}(M)$, of the bundle of bilinear forms such that, for each $x \in M$, $s(x)$ is a positive definite symmetric bilinear form. By a partition of unity argument such Riemannian metrics exist on every smooth manifold.

Similarly, a *symplectic structure* on a smooth manifold M is a closed, nondegenerate 2-form ω on M. Thus ω defines for each $x \in M$ a nondegenerate symplectic form $\omega : T_x M \times T_x M \to \mathbf{R}$ such that $d\omega = 0$. Thus, symplectic structures exist only on even dimensional manifolds.

In the sequel we will occasionally ignore the integrability condition $d\omega = 0$ for a symplectic structure. We will then refer to this as a *formal symplectic structure*.

7.1. Riemannian Metrics

To construct a Riemannian metric on $\mathrm{Rat}(n, m, p)$ we have to specify an inner product on each tangent space. Let X_+ and Y_- denote the stabilizing and antistabilizing solutions of the algebra Riccati equations

$$A^* X_+ + X_+ A - K_+ BB^* X_+ + C^* C = 0,$$
$$AY_- + Y_- A^* + BB^* Y_- - Y_- C^* C Y_- = 0.$$

Thus $X_+ = X_+^* > 0$ and $Y_- = Y_+^* < 0$. Let $Y_+ := -Y_- > 0$. Thus $Y_+ = X_+^{-1}$. By Theorem 3.1, the tangent space $T_G \ \mathrm{Rat}(n, m, p)$ decomposes as

$$T_G \ \mathrm{Rat}(n, m, p) = \mathscr{U}_1 \oplus \mathscr{U}_2,$$

where $\mathscr{U}_1 = \{\bar{\Gamma} L \Phi D^{-1} \mid L \in \mathbf{R}^{p \times n}\}$ and $\mathscr{U}_2 = \{\bar{D}^{-1} \Psi K \Gamma \mid K \in \mathbf{R}^{n \times m}\}$. We define

$$\begin{aligned} &\langle \bar{\Gamma} S_1 \Phi D^{-1} + \bar{D}^{-1} \Psi T_1 T, \bar{\Gamma} S_2 \Phi D^{-1} + \bar{D}^{-1} \Psi T_2 \Gamma \rangle \\ &\quad := \mathrm{Tr}\left(S_1 X_+^{-1} \tilde{S}_2\right) + \mathrm{tr}\left(\tilde{T}_1 X_+ T_2\right). \end{aligned} \tag{37}$$

Obviously, this defines a positive definite inner product on the tangent space TG Rat(n, m, p). Since X_+ depends smoothly on the realizations (A, B, C), this inner product depends smoothly on $G \in$ Rat(n, m, p). Thus it defines a Riemannian metric. This metric coincides with the Riemannian metric proposed by Delchamps (1986). We will refer to this as the *Riccati based Riemannian metric* on Rat(n, m, p). Note that the subspaces $\mathscr{U}_1$ and $\mathscr{U}_2$ are orthogonal to each other with respect to this Riemannian metric.

7.2. *Symplectic Structure*

We show now that, for the case $m = p$, each tangent space of Rat(n, m, p) can be endowed with a nondegenerate symplectic form. Since $\dim$ Rat$(n, m, m) = 2nm$, the space Rat(n, m, m) is even dimensional. Thus, in this case, there is no dimensional obstruction to the existence of a symplectic structure.

To define a formal symplectic structure on Rat(n, m, m) we once again consider the direct sum decomposition (21) of T_G Rat(n, m, m). We then define, for $m = p$,

$$\langle \bar{\Gamma} S_1 \Phi D^{-1} + \bar{D}^{-1} \Psi T_1 T, \bar{\Gamma} S_2 \Phi D^{-1} + \bar{D}^{-1} \Psi T_2 \Gamma \rangle := \operatorname{tr}(S_1 T_2 - S_2 T_1). \tag{38}$$

Note that this yields a well-defined, nondegenerate bilinear form on T_G Rat(n, m, m) which is alternating. Thus we have a symplectic structure on T_G Rat(n, m, m) which varies smoothly with $G \in$ Rat(n, m, m). We refer to this as the *constant symplectic structure* on Rat(n, m, m). Note that we have not verified that this actually defines a symplectic structure, i.e. that the integrability condition $d\omega = 0$ holds.

