Algebraic System Theory: An Analyst's Point of View

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ABSTRACT: *A systematic development of the realization theory of finite dimensional constant linear systems is presented which synthesizes the various currently available approaches.*

Based on representation theorems for submodules and quotient modules of spaces of polynomial matrices and vectors, this paper combines the abstract algebraic ideas centering around module theory, the use of coprime factorizations of rational transfer functions and state space equations into a unified theory.

I. Introduction

The relevance of algebraic ideas in the study of linear systems has been recognized and stressed by Kalman in various publications culminating in his excellent exposition in Part Four of Ref. (**8**). This paper is the result of a re-reading of that part. No doubt this is one of the finest achievements in Mathematical System Theory and it seems that Kalman's recognition of the module structure as the basic structure in linear system theory is bound eventually to become the standard way of exposition for the subject. (Personally this author owes a great deal to this book as the first reading of it aroused his interest in System Theory.) What is there to add to Kalman's exposition which could throw some more light on a well-documented subject? It seems that there is an omission in one important direction inasmuch as not sufficient contact is made with Rosenbrock's approach to linear system theory (**1**). In fact the coprime factorizations playing such an important role in (**1**) do not appear at all in (**3**). The aim of this paper is to produce an approach which would synthesize the algebraic approach of Kalman, the state space approach as well as the polynomial matrix methods of Rosenbrock.

The need for such a synthesis has been recognized by Eckberg in his doctoral thesis (**2**) and this paper has a lot in common with the ideas introduced there. The differences are mainly in that Eckberg emphasizes uniqueness via a choice of canonical matrices whereas we take a more abstract and coordinate free route whenever possible. We are left with uniqueness modulo similarity in line with the philosophy of the state space isomorphism theorem.

A word of justification for the choice of title is in order. Indeed, it seems presumptuous for a self-declared analyst to try to add on an essentially algebraic subject. The reason for such an attempt is based on the author's experiences and work on infinite dimensional linear systems [see (**5**)], and the further references therein. The techniques of operator theory used in that context are borrowed mostly from the theory of invariant subspaces (**6**, **14**) and are imbued with algebraic ideas and concepts. In fact for the person working in the field of infinite dimensional linear systems there are ready mathematical tools, like the Beurling–Lax representation theorem for invariant subspaces, the Sz.-Nagy–Foias lifting theorem, the spectral analysis of shift intertwining operators (**8**, **4**, **15**) to quote some, for which no equivalent can be readily found in the standard algebraic literature. This gives the operator theorist a certain advantage when applying himself to system theory problems. In this paper, we hope to apply the methods of operator theory in a purely algebraic context.

The structure of this paper is as follows: In Section II we discuss polynomial matrices, coprimeness and ideal structure in the ring of polynomial matrices. In Section III we obtain representations of submodules and quotient modules of the module of vector polynomials. The next section is devoted to the study of canonical models being restricted shifts in quotient modules. An important role is played by the determination of all $F[\lambda]$-module homomorphisms between canonical models and criteria of their invertibility. Section V treats the numerator denominator representation of transfer functions. The following section is devoted to realization theory and we devote the last section to an abstract generalization of the resultant theorem.

II. Polynomial Matrices and Coprimeness

Let F be an arbitrary field. We denote by $F[\lambda]$ the ring of polynomials over the field F. As a consequence of the Euclidean division algorithm $F[\lambda]$ is a principal ideal domain (**9**, **11**). F^n denotes the vector space of all n-tuples of elements in F with the usual definition of algebraic operations. By $F^n[\lambda]$ we denote the vector space, over F, of all vector polynomials with coefficients in F^n. $F^n[\lambda]$ is clearly isomorphic to the set of all n-tuples with $F[\lambda]$ coordinates. We will make no distinction between the two representations of $F^n[\lambda]$ and use them interchangeably. $F^n[\lambda]$ is clearly a module over the commutative ring $F[\lambda]$.

Similarly $F^{m\times n}$ will denote the vector space of all $m\times n$ matrices with entries from F and $F^{m\times n}[\lambda]$ the vector space of all matrix polynomials with $F^{m\times n}$ coefficients. Again there is an isomorphism between the matrix polynomials with $F^{m\times n}$ coefficients and the vector space of all $m\times n$ polynomial matrices, i.e. matrices with $F[\lambda]$ entries. Again it is clear that $F^{m\times n}[\lambda]$ is a module over the ring $F[\lambda]$. For $m = n$ the vector space $F^{n\times n}[\lambda]$ is actually a ring, noncommutative when $n > 1$. Furthermore, $F^{m\times n}[\lambda]$ is a left module over $F^{m\times m}[\lambda]$ and a right module over $F^{n\times n}[\lambda]$.

Since $F[\lambda]$ is an entire ring, i.e. has no zero divisors, then we can embed $F[\lambda]$ in its field of quotients, the field of all rational functions. We denote the field of quotients of $F[\lambda]$ by $F(\lambda)$. An element f of $F(\lambda)$ can be written as a quotient $f = p/q$ of polynomials. f is called a proper rational function if $\deg p < \deg q$. This is clearly independent of the representative used. Analogously we denote by $F^{m\times n}(\lambda)$ the set of all $m \times n$ matrix rational functions, i.e. the set of all $m \times n$ matrices with $F(\lambda)$ entries. $W \in F^{m\times n}(\lambda)$ is called a proper rational matrix function if all its elements are proper rational functions.

Next we introduce some terminology. Our basic reference for the following material is (**10**).

An element $U \in F^{n\times n}[\lambda]$ is called a unimodular matrix if $\det U$ is a nonzero element of F. This is equivalent to the existence of $V \in F^{n\times n}[\lambda]$ such that $UV = VU = I$. Let A, B, C be polynomial matrices such that $A = BC$ then C is called a right divisor of A and B a left divisor of A. Similarly we say A is a left multiple of C and a right multiple of B. A greatest common right divisor of two polynomial matrices A and B is a common right divisor of A and B which is a left multiple of any other common right divisor of A and B.

Our primary object is to show that the ring $F^{n\times n}[\lambda]$ is a principal ideal ring, i.e. that every one-sided ideal is generated by a single element. The basic result needed is provided by the following theorem quoted from (**10**).

Theorem 2.1

Every two polynomial matrices A and B have a greatest common right divisor D which can be expressed as

$$D = PA + QB \tag{2.1}$$

for some polynomial matrices P and Q.

Corollary 2.2

If two polynomial matrices A and B have a nonsingular greatest common right divisor D then every other greatest common right divisor is given by UD for some unimodular polynomial matrix U.

The last results can be easily generalized to the case of p polynomial matrices. Thus any p polynomial matrices A_i have a greatest common right divisor D which is expressible in the form $D = \sum_{i=1}^{p} P_i A_i$.

