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Duality in Polynomial Models with Some Applications to Geometric Control Theory

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***Abstract*—Duality is studied in the context of polynomial models for linear systems. The output injection group, the dual of the feedback group, is studied and a polynomial characterization of (C, A)-invariant subspaces as well as of the maximal reachability subspace contained in ker C is given.**

I. Introduction

THE QUESTION of duality in linear system theory has remained so far unclarified and is used mostly by transposing matrices. While this may yield results it is far from satisfactory from a theoretical point of view.

In a series of papers [1]–[6] there was an attempt to study finite-dimensional time-invariant systems using the polynomial model approach developed by the author [2]. The use of polynomial models rather than dealing with matrix representations has the advantage of a richer structure which naturally accommodates any study of zeros, poles and system structure, and isomorphism.

Our object in this paper is to study problems of duality in the context of polynomial models and their associated rational models. The advantage of this approach is that the dual space is not defined abstractly but is naturally equipped with a suitable polynomial module structure. Thus, the dual of a polynomial model system is again a polynomial model system.

While, theoretically, given a system (A, B, C) one can study the pair (C, A) by dualizing results obtained studying pairs (A, B), this does not seem to be always the best approach. One can find this observation substantiated in [11]. In fact, sometimes a direct study of the pair (C, A) is easier and yields cleaner results. In retrospect this, at least from the polynomial point of view, is natural. If one studies the input/output behavior of a system through the restricted input/output map f, where f: $U[\lambda]\rightarrow \lambda^{-1}Y[[\lambda^{-1}]]$ is a homomorphism over the ring of polynomials, then the input space $U[\lambda]$ and output space $\lambda^{-1}Y[[\lambda^{-1}]]$ have different structures. Thus, it is possible that for some problems it is more convenient to use realizations based on submodules of $\lambda^{-1}Y[[\lambda^{-1}]]$ whereas for other problems it seems preferable to work with quotient modules of $U[\lambda]$. One typical example is the characterization of (A, B)- and (C, A)-invariant subspaces. For the case of (A, B)-invariant subspaces the cleanest characterization seems to be [6, Theorem 4.6] and the setting is $\lambda^{-1}U[[\lambda^{-1}]]$. The analogous characterization of (C, A)-invariant subspaces, Theorem 3.3 of this paper, uses $Y[\lambda]$ as the setting.

It would be natural to expect that a characterization of (A, B)-invariant subspaces, which are associated with the input map, would use the space of input functions $U[\lambda]$ and quotient modules of it, and similarly that (C, A)-invariant subspaces would be best characterized in terms of submodules of the space of output functions $\lambda^{-1}Y[[\lambda^{-1}]]$. However, in both cases the setting that turned out to be the best choice from the technical point of view was not the natural choice and the reason for this is not clear at present.

The use of (C, A)-invariant subspaces is important in observation problems. In fact, the dual of the disturbance decoupling problem (DDP), the simplest application of

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the geometric control theory [12] is the disturbance decoupled estimation problem (DDEP), studied by Schumacher [11]. However, making one further step to the problem of disturbance decoupling by observation feedback (PDDOF) already forces one to study (A, B)- and (C, A)-invariant subspaces simultaneously [11], [13]. Thus, it seems important to be able to give polynomial characterizations of these subspaces and this is done through a study of the output injection group.

Finally, we study in the polynomial framework the maximal reachability subspace in ker C and obtain a nice characterization easily computable using the invariant factor algorithm, which gives insight to the nature of the transmission zeros of a system, without recourse to the Smith–McMillan form.

The structure of the paper is as follows. Section II is devoted to a general study of duality in polynomial models. In Section III we analyze the dual of the feedback group, namely, the output injection group as well as give a polynomial characterization of (C, A)-invariant subspaces. Section IV is devoted to a polynomial characterization of the maximal reachability subspace in ker C.

The results on duality owe much to many discussions on this subject with S. K. Mitter. Some of the results on (C, A)-invariant subspaces have been independently discovered by M. Kaashoek.