7.3. *Symplectic Structure on* Rat(n)

A problem with the above formal symplectic structures on Rat(n, m, m) is that it is difficult to compute them more explicitly. Thus, in this subsection we explore some possibilities to define a symplectic manifold structure on the set Rat(n) of scalar real rational transfer functions. This makes contact with the work of Brockett and Faybusovich [1991] as well as Atiyah and Hitchin [1988]. In these papers a certain symplectic manifold structure on Rat(n) is defined, which has a resemblance to the symplectic structure (55) defined below. As we shall see, Bezoutians play a significant role in defining symplectic structures.

There are at least two different ways to define a symplectic structure on Rat(n). The first construction uses the fact that the tangent bundle T Rat(n) is trivial. The construction goes as follows.

7.3.1. Trivialization Method. Consider the smooth trivialization of the tangent bundle

$$T\,\mathrm{Rat}(n) = \bigcup_{g \in \mathrm{Rat}(n)} \{g\} \times T_g\,\mathrm{Rat}(n)$$

via

$$\tau : \mathrm{Rat}(n) \times \mathbf{R}^{2n} \to T\,\mathrm{Rat}(n)$$

$$\tau(g, [\tau]) := (g, \pi/d^2).$$

Here $[\pi] = (\pi_0, \dots, \pi_{2n-1})^T$ denotes the coefficient vector of any polynomial π of degree $\leqslant 2n - 1$. It is easily seen that τ defines a diffeomorphism which maps each fiber $\{g\} \times \mathbf{R}^{2n}$ of the projection map $\mathrm{pr}_1 : \mathrm{Rat}(n) \times \mathbf{R}^{2n} \to \mathrm{Rat}(n)$ linearly isomorphically onto $\{g\} \times T_g$ Rat(n). Using this trivialization of the tangent bundle, it follows that every formal symplectic structure on Rat(n) is uniquely given by a smooth map

$$\Omega : \mathrm{Rat}(n) \to \Lambda^2(\mathbf{R}^{2n})$$

of Rat(n) into the linear vector space of $2n \times 2n$ skew symmetric matrices. Actually, in order to define a formal symplectic structure, we have to impose the additional nondegeneracy condition that $\Omega(g)$ is invertible for all $g \in$ Rat(n). Given such a smooth map $\Omega : \mathrm{Rat}(n) \to \Lambda^2(\mathbf{R}^{2n})$, satisfying the nondegeneracy condition, we define a symplectic structure on Rat(n) as

$$\left\langle \frac{\pi_1}{d^2}, \frac{\pi_2}{d^2} \right\rangle := [\pi_1]^T \Omega(g)[\pi_2]. \tag{39}$$

It is immediately checked that this defines a nondegenerate, alternating bilinear form on T_gRat(n) and thus a formal symplectic structure on Rat(n). Conversely, every formal symplectic structure on Rat(n) is of this form.

7.3.2. Diffeomorphism Approach. This approach depends on the fact that $\mathrm{Rat}(n)$ can be identified with an open subset of $\mathbf{R}^{2n}$. In this way a splitting of the tangent bundle

$$T\,\mathrm{Rat}(n) \simeq X \oplus X^*$$

is induced from any decomposition $\mathbf{R}^{2n} \simeq \mathbf{R}^n \times \mathbf{R}^n$. Explicitly, consider the diffeomorphism

$$f : \mathrm{Rat}(n) \to \mathbf{R}^n \times \mathbf{R}^n,$$

$$\frac{n}{d} \mapsto \begin{pmatrix} [n] \\ [d] \end{pmatrix}$$

which assigns to every rational function $g = n/d$ the coefficient vectors $[n], [d]$ of n and d. Thus for $n(z) = \sum_{j=0}^{n-1} n_j z^j$, $d(z) = \sum_{j=0}^{n} d_j z^j$, $d_n = 1$, we have $f = (f_1, f_2)$ with

$$f_1\left(\frac{n}{d}\right) := [n] = (n_1, \dots, n_{n-1})^T,$$

$$f_1\left(\frac{n}{d}\right) := [d] = (d_0, \dots, d_{n-1})^T.$$

Obviously the image $M = f(\mathrm{Rat}(n))$ of f is an open subset of $\mathbf{R}^{2n}$. To compute the derivative of the map

$$T_g f : T_g\,\mathrm{Rat}(n) \to \mathbf{R}^{2n}$$

at a tangent vector $\pi/d^2 \in T_g\,\mathrm{Rat}(n)$, we choose any smooth curve $t \mapsto n(t)/d(t) \in \mathrm{Rat}(n)$ such that

$$\frac{n(0)}{d(0)} = \frac{n}{d} = g, \qquad \text{(i)}$$

$$\frac{\dot{n}d - n\dot{d}}{d^2} = \frac{n'(0)d(0) - n(0)d'(0)}{d(0)^2} = \frac{\pi}{d^2}. \qquad \text{(ii)}$$