A set of polynomial matrices $A_1, \ldots, A_p$ is said to be right coprime if they do not have a nontrivial greatest common right divisor. Thus we clearly have the following (**1**).

Theorem 2.3

The polynomial matrices $A_1, \ldots, A_p$ are right coprime if and only if there exists polynomial matrices $B_1, \ldots, B_p$ such that

$$\sum_{i=1}^{p} P_i A_i = I. \tag{2.2}$$

Let us now consider a right ideal J in $F^{n\times n}[\lambda]$, that is $JF^{n\times n}[\lambda] \subset J$ and $J + J \subset J$.

Theorem 2.4

A subset J of $F^{n\times n}[\lambda]$ is a right ideal if and only if $J = DF^{n\times n}[\lambda]$ for some polynomial matrix D in $F^{n\times n}[\lambda]$.

Proof: Given D in $F^{n\times n}[\lambda]$ it is obvious that $DF^{n\times n}[\lambda]$ is a right ideal. To prove the converse let us assume that J is a right ideal in $F^{n\times n}[\lambda]$, then J is also a submodule of $F^{n\times n}[\lambda]$ and hence it is finitely generated. Let $A_1, \ldots, A_k$ be a set of generators and let D be their greatest common left divisor. Then clearly $J = DF^{n\times n}[\lambda]$.

III. Modules and Submodules of $F^n[\lambda]$

As pointed out by Kalman the module structure seems basic to the study of linear systems. A special role is played by finitely generated modules over the ring of polynomials over a field. We assume the reader's familiarity with the basic facts about modules as presented for example in (**9, 11**) or in Appendix A of (**3**).

We consider for a field F $F^n[\lambda]$ as a module over $F[\lambda]$. $F[\lambda]$ is a free module over $F[\lambda]$ having n generators. A subset M of a module K over a ring R is a submodule of K if $M \subset K$ and M is a module over K. Our interest lies in the structure of submodules of $F^n[\lambda]$. The situation is similar to that of Theorem 2.4.

Theorem 3.1

A subset M of $F^n[\lambda]$ is a submodule of $F^n[\lambda]$ if and only if $M = DF^n[\lambda]$ for some polynomial matrix D in $F^{n\times n}[\lambda]$.

The proof is similar to that of Theorem 2.4 and is omitted.

Corollary 3.2

Let M be a submodule of $F^n[\lambda]$ given by $M = DF^n[\lambda]$. If D is nonsingular then in any other representation $M = EF^n[\lambda]$, D and E differ by at most a right unimodular factor.

Proof: If D and E differ by a right unimodular factor then the modules generated by D and E are clearly equal. Conversely assume $DF^n[\lambda] = EF^n[\lambda]$ and that D is nonsingular. If $e_1, \ldots, e_n$ are the standard basis elements in F^n considered as elements of $F^n[\lambda]$ then De_i is the ith column of D. There exists therefore elements $g_1, \ldots, g_n$ in $F^n[\lambda]$ such that $De_i = Eg_i$. Combining these equalities into a matrix equality we get $D = EG$ for some, obviously nonsingular matrix G. Similarly $E = DH$ and hence $D = DHG$. The nonsingularity of D implies $I = HG$ and hence both G and H are unimodular.

The above corollary leads to the following definition. A submodule of $F^n[\lambda]$ will be called a full submodule if it has a representation $M = DF^n[\lambda]$ for some nonsingular matrix D. It is clear that M is a full submodule if and only if it is generated by n independent generators. Since D is nonsingular if and only if $\det D \neq 0$ we have the following trivial corollary.

Corollary 3.3

A submodule $M = DF^n[\lambda]$ of $F^n[\lambda]$ is a full submodule if and only if $\det D \neq 0$.

It should be pointed out that $\det D \neq 0$ means that $\det D$ as an element of $F[\lambda]$ is not the zero polynomial. It might be identically equal to zero as a function on F.

The inclusion of submodules is reflected in the corresponding polynomial matrices.

Lemma 3.4

Let M_1, M_2 be submodules of $F^n[\lambda]$ given by $M_i = D_i F^n[\lambda]$ then $M_1 \subset M_2$ if and only if $D_1 = D_2 E$ for some polynomial matrix E.

Corollary 3.5

Let D be an invertible element of $F^{n\times n}[\lambda]$ then $(\det D) F^n[\lambda] \subset DF^n[\lambda]$.

Proof: This follows from Cramer's rule.

Next we pass to the study of quotient modules of $F^n[\lambda]$. Let M be a submodule of $F^n[\lambda]$. A submodule M induces a natural equivalence relation in $F^n[\lambda]$ whereby

$$f \sim_{\mathrm{M}} g \quad \text{if } f - g \in M.$$

The set of all equivalence classes with the naturally induced algebraic operations is also a module, called the quotient module, over $F[\lambda]$ and we denote it by $F[\lambda]/M$.

As far as the study of finite dimensional linear systems is concerned we will be interested in a special class of quotient modules, the torsion quotient modules. In general, a module M over a ring R is called a torsion module if for each element m of M there exists a nonzero r in R such that $rm = 0$. Of special interest for the sequel is the following result.

Lemma 3.6

Let M be a submodule of $F^n[\lambda]$, then the quotient module $F^n[\lambda]/M$ is a torsion module if and only if M is a full submodule of $F^n[\lambda]$.

Proof: Assume M is a full submodule. Then $M = DF^n[\lambda]$ for a nonsingular D. As $(\det D) F^n[\lambda] \subset DF^n[\lambda]$ by Lemma 3.5 $\det D$ annihilates the quotient module $F^n[\lambda]/M$. Conversely assume that $F^n[\lambda]/M$ is a torsion module. Since it is finitely generated there exists a polynomial p annihilating all of $F^n[\lambda]/M$. This implies $pF^n[\lambda] \subset DF^n[\lambda]$ where $M = DF^n[\lambda]$. Thus $pI = DE$ and hence necessarily $\det D$ is nontrivial and M is full.

Every module over $F[\lambda]$ is at the same time a vector space over F. The next lemma characterizes those quotient modules of $F^n[\lambda]$ which are finite dimensional as vector spaces over F. The simple proof is omitted.

Lemma 3.7

Let M be a submodule of $F^n[\lambda]$. Then the quotient module $F^n[\lambda]/M$ is a finite dimensional vector space over F if and only if M is a full submodule of $F^n[\lambda]$.

If $n = 1$, that is we deal with the ring $F[\lambda]$, then a submodule, an ideal in this case, is generated by a unique monic polynomial m of minimal degree and using the division rule of polynomials we may represent each equivalence class of $F[\lambda]/M$ by a unique polynomial of degree less than the degree of n.