II. Duality in Polynomial Models

Let F be an arbitrary field, $F[\lambda]$ being the ring of polynomials. An m-dimensional vector space over F will be generally identified with F^m. $F^m((\lambda^{-1}))$ is the $F[\lambda]$-module of truncated Laurent series with coefficients in F^m, i.e., the set of series of the form $f(x)=\sum_{j=-\infty}^{n_f} f_j\lambda^j$. The quotient module $F^m((\lambda^{-1}))/F^m[\lambda]$ will be identified with $\lambda^{-1}F^m[[\lambda^{-1}]]$ the space of formal power series in λ^{-1} with coefficients in F^m and vanishing constant term. As usual π_+ and π_- will denote the projections of $F^m((\lambda^{-1}))$ on $F^m[\lambda]$ and $\lambda^{-1}F^m[[\lambda^{-1}]]$, respectively. Given a column vector $\xi\in F^m$, then $\tilde{\xi}$ will denote its transpose. If we define

$$[\xi,\eta]=\tilde{\eta}\xi, \tag{2.1}$$

then F^m is identified with its dual space. Given a polynomial matrix $P\in F^{p\times m}[\lambda]$, with $P(\lambda)=\sum_{j=0}^{n} P_j\lambda^j$, we define $\tilde{P}\in F^{m\times p}[\lambda]$ by

$$\tilde{P}(\lambda)=\sum_{j=0}^{n}\tilde{P}_j\lambda^j.$$

Next we define a pairing between elements of $F^m((\lambda^{-1}))$. To this end let $f, g\in F^m((\lambda^{-1}))$ be given by $f(\lambda)=\sum_{j=-\infty}^{n_f} f_j\lambda^j$ and $g(\lambda)=\sum_{j=-\infty}^{n_g} g_j\lambda^j$. We define $[f,g]$ by

$$[f,g]=\sum_{j=-\infty}^{\infty}\tilde{g}_j f_{-j-1}. \tag{2.2}$$

It is clear that $[f,g]$ is a bilinear form on $F^m((\lambda^{-1}))$. That $[f,g]$ is well defined follows from the fact that the sum in (2.2) has always at most a finite number of nonzero terms. We also note that $[f,g]=0$ for all $g\in F^m((\lambda^{-1}))$ if and only if $f=0$.

Given a subset M of $F^m((\lambda^{-1}))$ we define $M^\perp\subset F^m((\lambda^{-1}))$ by

$$M^\perp=\{g\in F^m((\lambda^{-1}))\,|\,[f,g]=0 \text{ for all } f\in M\}. \tag{2.3}$$

In particular, we have the following simple result:

$$(F^m[\lambda])^\perp=F^m[\lambda]. \tag{2.4}$$

The dual space of $F^m[\lambda]$, i.e., the space of F-linear functionals, is easily characterized.

Theorem 2.1: The dual space of $F^m[\lambda]$ is isomorphic to $\lambda^{-1}F^m[[\lambda^{-1}]]$.

Proof: Clearly, given $h\in\lambda^{-1}F^m[[\lambda^{-1}]]$, then the pairing $[f,h]$ of (2.2) defines a linear functional on $F^m[\lambda]$. Conversely, if ϕ: $F^m[\lambda]\to F$ is a linear functional, then ϕ is uniquely determined by its action on elements of the form $\xi\lambda^n$. As $\phi(\xi\lambda^n)$ is, with n fixed, a linear functional on F^m, we have the existence of η_n such that $\phi(\xi\lambda^n)=\tilde{\eta}_n\xi$. It is now easily checked that

$$\phi(f)=[f,h] \tag{2.5}$$

with $h(\lambda)=\sum_{j=1}^{\infty}\eta_j\lambda^{-j-1}$.