Thus $n(t)/d(t) \in \mathrm{Rat}(n)$ is chosen such that its derivation at $t = 0$ coincides with π/d^2. Such curves clearly exist. Then we compute

$$\begin{aligned}
T_g f_1\left(\frac{\pi}{d^2}\right) &= T_g f_1\left(\frac{\dot{n}d - n\dot{d}}{d^2}\right) \\
&= \frac{d}{dt} f_1\left(\frac{n(t)}{d(t)}\right)\bigg|_{t=0} \\
&= \frac{d}{dt}[n(t)]\bigg|_{t=0} = \left[\frac{d}{dt}n(t)\right]\bigg|_{t=0} \\
&= [\dot{n}], \\
T_g f_2\left(\frac{\pi}{d^2}\right) &= T_g f_2\left(\frac{\dot{n}d - n\dot{d}}{d^2}\right) \\
&= \frac{d}{dt} f_2\left(\frac{n(t)}{d(t)}\right)\bigg|_{t=0} \\
&= \frac{d}{dt}[d(t)]\bigg|_{t=0} = \left[\frac{d}{dt}d(t)\right]\bigg|_{t=0} \\
&= [\dot{d}],
\end{aligned}$$

Thus the derivative $T_g f : T_g \mathrm{Rat}(n) \to \mathbf{R}^{2n}$ assigns to each tangent vector π/d^2 the pair of coefficient vectors $([\dot{n}], [\dot{d}])$ of the unique polynomials $\dot{n}$, $\dot{d}$ of degree $< n$ such that $\pi = \dot{n}d - n\dot{d}$ holds. Equivalently,

$$T_g f\left(\frac{\pi}{d^2}\right) = \begin{pmatrix} [\dot{n}] \\ [\dot{d}] \end{pmatrix} = \mathrm{Res}(d, -n)^{-1}[\pi].$$

Pick any smooth map $\Omega : M \to \Lambda^2(\mathbf{R}^{2n})$ such that, for any $g \in M$, $\Omega(g)$ defines a nondegenerate, alternating bilinear form on $\mathbf{R}^{2n}$. Thus Ω defines a formal symplectic structure on the open subset $M \subset \mathbf{R}^{2n}$. We can pull back this structure, via f, to a formal symplectic structure on $\mathrm{Rat}(n)$. Explicitly, define

$$\left\langle \frac{\pi_1}{d^2}, \frac{\pi_2}{d^2} \right\rangle := [\pi_1]^T \widetilde{\mathrm{Res}(d, -n)}^{-1} \Omega(g) \mathrm{Res}(d, -n)^{-1} [\pi_2].$$

Again this defines a formal symplectic structure on Rat(n). For example, if

$$\Omega(g) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad g \in \mathrm{Rat}(n),$$

then we obtain the induced symplectic structure on Rat(n) given by

$$\widetilde{\mathrm{Res}(d, -n)}^{-1} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \mathrm{Res}(d, -n)^{-1},$$

Now, using wedge product notation, the skew symmetric matrix

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

corresponds to the 2-form

$$\sum_{i=1}^{n} dp_i \wedge dq_i,$$

while $\begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}$, with J given by (50), corresponds to

$$\sum_{i=1}^{n} dp_i \wedge dq_{n-i+1}.$$

In general, every $2n \times 2n$ skew symmetric matrix

$$\Omega = \begin{pmatrix} A & B \\ -B^* & C^* \end{pmatrix}, \qquad A^* = -A, \quad C^* = -C,$$

we have the associated 2-form

$$\sum_{i<j}^{n} a_{i,j}(p, q)\, dp_i \wedge dp_j + \sum_{i=1}^{n} \sum_{j=1}^{n} b_{i,j}(p, q)\, dp_i \wedge dq_j$$
$$+ \sum_{i<j}^{n} c_{i,j}(p, q)\, dq_i \wedge dq_j.$$

So it is easy to change language, if necessary.