Since it is easier in general to work with representatives rather than equivalence classes we would like to imitate the scalar construction in some way. The difficulty arises mostly through the nonuniqueness of such a representation. One way to overcome this difficulty is through the use of canonical matrices as was done by Eckberg in (**2**). We will proceed differently and study the whole set of such possible representations. We recall that $F(\lambda)$ denotes the set of all rational functions over F. By $F^n(\lambda)$ we denote the set of all n-vectors with rational entries. $F^n(\lambda)$ is a vector space over F but also a module over $F[\lambda]$. $F^n(\lambda)$ contains a subspace consisting of all n-vectors with proper rational entries. Now any rational function has a unique decomposition into the sum of a polynomial and a proper rational function. In fact if $f = p/q$ where p and q are coprime polynomials and q is monic then there exist unique polynomials s and r such that $p = sq+r$ and $\deg r < \deg q$. Given $\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$ in $F^n(\lambda)$ and $f_i = p_i/q_i$ with p_i and q_i as above we have $p_i = s_i q_i + r_i$ and $\deg r_i < \deg q_i$.

We now define a map $\Pi : F^n(\lambda) \to F^n(\lambda)$ by

$$\Pi \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} r_1/q_1 \\ \vdots \\ r_n/q_n \end{pmatrix}. \tag{3.1}$$

The map Π is a projection in $F^n(\lambda)$ whose kernel is $F^n[\lambda]$. Consider now a full submodule M of $F^n[\lambda]$ as embedded in $F^n(\lambda)$. By Theorem 3.1 $M = DF^n[\lambda]$ for some nonsingular D. We now define a map π_D in $F^n[\lambda]$ by

$$\pi_D f = D\Pi(D^{-1}f) \quad \text{for } f \in F^n[\lambda]. \tag{3.2}$$

Lemma 3.8

Let D be a nonsingular matrix in $F^{n\times n}[\lambda]$. Then π_D as defined by (3.2) is a projection map in $F^n[\lambda]$ and $\ker \pi_D = DF^n[\lambda]$.

Proof: Let $f \in F^n[\lambda]$. We consider $D^{-1}f$ which is an element of $F^n(\lambda)$. Decompose $D^{-1}f$ as $g+h$ where g is a proper rational function and h is in $F^n[\lambda]$. Thus $\Pi D^{-1}f = g$ and $D\Pi D^{-1}f = Dg = D(D^{-1}f - h) = f - Dh$. As $f - Dh \in F^n[\lambda]$ $\pi_D f$ is in $F^n[\lambda]$. π_D is a projection as

$$\pi_D^2 f = (D\Pi D^{-1})(D\Pi D^{-1})f = D\Pi^2 D^{-1} f = D\Pi D^{-1} f = \pi_D f.$$

Next we show that $\ker \pi_D = DF^n[\lambda]$. If f belongs to $DF^n[\lambda]$ then $f = Dg$ for some g in $F^n[\lambda]$. Hence

$$\pi_D f = D\Pi D^{-1} Dg = D\Pi g = 0.$$

Conversely if $\pi_D f = 0$ this implies $D\Pi D^{-1}f = 0$. Since D is nonsingular it follows that $\Pi(D^{-1}f) = 0$ or $D^{-1}f \in F^n[\lambda]$ and hence $f = Dg$ for some g or $f \in DF^n[\lambda]$.

We define now K_D by

$$K_D = \{\pi_D f \mid f \in F^n[\lambda]\}. \tag{3.3}$$

Clearly K_D is a vector space over F. From the above discussion it is clear that the following holds true.

Corollary 3.9

A polynomial vector f in $F^n[\lambda]$ belongs to K_D if and only if $D^{-1}f$ is proper rational.

In K_D we can induce a module structure by the following definition. For a polynomial p in $F[\lambda]$ and f in K_D we define

$$p \cdot f = \pi_D(pf). \tag{3.4}$$

It is easily checked that with this definition and the usual addition K_D becomes a module over $F[\lambda]$. The following sums up the situation.

Theorem 3.10

Let $M = DF^n[\lambda]$ be a full submodule of $F^n[\lambda]$. Then K_D with the above definitions is a module over $F[\lambda]$ isomorphic to the quotient module $F^n[\lambda]/M$.

IV. Canonical Models

A standard problem in linear algebra is the reduction of a matrix to a canonical form. This is done by way of a proper choice of basis which gives the desired matrix representation. An alternative approach to the problem is to find a canonical operator similar to a given one. The proper choice of basis can be accomplished as the next step but for a variety of purposes this will be redundant.

We introduce first some notation. In $F^n[\lambda]$ we define a linear map S by

$$Sf = \chi f \quad \text{for } f \in F^n[\lambda], \tag{4.1}$$

where χ is the identity polynomial, i.e. $\chi(\lambda) = \lambda$. Thus (4.1) is equivalent to

$$(Sf)(\lambda) = \lambda f(\lambda). \tag{4.2}$$

The operator S is called the shift operator and is of central importance in linear system theory. This is not surprising inasmuch as time invariance is expressed in terms of commutation properties with the shift operator.

The module $F^n[\lambda]$ is a module over $F[\lambda]$ but it also admits the ring $F^{n\times n}[\lambda]$ as a left operator ring. That is, there exists a map from $F^{n\times n}[\lambda] \times F^n[\lambda]$ into $F[\lambda]$ defined by

$$(A, f) \to Af, \tag{4.3}$$

where

$$(Af)(\lambda) = A(\lambda)f(\lambda). \tag{4.4}$$

Clearly for a fixed A the map defined by (4.3) is an $F[\lambda]$-homomorphism of $F^n[\lambda]$. In fact the converse is true. This is the content of the text theorem whose simple proof is omitted.

Theorem 4.1

A map in $F^n[\lambda]$ is an $F[\lambda]$-homomorphism if and only if it has the form (4.4) for some A in $F^{n\times n}[\lambda]$.

Corollary 4.2

A linear transformation $\mathfrak{A}$ in $F^n[\lambda]$ commutes with S if and only if

$$(\mathfrak{A}f)(\lambda) = A(\lambda)f(\lambda) \tag{4.5}$$

for some A in $F^{n\times n}[\lambda]$.

Proof: A linear transformation in $F^n[\lambda]$ commutes with S if and only if it commutes with any polynomial in S, i.e. if and only if it is an $F[\lambda]$-homomorphism.

The canonical models we are after for the representation of finite dimensional linear systems will be the set of all torsion quotient modules K_D where

$$K_D = \pi_D F^n[\lambda]. \tag{4.6}$$

Here D is nonsingular by assumption. In Section III we introduced already an $F[\lambda]$ module structure on K_D. Now we single out one operator $S(D)$ defined by

$$S(D)f = \pi_D(\chi f), \quad f \in K_D. \tag{4.7}$$

Obviously (4.7) is a special case of (3.4) and hence $S(D)$ is actually an $F[\lambda]$-homomorphism. We refer to $S(D)$ as the restricted shift in K_D or just the restricted shift when its domain is clear from the context.