Consider now the two shift operators S_+ and S_- acting in $F^m[\lambda]$ and $\lambda^{-1}F^m[[\lambda^{-1}]]$, respectively, and define by

$$(S_+f)(\lambda)=\lambda f(\lambda) \qquad \text{for } f\in F^m[\lambda] \tag{2.6}$$

and

$$S_-h=\pi_-(\lambda h) \qquad \text{for } h\in\lambda^{-1}F^m[[\lambda^{-1}]]. \tag{2.7}$$

Given a linear transformation A: $F^m[\lambda]\to F^p[\lambda]$ its dual or adjoint, denoted by A^*, is the unique transformation A^*: $\lambda^{-1}F^p[[\lambda^{-1}]]\to\lambda^{-1}F^m[[\lambda^{-1}]]$ that satisfies

$$[Af,h]=[f,A^*h] \tag{2.6}$$

for all $f\in F^m[\lambda]$ and $h\in\lambda^{-1}F^p[[\lambda^{-1}]]$.

Lemma 2.2: The dual of S_+ is S_-.

Proof: This follows from the easily checked fact that

$$[S_+f,h]=[f,S_-h] \tag{2.7}$$

holds for all $f\in F^M[\lambda]$ and $h\in\lambda^{-1}F^m[[\lambda^{-1}]]$.

The way we identified $F^m[\lambda]^*$ is compatible with the $F[\lambda]$-module structures on $F^m[\lambda]$ and $\lambda^{-1}F^m[[\lambda^{-1}]]$.

Lemma 2.3: Let $v\subset F^m[\lambda]$ be an $F[\lambda]$-submodule; then $V^\perp\subset\lambda^{-1}F^m[[\lambda^{-1}]]$ is also a submodule.

Proof: This follows from (2.7).

The next two lemmas provide simple computational rules.

Lemma 2.4: Given the projections π_+ and π_- we have for all $f, g\in F^m((\lambda^{-1}))$ that

$$[\pi_+f,g]=[f,\pi_-g]. \tag{2.8}$$

Lemma 2.5: Given $A\in U^{p\times m}[\lambda]$, $f\in F^m[\lambda]$, and $h\in\lambda^{-1}F^p[[\lambda^{-1}]]$, then

$$[Af, h] = [f, \pi_- \tilde{A} h]. \tag{2.9}$$

Since multiplication by elements of $F^{p\times m}[\lambda]$ represent all $F[\lambda]$-module homomorphisms from $F^m[\lambda]$ into $F^p[\lambda]$, then Lemma 2.5 describes a class of $F[\lambda]$-module homomorphisms from $\lambda^{-1}F^p[[\lambda^{-1}]]$ into $\lambda^{-1}F^m[[\lambda^{-1}]]$. For some results related to this one can refer to [4].

In some cases, given a submodule $V \subset F^m[\lambda]$ the submodule $V^\perp$ of $\lambda^{-1}F^m[[\lambda^{-1}]]$ can be identified. To this end we recall that a submodule V of $F^m[\lambda]$ is called a full submodule if $F^m[\lambda]/V$ is a torsion module or equivalently if V has a representation

$$V = DF^m[\lambda] \tag{2.10}$$

with $D \in F^{m\times m}[\lambda]$ a nonsingular polynomial matrix. Next we recall [2], [4], [6] that given a nonsingular $D \in F^{m\times m}[\lambda]$ we can define two projections, π_D; $F^m[\lambda] \to F^m[\lambda]$ and π^D: $\lambda^{-1}F^m[[\lambda^{-1}]] \to \lambda^{-1}F^m[[\lambda^{-1}]]$, by

$$\pi_D f = D\pi_- D^{-1} f \qquad \text{for } f \in F^m[\lambda] \tag{2.11}$$

and

$$\pi^D h = \pi_- D^{-1} \pi_+ Dh \qquad \text{for } h \in \lambda^{-1}F^m[[\lambda^{-1}]]. \tag{2.12}$$

We denote by K_D and L_D the ranges of π_D and π^D, respectively, and note the equality

$$D^{-1}K_D = L_D. \tag{2.13}$$

Theorem 2.6: Let $V = DF^m[\lambda]$ *with* D *nonsingular in* $F^{m\times m}[\lambda]$. *Then*

$$V^\perp = L_{\tilde{D}}. \tag{2.14}$$

Proof: Let $f \in F^m[\lambda]$ and $h \in V^\perp$; then $0 = [Df, h] = [f, \tilde{D}h] = [f, \pi_- \tilde{D}h]$. But this implies $h \in L_{\tilde{D}}$. The converse follows from the same formulas.