7.3.3. *Symplectic Structures and Bezoutians.* Here we show how to construct an honest symplectic structure (i.e. one satisfying the closedness condition $d\omega = 0$) on Rat(n) via Bezoutians. Let $g = n/d \in \mathrm{Rat}(n)$ denote a strictly proper transfer function of McMillan degree n. Let

$$X^d := \left\{ \frac{\pi}{d} \in \mathbf{R}(z) \middle| \deg \pi < n \right\} \tag{40}$$

denote the rational model space associated with the monic polynomial d. Thus X^d is a real vector space of dimension n. From Theorem 3.1 we know that the tangent space of Rat(n) at g is T_g Rat(n) $= X^{d^2}$. We have already seen how to split the tangent space X^{d^2} into a direct sum of two n-dimensional subspaces. Here we give a different, more direct, construction.

PROPOSITION 7.1. *For any $g \in$ Rat(n) there is a direct sum decomposition of the tangent space T_g Rat(n) as*

$$T_g \operatorname{Rat}(n) = X^d \oplus g \cdot X^d. \tag{41}$$

Proof. Every tangent vector of T_g Rat(n) is of the form π/d^2 for a unique polynomial $\pi \in \mathbf{R}[z]$ of degree $< 2n$. By coprimeness of n and d there exist unique polynomials $a_\pi, b_\pi \in \mathbf{R}[z]$, both of degree $< n$, with

$$\pi = a_\pi n + b_\pi d. \tag{42}$$

Thus

$$\frac{\pi}{d^2} = \frac{b_\pi}{d} + g\frac{a_\pi}{d^2} \in X^d \oplus gX^d. \tag{43}$$

Conversely, if $a/d, b/d \in X^d$ then $(an + bd)/d^2 \in X^{d^2}$. The result follows. ∎

Actually, as the polynomials a_π, b_π appearing in the above proof vary continuously with π, the splitting of T_g Rat(n) into subspaces X^d and gX^d is continuous in $g \in$ Rat(n). Thus this decomposition defines a topological splitting of the tangent bundle T Rat(n) into subbundles. There is a natural symplectic structure associated with this tangent bundle decomposition. This is where the Bezoutian enters the stage. By coprimeness of n and d, for each pair of polynomials $\pi_1, \pi_2 \in \mathbf{R}[z]$ of degree $< 2n$, there exist polynomials a_i, b_i of degree $< n$ such that

$$\pi_i = a_i d - b_i n \tag{44}$$

for $i = 1, 2$.

Let Res(d, n) denote the resultant matrix. Thus Res(d, n) is a $2n \times 2n$ matrix of the form

$$
\mathrm{Res}(d, n) = \begin{pmatrix}
d_0 & & & & & n_0 & & & & \\
\cdot & \cdot & & & & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot & & \\
\cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \\
d_m & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & n_0 \\
 & \cdot & \cdot & \cdot & d_0 & n_n & \cdot & \cdot & \cdot & \cdot \\
 & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\
 & & & \cdot & \cdot & & & \cdot & \cdot & \cdot \\
 & & & & \cdot & & & & \cdot & \cdot \\
 & & & & d_m & & & & & n_n
\end{pmatrix}. \tag{45}
$$

For any polynomial $a(z) = \sum_{i=0}^{k} a_i z^i$ let $[a] := (a_0, \dots, a_k)^T \in \mathbf{R}^{k+1}$ denote the vector of coefficients. Then from (44) we obtain

$$
[\pi_i] = \mathrm{Res}(d, -n)\begin{pmatrix} [a_i] \\ [b_i] \end{pmatrix}, \qquad i = 1, 2. \tag{46}
$$

Therefore, for $i = 1, 2$

$$
\begin{pmatrix} [a_i] \\ [b_i] \end{pmatrix} = \mathrm{Res}(d, -n)^{-1}[\pi_i]. \tag{47}
$$

Let $J \in \mathbf{R}^{n \times n}$ denote an arbitrary invertible matrix. Then

$$
\Omega_J := \mathrm{Res}\widetilde{(d, -n)}^{-1}\begin{pmatrix} 0 & J \\ -J^T & 0 \end{pmatrix}\mathrm{Res}(d, -n)^{-1} \tag{48}
$$

is an invertible $2n \times 2n$ skew symmetric matrix. Thus

$$
\left\langle \frac{\pi_1}{d^2}, \frac{\pi_2}{d^2} \right\rangle := [\pi_1]^T \Omega_J [\pi_2] = [a_1]^T J [b_2] - [b_1]^T J [a_2] \tag{49}
$$

defines a nondegenerate, alternating bilinear form on X^{d^2} and therefore a symplectic structure on the tangent space.