The restricted shifts as defined by (4.7) will serve as our canonical models for the general linear transformation in a finite dimensional vector space over F.

We now proceed with a more detailed study of the transformation $S(D)$.

Theorem 4.3

A number $\lambda_0 \in F$ is an eigenvalue of $S(D)$ if and only if $\ker D(\lambda_0) \neq \{0\}$. In that case the eigenvectors of $S(D)$ have the form $(\chi - \lambda_0)^{-1} D\xi$ for $\xi \in \ker D(\lambda_0)$.

Proof: Assume $D(\lambda_0)\,\xi = 0$ and define f by $f = (\chi - \lambda_0)^{-1} D\xi$ then clearly $\pi_D f = f$, that is $f \in K_D$ and

$$[S(D) - \lambda_0 I]f = \pi_D(\chi - \lambda_0)f = \pi_D D\xi = 0,$$

that is f is an eigenvector of $S(D)$ corresponding to the eigenvalue λ_0. Conversely assume f is an eigenvector of $S(D)$ which corresponds to the eigenvalue λ_0 then $\pi_D(\chi-\lambda_0)f = 0$ or $(\chi-\lambda_0)f \in \ker \pi_D = DF^n[\lambda]$. Therefore, $(\chi-\lambda_0)f = Dg$ for some g in $F^n[\lambda]$, or $f = (\chi-\lambda_0)^{-1}Dg$. It remains to show that g is a constant vector. Since $f \in K_D$ it follows from Lemma 3.7 that $D^{-1}f = (\chi-\lambda_0)^{-1}g$ is proper rational and hence g is necessarily constant.

Corollary 4.4

A number $\lambda_0 \in F$ is an eigenvalue of $S(D)$ if and only if $\chi-\lambda_0$ divides $d = \det D$.

Proof: The polynomial $d = \det D$ is divisible by $\chi-\lambda_0$ if and only if $d(\lambda_0) = 0$ which is equivalent to $\ker D(\lambda_0) \neq \{0\}$.

The above corollary indicates the direction for generalizing Theorem 4.3.

Theorem 4.4

Given a polynomial p in $F[\lambda]$ then $p[S(D)]$ is invertible if and only if p and d are coprime.

We omit the direct proof. This theorem follows also as a corollary to the more general result given by Theorem 4.7.

Since we are interested in the relationship between different canonical models it is of importance to characterize the conditions guaranteeing the similarity of two transformations of the form $S(D)$. For this we introduce the notion of intertwining operators. Let K and K_1 be vector spaces over F and let T and T_1 be two linear transformations meeting in K and K_1 respectively. We say that a linear map $X: K \to K_1$ intertwines T and T_1 if $XT = T_1X$. If X happens to be invertible then T and T_1 are similar. In the special case that the spaces are K_D and K_{D_1} and the maps are $S(D)$ and $S(D_1)$, respectively, then a map $X: K_D \to K_{D_1}$ intertwines $S(D)$ and $S(D_1)$ if and only if it is an $F[\lambda]$-module homomorphism. Thus the set of all $F[\lambda]$-module homomorphisms from K_D into K_{D_1} is the one we wish to characterize and in particular the subclass of isomorphisms. The characterization is a simple version of the Sz.-Nagy–Foias lifting theorem [(**14**), p. 64] adapted to this context.

Theorem 4.5

Let D and D_1 be invertible elements of $F^{n\times n}[\lambda]$ and $F^{m\times m}[\lambda]$, respectively. A map $X: K_D \to K_{D_1}$ is an $F[\lambda]$-module homomorphism if and only if there exist Ξ and Ξ_1 in $F^{m\times n}[\lambda]$ satisfying

$$\Xi D = D_1 \Xi_1 \tag{4.8}$$

and X is defined by

$$Xf = \pi_{D_1}\Xi f \quad \text{for } f \in K_D. \tag{4.9}$$

Before proving Theorem 4.5 we will prove the following lemma.

Lemma 4.6

Let D_1 be an invertible element of $F^{m\times m}[\lambda]$. A map $X\colon F^n[\lambda]\to K_{D_1}$ is an $F[\lambda]$-module homomorphism if and only if for some Ξ in $F^{m\times n}[\lambda]$ X is given by

$$Xf = \pi_{D_1}\Xi f. \tag{4.10}$$

Proof: Assume $X\colon F^n[\lambda]\to K_{D_1}$ is an $F[\lambda]$-module homomorphism. Let $e_1,\ldots,e_n$ be the standard basis elements of F^n, they serve also as a set of generators of the free module $F^n[\lambda]$. Let $Xe_i = \xi_i \in K_D$. Let Ξ be the $n\times n$ polynomial matrix whose columns are $\xi_1,\ldots,\xi_n$. It follows by linearity that for any $\eta\in F^n$ we have $(X\eta)(\lambda) = \Xi(\lambda)\eta$. Since X is an $F[\lambda]$-module homomorphism we have for any polynomial p in $F[\lambda]$ that

$$X(p\eta) = \pi_{D_1}p(\Xi\eta) = \pi_{D_1}\Xi(p\eta).$$

Thus (4.10) follows. The converse is obvious.

Proof of Theorem 4.5: If $X\colon K_D\to K_{D_1}$ is defined through (4.9) and (4.8) then it is clearly an $F[\lambda]$-module homomorphism. Conversely let $X\colon K_D\to K_{D_1}$ be an $F[\lambda]$-module homomorphism. Thus

$$XS(D) = S(D_1)X. \tag{4.11}$$

Right multiplying (4.11) by π_D we obtain

$$XS(D)\pi_D = S(D_1)X\pi_D$$

and this implies

$$(X\pi_D)S = S(D_1)(X\pi_D), \tag{4.12}$$

where $S\colon F^n[\lambda]\to F^n[\lambda]$ is defined by (4.1). Thus $X\pi_D$ satisfies the conditions of Lemma 4.6 and hence

$$X\pi_D f = \pi_{D_1}\Xi f \tag{4.13}$$

for some polynomial matrix Ξ. Now $X\pi_D$ and X act equally on K_D and hence (4.13) implies (4.9). Also $X\pi_D Dg = 0$ for any $g\in F^n[\lambda]$ hence $\pi_{D_1}\Xi Dg = 0$ or

$$\Xi DF^n[\lambda]\subset D_1F^n[\lambda]. \tag{4.14}$$

But (4.14) implies the existence of a Ξ_1 for which (4.8) holds. The following theorem characterizes the invertibility properties of transformations that intertwine restricted shifts.

Theorem 4.7

Let D and D_1 be invertible polynomial matrices in $F^{n\times n}[\lambda]$ and $F^{m\times m}[\lambda]$, respectively, and let $K\colon K_D\to K_{D_1}$ be defined by (4.9) with (4.8) holding.