Next we compute the adjoint of the projection π_D.

Theorem 2.7: The adjoint of the projection π_D *is* $\pi^{\tilde{D}}$.

Proof: Let $f \in F^m[\lambda]$ and $h \in \lambda^{-1}F^m[[\lambda^{-1}]]$; then

$$\begin{aligned}
[\pi_D f, h] &= [D\pi_- D^{-1} f, h] = [\pi_- D^{-1} f, \tilde{D}h] \\
&= [D^{-1} f, \pi_+ \tilde{D} h] = [f, \tilde{D}^{-1}\pi_+ \tilde{D} h] \\
&= [\pi_+ f, \tilde{D}^{-1}\pi_+ \tilde{D} h] = [f, \pi_- \tilde{D}^{-1}\pi_+ \tilde{D} h] \\
&= \left[f, \pi^{\tilde{D}} h\right].
\end{aligned}$$

Our main interest is to get a convenient and useful representation for K_D^*. To this end we note that, in general, given a linear space X and a subspace M, then if X^* is the dual space of X, we have the isomorphism

$$(X/M)^* = M^\perp. \tag{2.15}$$

Recall also [4] that S_D: $K_D \to D_D$ and S^D: $L_D \to L_D$ are defined by

$$S_D f = \pi_D \lambda f \quad \text{and} \quad S^D = S_- | L_D, \tag{2.16}$$

respectively.

Theorem 2.8: Let $D \in F^{m\times m}[\lambda]$ *be nonsingular; then*

$$K_D^* = L_{\tilde{D}} \tag{2.17}$$

and

$$S_D^* = S^{\tilde{D}}. \tag{2.18}$$

Proof: Since K_D is isomorphic to $F^m[\lambda]/DF^m[\lambda]$, then K_D^* is isomorphic to $(F^m[\lambda]/DF^m[\lambda])^*$ which by the previous remark is isomorphic to $(DF^m]\lambda])^\perp$. By Theorem 2.6, this is equal to $L_{\tilde{D}}$. It is now easily checked that under the pairing (2.2) we actually have (2.17).

Finally, let $f \in K_D$ and $h \in L_{\tilde{D}}$; then

$$\begin{aligned}
[S_D f, h] &= [\pi_D \lambda f, h] = \left[\lambda f, \pi^{\tilde{D}} h\right] \\
&= [\lambda f, h] = [f, \lambda h] \\
&= [\pi_+ f, \lambda h] = [f, \pi_- \lambda h] = \left[f, S^{\tilde{D}} h\right].
\end{aligned}$$

Now the $F[\lambda]$-module $L_{\tilde{D}}$ is isomorphic to $K_{\tilde{D}}$; hence, we can identify K_D^* with $K_{\tilde{D}}$ by defining for all $f \in K_D$ and all $g \in K_{\tilde{D}}$

$$\langle f, g \rangle = [D^{-1} f, g] = [f, \tilde{D}^{-1} g]. \tag{2.19}$$

As a direct corollary of Theorem 2.8 we have the following.

Theorem 2.9: The dual space of K_D *can be identified, under the pairing* (2.19), *with* $K_{\tilde{D}}$. *Moreover, we have*

$$S_D^* = S_{\tilde{D}}, \tag{2.20}$$

i.e.,

$$\langle S_D f, g \rangle = \langle f, S_{\tilde{D}} g \rangle \tag{2.21}$$

for all $f \in K_D$ *and* $g \in K_{\tilde{D}}$.