By specializing the $n \times n$ matrix J, various choices of symplectic structures on T_g Rat(n) are obtained. To connect up with Bezoutians we choose

$$J = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{50}$$

Using a result of Kravitsky (1980) (see also Helmke and Fuhrmann, 1989), we have

$$\operatorname{Res}(d,n)\begin{pmatrix} 0 & J \\ -J^T & 0 \end{pmatrix}\widetilde{\operatorname{Res}(d,n)} = \begin{pmatrix} 0 & B(d,n) \\ -B(d,n) & 0 \end{pmatrix}, \tag{51}$$

where $B(d, n)$ denotes the Bezoutian (see Helmke and Fuhrmann, 1989). Therefore, in this case

$$\Omega_J = \begin{pmatrix} 0 & B(n,d)^{-1} \\ -B(n,d)^{-1} & 0 \end{pmatrix} \tag{52}$$

and the above symplectic structure on T_g Rat(n) is

$$\left\langle \frac{\pi_1}{d^2}, \frac{\pi_2}{d^2} \right\rangle := [\pi_1]^T \begin{pmatrix} 0 & B(n,d)^{-1} \\ -B(n,d)^{-1} & 0 \end{pmatrix} [\pi_2]. \tag{53}$$

Note that the inverse of a Bezoutian is a Hankel matrix. Explicitly

$$B(d,n)^{-1} = H_n\left(\frac{a}{d}\right),$$

where $H_n(a/d)$ denotes the $n \times n$ Hankel matrix associated with the rational function a/d. Here a is the uniquely determined solution of the Bezout equation $an + bd = 1$ which has degree $< n$. Thus we could also express the symplectic form as

$$\left\langle \frac{\pi_1}{d^2}, \frac{\pi_2}{d^2} \right\rangle := [\pi_1]^T \begin{pmatrix} 0 & H_n(a/d) \\ -H_n(a/d) & 0 \end{pmatrix} [\pi_2]. \tag{54}$$

We can now state and prove our main result of this section.

THEOREM 7.1. *The $2n \times 2n$ Bezoutian type matrix*

$$\Omega_1 = \begin{pmatrix} 0 & B(n,d)^{-1} \\ -B(n,d)^{-1} & 0 \end{pmatrix} \tag{55}$$

defines a symplectic structure on the tangent spaces T_g Rat(n), $g = n/d$. Moreover, it defines an exact, and hence closed, nondegenerate differential 2-form on Rat(n).

Proof. We have already seen that Ω_1 defines a nondegenerate symplectic structure on T_g Rat(n) and thus defines a nondegenerate differential 2-form ω_1 on Rat(n). It remains to show that ω_1 is exact, that is, $\omega_1 = d\alpha$, for a smooth 1-form α on Rat(n). Let $p_i : \text{Rat}(n) \to \mathbf{R}$, $q_j : \text{Rat}(n) \to \mathbf{R}$, where

$$\begin{aligned} p_i(n/d) &:= \text{coefficient of } z^{i-1} \text{ in } n(z), \\ q_i(n/d) &:= \text{coefficient of } z^{j-1} \text{ in } d(z) \end{aligned} \tag{56}$$

for $i, j = 1, \dots, n$. By differentiation of these functions we obtain the smooth 1-forms $dp_i : T_g \text{ Rat}(n) \to \mathbf{R}$ defined by

$$dp_i\left(\frac{z^{j-1}}{d}\right) = \begin{cases} 1, & j = i, \\ 0, & j \neq i, \end{cases} \tag{57}$$

and $dq_i : T_g \text{ Rat}(n) \to \mathbf{R}$ defined by

$$dq_i\left(\frac{z^{j-1}n}{d}\right) = \begin{cases} 1, & j = i, \\ 0, & j \neq i. \end{cases} \tag{58}$$

Thus $dp_1, \dots, dp_n, dq_1, \dots, dq_n$ form a basis of smooth exact 1-forms on Rat(n) which is dual to the basis $\{d^{-1}, zd^{-1}, \dots, z^{n-1}d^{-1}, nd^{-2}, nzd^{-2}, \dots, nz^{n-1}d^{-2}\}$ of the tangent space.