(a) X is onto K_{D_1} if and only if Ξ and D_1 are left coprime.

(b) X is one-to-one if and only if Ξ_1 and D are right coprime.

Proof: (a) Consider the range of $X = \{\pi_{D_1}\Xi f \mid f \in K_D\}$, this is clearly a submodule of K_{D_1}. X is not onto if and only if $\{\pi_{D_1}\Xi f \mid f \in K_D\} + D_1 F^m[\lambda]$ which is equal to $\Xi F^n[\lambda] + D_1 F^m[\lambda]$ differs from $F^m[\lambda]$. Since

$$\Xi F^n[\lambda] + D_1 F^m[\lambda] = \Delta F^m[\lambda]$$

for some Δ it follows that Δ, being the greatest common left divisor of Ξ and D_1, is not unimodular if and only if Ξ and D_1 are left coprime.

(b) Let $f \in K_D$ be in the kernel of X. Since $f \in K_D$ we can write $f = Dg$ for some proper rational vector function g. Now $Xf = 0$ implies $\pi_{D_1}\Xi f = 0$ or $\Xi Dg = D_1 p$ for some $p \in F^m[\lambda]$. Using (4.8) we obtain $\Xi_1 g = p$. Let us define $J_g = \{A \in F^{m\times m}[\lambda] \mid Ag \in F^n[\lambda]\}$ then clearly J_g is a left ideal and hence $J_g = F^{m\times m}[\lambda]\,\Delta$ for some $\Delta \in F^{m\times m}[\lambda]$. Since D and Ξ_1 belong to J_g they have a greatest common right divisor given by Δ.

Conversely assume Ξ_1 and D are not right coprime and let Δ be their greatest common right divisor. Let g be a proper rational function for which Δg is in $F^m[\lambda]$. Such a g certainty exists for we can take $g = \Delta^{-1}h$ for any $h \in K_\Delta$. Let $f = Dg$ then

$$Xf = \pi_{D_1}\Xi f = \pi_{D_1}\Xi Dg = \pi_{D_1} D_1 \Xi_1 g = 0$$

for $\Xi_1 g \in F^m[\lambda]$ as Δ is a right divisor of Ξ_1. Thus X is not one-to-one.

Using Theorem 4.7 we can go now into a discussion of canonical forms. First we characterize similarity in terms of equivalence of polynomial matrices. Let A and B be elements of $F^{n\times n}[\lambda]$. We say that A and B are equivalent if there exist unimodular matrices P and R for which $B = PAR$. This is clearly an equivalence relation.

Theorem 4.8

Let D and D_1 be invertible elements of $F^{n\times n}[\lambda]$. Then $S(D)$ and $S(D_1)$ are similar if and only if D and D_1 are equivalent.

Proof: Assume D and D_1 are equivalent then $D_1 = PDR$ for some unimodular matrices P and R. This is equivalent to

$$PD = D_1 Q \tag{4.15}$$

with $Q = R^{-1}$ also unimodular. Since a unimodular matrix is left and right coprime with any other matrix we can apply Theorem 4.7 to obtain the similarity of $S(D)$ and $S(D_1)$.

Conversely assume $S(D)$ and $S(D_1)$ are similar. It is well known that any matrix over a principal ideal ring is equivalent to its Smith canonical forms, that is to a diagonal matrix having the invariant factors on the diagonal (7). Let Δ and Δ_1 be the Smith canonical forms of D and D_1. By the first part of the proof $S(D)$ and $S(D_1)$ are similar to $S(\Delta)$ and $S(\Delta_1)$, respectively, and hence by transitivity $S(\Delta)$ and $S(\Delta_1)$ are similar. Thus Δ and Δ_1 are equal which implies, again by transitivity, that D and D_1 are equivalent.

Corollary 4.9

Let D be an invertible matrix in $F^{n\times n}[\lambda]$. Then up to a constant factor $\det D$ is the characteristic polynomial of $S(D)$. The degree of $\det D$ is the dimension of K_D as a linear space.

V. Transfer Functions and their Factorization

An element W of $F^{m\times n}(\lambda)$ is called a transfer function if it is a proper rational matrix function. Being proper is related to the causality of the system. We will omit the physical considerations concerning the relation between transfer functions and the input–output relations of a system. This topic is well covered in the literature and we refer to (**1**, **3**, **12**). We proceed with the mathematical analysis. The important results are summed up by the following theorem.

Theorem 5.1

Let W be a proper rational function in $F^{m\times n}(\lambda)$ then W has the following representation:

$$W = \theta/\psi, \tag{5.1}$$

where $\theta \in F^{m\times n}[\lambda]$ and $\psi \in F[\lambda]$,

$$W = D^{-1}N, \tag{5.2}$$

where D is an invertible element in $F^{m\times m}[\lambda]$ and $N \in F^{m\times n}[\lambda]$ and

$$W = N_1 D_1^{-1}, \tag{5.3}$$

where D_1 is an invertible element in $F^{n\times n}[\lambda]$ and $N_1 \in F^{m\times n}[\lambda]$.

If we assume ψ to be monic and coprime with all elements of θ then it is unique. If D and N are left coprime then they are unique up to a common left unimodular factor, and so analogously for D_1 and N_1.

Proof: Consider the following three sets:

$$J = \{\phi \in F[\lambda] \mid \phi W \in F^{m\times n}[\lambda]\},$$

$$J_L = \{P \in F^{m\times m}[\lambda] \mid PW \in F^{m\times n}[\lambda]\}$$

and

$$J_R = \{Q \in F^{n\times n}[\lambda] \mid WQ \in F^{n\times n}[\lambda]\}.$$

Obviously J is an ideal in $F[\lambda]$, J_L a left ideal in $F^{m\times m}[\lambda]$ and J_R a right ideal in $F^{n\times n}[\lambda]$. We claim all three ideals are nontrivial. In fact if ϕ_0 is the least common multiple of all the denominators of the entries of W then $\phi_0 \in J$. Since $F[\lambda]$ is a principal ideal domain we have $J = \psi F[\lambda]$ for some ψ, unique up to a constant factor. Thus $\psi W = \theta$ for some θ in $F^{m\times n}[\lambda]$ and hence the factorization (5.1). Now ψI_m belongs to J_L and ψI_n to J_R. Hence J_L and J_R

are full one-sided ideals. By Theorem 2.4 we have $J_L = F^{m\times n}[\lambda]D$ and $J_R = D_1 F^{n\times n}[\lambda]$. Since $\psi I_m = ED$ it follows that $d = \det D$ divides ψ^m and hence $d \neq 0$. Thus D is invertible in $F^{m\times m}(\lambda)$ and its inverse is given by E/ψ. Since $DW = N$ for some N it follows that (5.2) holds. The uniqueness part follows from the uniqueness part of Theorem 2.4. The statement about the factorization (5.3) is proved analogously.