In [2, Theorem 4.5] the homomorphisms between two models K_D and K_{D_1} were characterized in the following way. A map X: $K_D \to K_{D_1}$ is an $F[\lambda]$-homomorphism, i.e., satisfies

$$XS_D = S_{D_1} X \tag{2.22}$$

if and only if

$$Xf = \pi_{D_1} \Xi f \qquad \text{for } f \in K_D \tag{2.23}$$

where Ξ, Ξ_1 are polynomial matrices satisfying

$$\Xi D = D_1 \Xi_1. \tag{2.24}$$

Moreover, X is injective if and only if D and Ξ_1 are right coprime and surjective if and only if Ξ and D_1 are left coprime. It is of interest to find a simple expression for the dual map X^*: $K_{\tilde{D}_1} \to K_{\tilde{D}}$. We have the following.

Theorem 2.10: If X: $K_D \to K_{D_1}$ *is the map defined by* (2.22) *and* (2.23), then X^*: $K_{\tilde{D}_1} \to \tilde{K}$ is given by

$$X^* g = \pi_{\tilde{D}} \tilde{\Xi}_1 g \qquad \text{for } g \in K_{\tilde{D}_1} \tag{2.25}$$

where

$$\tilde{\Xi}_1 \tilde{D}_1 = \tilde{D} \tilde{\Xi}. \tag{2.26}$$

Proof: Let $f \in K_D$ and $g \in K_{\tilde{D}_1}$; then

$$\begin{aligned}\langle Xf, g\rangle &= \langle \pi_{D_1}\Xi f, g\rangle \\ &= \left[D_1^{-1}\pi_{D_1}\Xi_1 f, g\right] = \left[D_1^{-1}D_1\pi_- D_1^{-1}\Xi f, g\right] \\ &= \left[f, \tilde{\Xi}\tilde{D}_1^{-1}g\right] = \left[f, \tilde{D}^{-1}\tilde{\Xi}_1 g\right] \\ &= \left[f, \pi_-\tilde{D}^{-1}\tilde{\Xi}_1 g\right] = \left[f, \tilde{D}^{-1}\tilde{D}\pi_-\tilde{D}^{-1}\tilde{\Xi}_1 g\right] \\ &= \left[D^{-1}f, \pi_{\tilde{D}}\tilde{\Xi}_1 g\right] = \langle f, X^* g\rangle\end{aligned}$$

which proves the theorem.

It should be noted that the condition for injectivity of X^*, namely, the right coprimeness of $\tilde{\Xi}$ and $\tilde{D}_1$, which is the same as the left coprimeness of Ξ and D_1, coincides with the condition for surjectivity of X. Similarly, this is so for the other coprimeness conditions.

Submodules of K_D are associated with factorization of D. In fact, a subspace $V \subset K_D$ is a submodule if and only if $V = EK_F$ for some factorization $D = ED$ into nonsingular factors [6]. One is naturally interested in the corresponding representation of $V^\perp \subset K_{\tilde{D}}$.

Theorem 2.11: Let $V \subset K_D$ be a submodule with the representation $V = EK_F$. Then $V^\perp \subset K_{\tilde{D}}$ is also a submodule and is given by $V^\perp = \tilde{F}K_{\tilde{E}}$.

Proof: That $V^\perp$ is a submodule, or equivalently $S_{\tilde{D}}$-invariant follows from (2.21). Let now $f \in V^\perp$; then for every $g \in K_F$ we have

$$0 = \langle Eg, f\rangle = \left[D^{-1}Eg, f\right] = \left[F^{-1}g, f\right] = \left[g, \tilde{F}^{-1}f\right]$$

or $\tilde{F}^{-1}f \in K_F$. But clearly, $\tilde{F}^{-1}f \in [(F \cdot F^m[\lambda])^\perp$ as for any $g \in F^m[\lambda]$