We can decompose any tangent vectors $\xi, \eta \in T_g$ Rat(n) with respect to the above basis of T_g Rat(n) as

$$\xi = \sum_{i=1}^{n} \xi_i^{(1)} \frac{z^{i-1}}{d} - \sum_{j=1}^{n} \xi_j^{(2)} \frac{z^{i-1}n}{d^2},$$

$$\eta = \sum_{i=1}^{n} \eta_i^{(1)} \frac{z^{i-1}}{d} - \sum_{j=1}^{n} \eta_j^{(2)} \frac{z^{i-1}n}{d^2}.$$

Then, for $i \leqslant j$, the wedge product of dp_i and dq_j is

$$\begin{aligned} dp_i \wedge dq_j(\xi, \eta) &= dp_i(\xi)\, dq_j(\eta) - dp_i(\eta)\, dq_j(\xi) \\ &= \xi_i^{(1)}\eta_j^{(2)} - \xi_j^{(2)}\eta_i^{(1)}. \end{aligned}$$

On the other hand, by the definition of the symplectic form Ω_1, we have

$$\Omega_1(\xi, \eta) = [a_1]^T J [b_2] - [b_1]^T J [a_2],$$

with

$$[a_1] = \begin{pmatrix} \xi_1^{(1)} \\ \vdots \\ \xi_n^{(1)} \end{pmatrix}, \qquad [b_1] = \begin{pmatrix} \xi_1^{(2)} \\ \vdots \\ \xi_n^{(2)} \end{pmatrix},$$

$$[a_2] = \begin{pmatrix} \eta_1^{(1)} \\ \vdots \\ \eta_n^{(1)} \end{pmatrix}, \qquad [b_1] = \begin{pmatrix} \eta_1^{(2)} \\ \vdots \\ \eta_n^{(2)} \end{pmatrix},$$

where

$$\xi = \frac{a_1}{d} - \frac{b_1 n}{d^2}, \qquad \eta = \frac{a_2}{d} - \frac{b_2 n}{d^2}.$$

Thus

$$[a_1]^T J[b_2] - [b_1]^T J[a_2] = \sum_{i=1}^{n} \left(\xi_i^{(1)} \eta_{n-i+1}^{(2)} - \xi_i^{(2)} \eta_{n-i+1}^{(1)} \right)$$
$$= \sum_{i=1}^{n} dp_i \wedge dq_{n-j+1}(\xi, \eta).$$

Therefore

$$\Omega_1 = \sum_{i=1}^{n} dp_i \wedge dq_{n-j+1}$$
$$= \sum_{i=1}^{n} d(p_i \, dq_{n-j+1}) = d\alpha \tag{59}$$

is an exact 2-form on Rat(n) with $\alpha = \sum_{i=1}^{n} p_i \, dq_{n-j+1}$. ■

Of course, we can also consider other choices of symplectic forms on Rat(n). Whether or not these symplectic structures actually define closed differential 2-forms has to be seen in each case individually. A number of such possible choices for symplectic forms are

$$\Omega_1 = \begin{pmatrix} 0 & B(n,d)^{-1} \\ -B(n,d)^{-1} & 0 \end{pmatrix},$$

$$\Omega_2 = \begin{pmatrix} 0 & B(n,d) \\ -B(n,d) & 0 \end{pmatrix},$$

$$\Omega_3 = \begin{pmatrix} 0 & H(n/d) \\ -H(n/d) & 0 \end{pmatrix},$$

$$\Omega_4 = \begin{pmatrix} 0 & B(a,d) \\ -B(a,d) & 0 \end{pmatrix},$$

$$\Omega_5 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Of course, all these choices define nondegenerate alternating bilinear forms on each tangent space. Moreover, they vary smoothly with $g \in$ Rat(n). The problem remains to check if they yield all closed 2-forms on Rat(n).

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Received 18 October 1995; final manuscript accepted 24 March 1997