VI. Realization Theory

In this section we apply the previously outlined results about canonical models, transfer function factorization and intertwining operators to the question of realization. Let W be a transfer function matrix, that is a proper rational function in $F^{m\times n}(\lambda)$. Thus W has a formal expansion

$$W(\lambda) = \sum_{i=0}^{\infty} W_i \lambda^{-i-1}.$$

A triple of operators $\{A, B, C\}$ is a realization of W if

$$W_i = CA^i B \quad \text{for } i \geqslant 0. \tag{6.1}$$

We will not go into details of the physical motivation for the definitions as excellent accounts are available (**12**, **3**). We remark though that $\{A, B, C\}$ is the shorthand notation for the discrete time-invariant linear systems given by the equations

$$\left.\begin{aligned} x_{n+1} &= Ax_n + Bu_n, \\ y_n &= Cx_n. \end{aligned}\right\} \tag{6.2}$$

The system $\{A, B, C\}$ is controllable if the set of vectors of the form $\sum A^i Bu_i$ spans the state space, which is the domain of definition of A. The system is observable of $CA^i x = 0$ for all $i \geqslant 0$ implies that $x = 0$.

We use our canonical models to define a special class of systems. Let $D \in F^{n\times n}[\lambda]$ be nonsingular and let $B\colon F^p \to K_D$ be a linear map. Here F^p is the input space. Let $e_1, \ldots, e_p$ be the elements of the standard basis in F^p. Let $b_i = Be_i$, $i = 1, \ldots, p$. The elements $b_1, \ldots, b_p$ are called by Kalman (**3**) the accessible generators of the quotient module K_D. It is clear that the controllability of the pair $\{S(D), B\}$ where $S(D)$ is defined by (4.7) is equivalent to $(b_1, \ldots, b_p\}$ being a set of generators of 1_D. Let N be the element of $F^{n\times p}[\lambda]$ whose columns are $b_1, \ldots, b_p$. Then obviously we have

$$(B\xi)(\lambda) = N(\lambda)\,\xi \quad \text{for } \xi \in F^p. \tag{6.3}$$

The following is a characterization of the controllability of the system introduced above.

Theorem 6.1

Let $S(D)$ be defined in K_D by (4.7) and let B be defined by (6.3). Then $\{S(D), B\}$ is a controllable pair if and only if D and N are left coprime.

Proof: We consider the set L of all elements of the form $\sum S(D)^j N\xi_j$. L is clearly a submodule of K_D and $L+DF^n[\lambda]$ is a submodule of $F^n[\lambda]$. By Theorem 3.1 we have $L = EF^n[\lambda]$ for some E in $F^{n\times n}[\lambda]$ and the above-mentioned inclusion implies the factorization $D = EG$. Also $N = EM$ follows from the fact that $N\xi \in EF^n[\lambda]$. Thus controllability is equivalent to D, N being left coprime.

Let W $F^{n\times m}(\lambda)$ be proper. By Theorem 5.1 W has the coprime factorization $W = D^{-1}N$. The polynomial matrices D and N are unique up to a left unimodular factor. We proceed with the construction of a canonical realization of W. As state space we take the representation of the quotient module $F^n[\lambda]/DF^n[\lambda]$ given by K_D. Define $B\colon F^m \to K_D$ by (6.3). Clearly $B\xi \in K_D$ for every $\xi \in F^m$. This follows from Corollary 3.9 as $D^{-1}B\xi = D^{-1}N\xi$ is proper rational and hence

$$\pi_D N\xi = D\Pi D^{-1}N\xi = DD^{-1}N\xi = N\xi.$$

Let $S(D)$ be the right shift in K_D as defined by (4.7). We define a map $C\colon K_D \to F^n$ by

$$Cf = (D^{-1}f)_1, \tag{6.4}$$

where for any proper rational function Ω having the formal expansion $\Omega(\lambda) = \sum_{i=0}^{\infty}\Omega_i \lambda^{-i-1}$ we define $(\Omega)_i = \Omega_i$.

Clearly, the system $\{S(D), B, C\}$ is a realization of W as for $\xi \in F^m$

$$\begin{aligned} CS(D)^i B\xi &= (D^{-1}\pi_D \chi^i N\xi)_1 = (D^{-1}D\Pi D^{-1}\chi^i N\xi)_1 \\ &= (\Pi\chi^i D^{-1}N\xi)_1 = (\Pi\chi^i W\xi)_1 = W_i\,\xi. \end{aligned}$$

We claim that this realization is canonical, that is both controllable and observable. We begin with observability. Suppose for $f\in K_D$ we have $CS(D)^n f = 0$ for all $n \geqslant 0$. This means that

$$(D^{-1}\pi_D \chi^n f)_1 = (D^{-1}D\Pi D^{-1}\chi^n f)_1 = (\Pi\chi^n D^{-1}f)_1 = 0.$$

But this implies that $(D^{-1}f)_n = 0$ for all n and as, by Lemma 3.7, $D^{-1}f$ is proper rational it follows that $D^{-1}f = 0$ and hence also $f = 0$. The controllability of the system follows from Theorem 6.1.

The second factorization given by Theorem 5.1, that is the factorization (5.3), gives rise to a second realization.

The equality $D^{-1}N = N_1 D_1^{-1}$ is equivalent to

$$ND_1 = DN_1. \tag{6.5}$$

Since D, N are left coprime and D_1, N_1 are right coprime it follows from Theorem 4.5 that the map $X\colon K_{D_1} \to K_D$ defined by

$$Xf = \pi_D Nf \tag{6.6}$$

is invertible. Moreover X intertwines $S(D_1)$ and $S(D)$.

We now define maps $B_1\colon F^m \to K_{D_1}$ and $C_1\colon K_{D_1} \to F^n$ in such a way that the following diagram is commutative.

$$\begin{tikzcd}
& F^m \arrow[dl, "B_1"'] \arrow[dr, "B"] & \\
K_{D_1} \arrow[rr, "X"] \arrow[d, "S(D_1)"'] & & K_D \arrow[d, "S(D)"] \\
K_{D_1} \arrow[rr, "X"] \arrow[dr, "C_1"'] & & K_D \arrow[dl, "C"] \\
& F^n &
\end{tikzcd}$$