$$\left[Fg, \tilde{F}^{-1}f\right] = \left[g, f\right] = 0.$$

The two identities imply $\pi_-\tilde{F}^{-1}f = 0$ or $f = \tilde{F} \cdot f_1$ with $f_1 \in F^m(\lambda)$. Now $f \in K_D$ implying $\pi_+\tilde{D}^{-1}f = 0$. Hence, $\pi_+\tilde{E}^{-1}f_1 = 0$ or $f_1 \in K_{\tilde{E}}$, and consequently $f \in \tilde{F}K_{\tilde{E}}$. Conversely, if $f \in \tilde{F}K_{\tilde{E}}$ and $g \in EK_F$, then $f = Ff_1$, $g = Eg_1$ with $f_1 \in K_{\tilde{E}}$ and $g_1 \in KDF$. Then

$$\langle g, f\rangle = \left[D^{-1}Eg_1, \tilde{F}f_1\right] = \left[g_1, f_1\right] = 0.$$

It may be noted that $\dim V = \deg\det F$, $\dim V^\perp = \deg\det \tilde{E} = \deg\det E$ and so $\dim V + \dim V^\perp = \deg\det E + \deg\det F = \deg\det D = \dim K_D$.

So far our considerations were purely module theoretic. Our next step is to relate these concepts of duality to the study of systems. Suppose we are given a strictly proper $p \times m$ transfer G which we assume to have a representation of the form

$$G(\lambda) + N(\lambda)D(\lambda)^{-1}M(\lambda) + P(\lambda) \tag{2.27}$$

with N, M, and P polynomial matrices of appropriate sizes. As in [3] we associate with this representation of G a realization (A, B, C) in the following way. We let K_D be our state space and define the operators A, B, C by

$$A = S_D, \tag{2.28}$$

$$B\xi = \pi_D M\xi \qquad \text{for } \xi \in F^m, \tag{2.29}$$

and

$$Cf = (ND^{-1}f)_{-1} \qquad \text{for } f \in K_D. \tag{2.30}$$

We call this the realization associated with the representation (2.27). That it is indeed a realization is easily checked, the proof being given in [3].

It is of interest to compute the adjoints of the maps A, B, and C. For A the answer is given by Theorem 2.9.

Next we compute B^*: $K_{\tilde{D}} \to F^m$. Let $g \in K_{\tilde{D}}$ and $\xi \in F^m$. Then

$$\begin{aligned}\langle B\xi, g\rangle &= \left[D^{-1}\pi_D M\xi, g\right] = \left[D^{-1}D\pi_- D^{-1}M\xi, g\right] \\ &= \left[\xi, \tilde{M}\tilde{D}^{-1}g\right] = \tilde{\xi}(\tilde{M}\tilde{D}^{-1}g)_{-1}.\end{aligned}$$

Thus, we proved

$$B^* g = (\tilde{M}\tilde{D}^{-1}g)_{-1}. \tag{2.31}$$

Finally, we note that with $\eta \in F^p$ and $f \in K_D$ we have

$$\begin{aligned}\tilde{\eta}Cf &= \tilde{\eta}(ND^{-1}f)_{-1} = \left[ND^{-1}f, \eta\right] \\ &= \left[f, \tilde{D}^{-1}\tilde{N}\eta\right] = \left[f, \pi_-\tilde{D}^{-1}\tilde{N}\eta\right] \\ &= \left[D^{-1}f, \tilde{D}\pi_-\tilde{D}^{-1}N\eta\right] = \langle f, \pi_D\tilde{N}\eta\rangle\end{aligned}$$

or

$$C^*\eta = \pi_{\tilde{D}}\tilde{N}\eta. \tag{2.32}$$

Combining these results can be summarized by the following.

Theorem 2.12: The adjoint of the realization of the transfer function G associated with the representation $G = ND^{-1}M + P$ is the realization of $\tilde{G}$ associated with the representation $\tilde{G} = \tilde{M}\tilde{D}^{-1}\tilde{N} + \tilde{P}$.

In particular, this implies that the two associated polynomial system matrices are related by transposition.

One can look also at duality from the input/output point of view. To this end let $f: F^m[\lambda] \to \lambda^{-1}F^p[[\lambda^{-1}]]$ be a restricted input/output map, that is, an $F[\lambda]$-homomorphism. There exists a dual map f^*: $(\lambda^{-1}F^p[[\lambda^{-1}]])^* \to (F^m[\lambda])^*$. We already identified $(F^m[\lambda])^*$ with $\lambda^{-1}F^m[[\lambda^{-1}]]$. Now $(\lambda^{-1}F^p[[\lambda^{-1}]])^*$ is generally too big. However, it contains a copy of $F^p[\lambda]$ as each space is embedded in its double dual. If we restrict f^* to $F^p[\lambda]$ we obtain a module homomorphism from $F^p[\lambda]$ into $\lambda^{-1}F^m[[\lambda^{-1}]]$ which we still denote by f^*. This map will be called the dual input/output map.

If we assume the input/output map to have G as transfer function, then

$$f(u) = \pi_- Gu \qquad \text{for } u \in F^m[\lambda]. \tag{2.33}$$

Given any $v \in F^p[\lambda]$ and $g \in F^m[\lambda]$ we have $f^*(v) \in (F^m[\lambda])$ and computing

$$\begin{aligned}\left[f^*(v), g\right] &= \left[v, f(g)\right] = \left[v, \pi_- Gg\right] \\ &= \left[\pi_+ v, Gg\right] = \left[v, Gg\right] \\ &= \left[\tilde{G}v, g\right] = \left[\pi_-\tilde{G}v, g\right]\end{aligned}$$

and to

$$f^*(v)=\pi_-\tilde{G}v \quad \text{for } v\in F^p[\lambda]. \tag{2.34}$$

Hence, the transfer function associated with f^* is just G.

To conclude this section we establish how Toeplitz operators, playing such a prominent role in the study of feedback [5], transforms by duality.

Here we have two options. First given $A\in F^{p\times m}((\lambda^{-1}))$ we define the induced Toeplitz operator T_A: $F^m[\lambda]\to F^p[\lambda]$ by

$$T_A f=\pi_+ Af \quad \text{for } f\in F^m[\lambda]. \tag{2.35}$$

The adjoint map T_A: $\lambda^{-1}F^p[[\lambda^{-1}]]\to\lambda^{-1}F^m[[\lambda^{-1}]]$ is given by

$$T_A^* h=\pi_-\tilde{A}h \tag{2.36}$$

which operator we also denote by $T^{\tilde{A}}$. This is a direct consequence of the equality

$$[T_A f, h]=[\pi_+ Af, h]=[Af, h]$$
$$=[f, \tilde{A}h]=[f, \pi_-\tilde{A}h].$$

The second approach is to study the Toeplitz map from K_D into K_{D_1}. We deal only with the case that $\Gamma=D_1D^{-1}$ is a bicausal isomorphism. In that case we know that actually $T_{DD_1^{-1}}$ is an invertible map from K_{D_1} onto K_D [5, Theorem 4.3].

Theorem 2.13: The dual map $T^*_{DD_1}-1$ of $T_{DD_1}-1$ is the map from K_D onto $K_{\tilde{D}_1}$ given by

$$T^*_{DD_1^{-1}}f=f \quad \text{for all } f\in K_{\tilde{D}_1}. \tag{2.37}$$

Proof: First we note that the map X: $K_{\tilde{D}_1}\to K_{\tilde{D}}$ given by $XF=f$ is well defined. This is a consequence of the part [6, Lemma 5.5] that if $T_1^{-1}T$ is a bicausal isomorphism, then K_T and K_{T_1} contain the same elements (but differ in their structure).

To prove (2.37) let g and f be arbitrary elements of $K_{\tilde{D}}$ and $K_{\tilde{D}_1}$, respectively. Then

$$\langle f, T^*_{DD_1^{-1}}g\rangle=\langle T_{DD_1^{-1}}f, g\rangle$$
$$=[D^{-1}\pi_+DD_1^{-1}f, g]=[\pi_+DD_1^{-1}f, \tilde{D}^{-1}g]$$
$$=[DD_1^{-1}f, \tilde{D}^{-1}g]=[D_1^{-1}f, g]=\langle f, g\rangle$$

which proves the theorem.