The commutativity condition is equivalent to $XB_1 = B$ and $C_1 = CX$. We check that B_1 is necessarily given by

$$B_1 \xi = \pi_{D_1} \xi, \tag{6.7}$$

for

$$XB_1 \xi = \pi_D N \pi_{D_1} \xi = \pi_D N \xi = N\xi = B\xi.$$

Also we have for every $f \in K_{D_1}$ that

$$C_1 f = CXf = (D^{-1}Xf)_1 = (D^{-1}\pi_D Nf) = (D^{-1} D \Pi D^{-1} Nf)_1,$$

or

$$C_1 f = (\Pi W f)_1. \tag{6.8}$$

That $\{S(D_1), B_1, C_1\}$ is a realization and moreover a canonical one is clear from the invertibility of X and the commutativity of the diagram. This can also be verified directly as

$$C_1 S(D_1)^n B_1 \xi = (\Pi W \pi_{D_1} \chi_{D_1} \xi) = (\Pi W \xi)_n = W_n \xi.$$

We now consider two special cases. First, let $W = D^{-1} N$ and

$$N(\lambda) = N_0 + N_1 \lambda + \dots + N_{k-1} \lambda^{k-1}$$

and

$$D(\lambda) = D_0 + D_1 \lambda + \dots + D_{k-1} \lambda^{k-1} + I \lambda^k.$$

It is easily checked that

$$K_D = \pi_D F^n[\lambda] = \{\alpha_0 + \alpha_1 \lambda + \dots + \alpha_{k-1} \lambda^{k-1} \mid \alpha_i \in F^n\}.$$

If we make identification

$$\alpha_0+\alpha_1\lambda+\ldots+\alpha_{k-1}\lambda^{k-1}\leftrightarrow\begin{pmatrix}\alpha_0\\ \vdots\\ \alpha_{k-1}\end{pmatrix}$$

then we have the representation

$$S(D)\leftrightarrow\begin{pmatrix}0 & & & -D_0\\ I & & & \cdot\\ & \ddots & & \vdots\\ & & I & -D_{k-1}\end{pmatrix},$$

$$B\leftrightarrow\begin{pmatrix}N_0\\ \vdots\\ N_{k-1}\end{pmatrix}\quad\text{and}\quad C\leftrightarrow(0\ \ldots\ 0I)$$

and for that reason we call the realization $\{S(D),B,C\}$ the standard observable realization. In the same fashion if $W=N_1D_1^{-1}$ and

$$N_1(\lambda)=N_0'+\ldots+N_{k-1}'\lambda^{k-1}$$

and

$$D_1(\lambda)=D_0'+\ldots+D_{k-1}'\lambda^{k-1}+I\lambda^k,$$

then with the same coordinatization of K_{D_1} we have

$$S(D_1)\leftrightarrow\begin{pmatrix}0 & & & -D_0'\\ I & & & \cdot\\ & \ddots & & \vdots\\ & & I & -D_{k-1}'\end{pmatrix},$$

$$B_1\leftrightarrow\begin{pmatrix}I\\ 0\\ \vdots\\ 0\end{pmatrix}\quad\text{and}\quad C_1\leftrightarrow(W_0\ \ldots\ W_{k-1}).$$

We call the realization $\{S(D_1),B_1,C_1\}$ the standard controllable realization. The above construction should be compared for example with [(**12**), p. 106].

VII. The Generalized Resultant Theorem

A classical result of Sylvester [(**9**), p. 135] gives a simple criterion, in terms of the nonsingularity of the resultant matrix, for the coprimeness of two polynomials. This section is devoted to an abstract generalization of this result. For a different approach to the problem we refer to (**13**).

For motivation we review the classical result.

Lemma 7.1

Let $p, q \in F[\lambda]$ then p and q are coprime if and only if

$$F[\lambda]/pqF[\lambda] = p\{F[\lambda]/qF[\lambda]\} + q\{F[\lambda]/pF[\lambda]\}. \tag{7.1}$$

We identify the quotient ring elements with their unique representative of lowest degree.

Proof: Assume p and q are coprime, then for every $f \in F[\lambda]$ there exist $a, b \in F[\lambda]$ for which $f = ap + bq$. Thus

$$f \bmod (pq) = p(a \bmod q) + q(b \bmod p)$$

or

$$F[\lambda]/pqF[\lambda] \subset p\{F[\lambda]/qF[\lambda]\} + q\{F[\lambda]/pF[\lambda]\}.$$

The converse inclusion holds by a dimensionality argument. Conversely assume now the equality (7.1). In particular there exist polynomials a and b such that $1 = aq + bq$ but this is equivalent to the coprimeness of p and q.

Assume now that

$$p(\lambda) = p_0 \, p_1 \lambda + \ldots + p_n \lambda^n \quad \text{and} \quad q(\lambda) = q_0 + q_1 \lambda + \ldots + q_m \lambda^m \tag{7.2}$$

then $F[\lambda]/pF[\lambda]$ is isomorphic to F_{n-1} the set of all polynomials of degree less than n with the multiplication being modulo p. Similarly $F[\lambda]/qF[\lambda]$ is isomorphic to $F_{m-1}[\lambda]$. The following follows easily from Lemma 7.1.

Corollary 7.2

The polynomials p and q in $F[\lambda]$ be given by (7.2). Then p and q are coprime if and only if

$$F_{m+n}[\lambda] = pF_m[\lambda] + qF_n[\lambda].$$

Theorem 7.3

Let p and q be given by (7.2) then p and q are coprime if and only if $\det \Gamma(p,q) \neq 0$ where $\Gamma(p,q)$ is the resultant matrix

$$\Gamma(p,q) = \begin{array}{c} \left(\begin{array}{cccccc} p_0 & & \cdots & p_n & 0 & \\ & \ddots & \cdots & & \ddots & \\ & & p_0 & \cdots & & p_n \\ q_0 & & \cdots & q_m & & \\ & \ddots & \cdots & & \ddots & \\ & & q_0 & \cdots & & q_m \end{array} \right) \end{array} \begin{array}{l} \left.\vphantom{\begin{array}{c} a\\a\\a \end{array}}\right\} m \\ \left.\vphantom{\begin{array}{c} a\\a\\a \end{array}}\right\} n \end{array}.$$

Proof: By Corollary 7.2 p and q are coprime if and only if the set

$$B = \{\chi^i p \mid i = 0, \ldots, m-1\} \cup \{\chi^i q \mid j = 0, \ldots, n-1\}$$

is a basis for $F_{m+n-1}[\lambda]$. In terms of the polynomial coefficients this is equivalent to $\det \Gamma(p,q) \neq 0$.

We now pass to the generalized result. Let D_1 and D_2 be two nonsingular polynomial matrices in $F^{n \times n}[\lambda]$ and let $M_i = D_i F^n[\lambda]$ be the corresponding full submodules. Define M by $M = M_1 \cap M_2$ so M is also a full submodule and hence has a representation $M = DF^n[\lambda]$ for some nonsingular D. Since $M \subset M_i$ there exist polynomial matrices E_i for which the equalities

$$D = D_1 E_1 = D_2 E_2$$

hold.

Theorem 7.4

(a) The polynomial matrices D_1 and D_2 are left coprime if and only if the equality

$$\det D = \det D_1 \cdot \det D_2 \tag{7.4}$$

holds up to a constant factor on one side. The left coprimeness of D_1 and D_2 implies the right coprimeness of E_1 and E_2.

(b) The equality

$$F^n[\lambda]/DF^n[\lambda] = D_1\{F^n[\lambda]/E_1 F^n[\lambda]\} + D_2\{F^n[\lambda]/E_2 F^n[\lambda]\} \tag{7.5}$$

holds if and only if D_1 and D_2 are left coprime. That this generalizes the resultant theorem is obvious from a comparison with Lemma 7.1.

Proof: Suppose D_1 and D_2 are left coprime. By Theorem 2.1 there exist polynomial matrices G_1 and G_2 such that $I = D_1 G_1 + D_2 G_2$. Therefore every $f \in F^n[\lambda]$ has a representation

$$f = D_1 G_1 f + D_2 G_2 f = D_1 f_1 + D_2 f_2.$$

If we apply the projection π_D of $F^n[\lambda]$ onto K_D and use the equalities (7.3) then

$$\begin{aligned}\pi_D f = D\Pi D^{-1} f &= D_1 E_1 \Pi E_1^{-1} D_1^{-1} D_1 f_1 + D_2 E_2 \Pi E_2^{-1} D_2^{-1} D_2 f_2 \\ &= D_1 \pi_{E_1} f_1 + D_2 \pi_{E_2} f_2.\end{aligned}$$

Therefore, we set the inclusion $K_D \subset D_1 K_{E_1} + D_2 K_{E_2}$. To prove the converse inclusion it suffices, by symmetry, to show that $D_1 K_{E_1} \subset K_D$. Let $f \in D_1 K_{E_1}$, i.e. $f = D_1 g$ and $E_1^{-1} g$ is proper rational. So $D^{-1} f = (D_1 E_1)^{-1} f$, $E_1^{-1} D_1^{-1} D_1 g = E_1^{-1} g$. By Lemma 3.7 it follows that $f \in K_D$ and hence the equality (7.5) is proved. From the proof it is clear that the inclusion

$$D_1 K_{E_1} + D_2 K_{E_2} \subset K_D \tag{7.6}$$

holds always.

We now consider the rational function $D_2^{-1} D_1$ which to begin with we assume to be proper. By Theorem 5.1 there exist polynomial matrices F_1 and F_2 which are right coprime and for which

$$D_2^{-1} D_1 = F_2 F_1^{-1}$$

which is equivalent to

$$D_1 F_1 = D_2 F_2. \tag{7.8}$$

Since clearly $D_1 F_1 F^n[\lambda] \subset D_1 F^n[\lambda]$ and $D_2 F_2 F^n[\lambda] \subset D_2 F^n[\lambda]$ it follows from (7.8) that

$$D_i F_i F^n[\lambda] \subset D_1 F^n[\lambda] \cap D_2 F^n[\lambda] = DF^n[\lambda].$$

Thus for some polynomial matrix G we have

$$D_1 F_1 = D_2 F_2 = DG \quad \text{or} \quad DG = D_1 E_1 G = D_2 E_2 G$$

and hence $F_1 = E_1 G$ and $F_2 = E_2 G$. But F_1 and F_2 are assumed to be right coprime and hence necessarily G is unimodular. The unimodularity of G now implies also the right coprimeness of E_1 and E_2. We recall that we assume $D_2^{-1} D_1$ to be a proper rational matrix. We apply now the realization theory developed in Section VI to deduce the similarity of $S(D_2)$ and $S(E_1)$. This in turn implies the equivalence of D_2 and E_1 and hence in particular the equality

$$\det D_2 = \det E_1 \tag{7.9}$$

holds. Using (7.9) and (7.3) the equality (7.4) follows.

To prove the converse half of the theorem we assume D_1 and D_2 to have a nontrivial greatest common left divisor L. L is determined only up to a right unimodular matrix. Thus we have

$$D_1 = LC_1 \quad \text{and} \quad D_2 = LC_2 \tag{7.10}$$

and C_1, C_2 are left coprime. Now as

$$D_1 F^n[\lambda] \cap D_2 F^n[\lambda] = L\{C_1 F^n[\lambda] \cap C_2 F^n[\lambda]\} = LD' F^n[\lambda] = DF^n[\lambda]$$

and

$$\det D' = \det C_1 \cdot \det C_2$$

it follows that

$$\det D = \det L \cdot \det D' = \det L \cdot \det C_1 \cdot \det C_2 \neq \det D_1 \det D_2.$$

Similarly the equality (7.5) cannot hold by a dimensionality argument. As linear spaces the dimension of $F^n[\lambda]/DF^n[\lambda]$ is equal to the degree of the polynomial $\det D = \det L \cdot \det D'$ whereas the degree of

$$D_1\{F^n[\lambda]/E_1 F^n[\lambda]\} + D_2\{F^n[\lambda]/E_2 F^n[\lambda]\}$$

is equal to the degree of the $\det D'$. Since L is not unimodular there cannot be equality.

We now indicate how to remove the restriction that $D_2^{-1} D_1$ is proper rational. In general, given D in $F^{n\times n}[\lambda]$ there is an induced equivalence relation in $F^{n\times n}[\lambda]$ whereby two polynomial matrices A and B are left equivalent modulo D, denoted by $A \overset{\lambda}{\underset{D}{\sim}} B$ if D is a left divisor of $A-B$. Now given D and M there exists a unique N for which $M \overset{\lambda}{\underset{D}{\sim}} N$ and $D^{-1} N$ is proper rational. To obtain N we write $D^{-1} M = L + K$ where L is a proper rational matrix and K is a polynomial matrix. Let

$$N = DL = D(D^{-1} M + K) = M + DK.$$

Clearly $N \overset{\lambda}{\underset{D}{\sim}} M$ and $D^{-1} N$ is proper. Uniqueness of N follows easily from the unique decomposition of a rational matrix into a sum of a proper rational matrix and a polynomial matrix.

Now if $D_2^{-1} D_1$ is not proper rational there exists Δ_1, $\Delta_1 \underset{D}{\overset{\lambda}{\sim}} D_1$, for $D_2^{-1} \Delta$ is proper and $D_1 = \Delta_1 + D_2 R$. Thus there exist F_1 and F_2 right coprime for which

$$D_2^{-1} \Delta_1 = F_2 F_1^{-1} \tag{7.11}$$

holds. Moreover, as in the first part of the proof we have $\det D_2 = \det F_1$. From (7.11) we get $\Delta_1 F_1 = D_2 F_2$ and hence

$$D_1 E_1 = D_2 E_2, \tag{7.12}$$

where $E_1 = F_1$ and $E_2 = F_2 + RF_1$. E_1 and E_2 are right coprime, this follows from the right coprimeness of F_1 and F_2.

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