This result indicates already that the study of the dual of the feedback groups and hence also the study of (C, A)-invariant subspaces may be substantially simpler than the study of feedback itself. This will be taken up in the next section.

III. The Output Injection Group and (C, A)-Invariant

Suppose (A, B, C) is an observable realization of a $p\times m$ transfer function G, i.e., $G(\lambda)=C(\lambda I-A)^{-1}B$. Since C and $(\lambda I-A)$ are right coprime it follows that G can be written as $G(\lambda)=T(\lambda)^{-1}U(\lambda)$ and the realization associated with this representation in the state space K_T is isomorphic to the original system. We define the output injection group as the group which acts on triples by $(A, B, C)\to(R^{-1}(A+HC)R, R^{-1}B, PCR)$ with P and R invertible. This is clearly the dual to the feedback group. Our main interest is to study the changes in the transfer function G by application of a group element.

The result that follows is a reformulation of a theorem of Hautus and Heymann [8], [5] in this context. Thus, one approach to prove the theorem is to dualize the corresponding feedback result. Since, however, a direct proof for the output injection case is easier than that of the feedback case it is of interest to give an independent derivation with the option of getting the Hautus–Heymann theorem by duality considerations. This we proceed to do adapting the argument in [5]. First we note the following standard result in linear algebra.

Lemma 3.1: Let V_0, V_1, V_2 be finite-dimensional linear spaces over a field F and let D: $V_0\to V_2$ and C: $V_0\to V_1$ be linear transformations. Then there exists a linear transformation H: $V_1\to V_2$ such that

$$D=HC \tag{3.1}$$

if and only if

$$\ker D\supset\ker C. \tag{3.2}$$

Theorem 3.2: Let (A, B, C) be an observable realization of the transfer function $G(\lambda)=T(\lambda)^{-1}U(\lambda)$. Then $G_1(\lambda)$ is the transfer function of a system (A_1, B_1, C_1) output injection equivalent to (A, B, C) if and only if $G_1(\lambda)=T_1(\lambda)^{-1}U(\lambda)$ and $T_1(\lambda)^{-1}T(\lambda)$ is a bicausal isomorphism.

Proof: Clearly, similarity transformations do not change the transfer function and a change of basis transformation in the output space changes the transfer function by left multiplication by the invertible map. Thus, we assume without loss of generality that $A_1=A+HC$, $B_1=B$, and $C_1=C$. Then

$$C_1(\lambda I-A_1)^{-1}=C(\lambda I-HC)^{-1}$$
$$=C[(I-HC(\lambda I-A)(\lambda I-A)]^{-1}$$
$$=C(\lambda I-A)^{-1}(I-HC(\lambda I-A)^{-1})^{-1}$$
$$=(I-C(I-A)^{-1}H)^{-1}C(\lambda I-A)^{-1}$$

which in turn implies that

$$G_1(\lambda)=C_1(\lambda I-A_i)^{-1}B_1$$
$$=\Gamma(\lambda)^{-1}G(\lambda)=\Gamma(\lambda)^{-1}T(\lambda)^{-1}U(\lambda)$$

where $\Gamma(\lambda)=(I-C(\lambda C-A)^{-1}H)$ is a bicausal isomorphism. Moreover,

$$T_1(\lambda)=T(\lambda)\Gamma(\lambda)=T(\lambda)+T(\lambda)C(\lambda I-A)^{-1}H$$
$$=T(\lambda)+Q(\lambda)$$

where $Q(\lambda)$ is a polynomial matrix such that $T(\lambda)^{-1}Q(\lambda)$ is strictly proper.

Conversely, assume $T_1(\lambda)=T(\lambda)+Q(\lambda)$ with $T^{-1}Q$ strictly proper. Then $\Gamma=T_1^{-1}\Gamma$ is a bicausal isomorphism with the constant term equal to the identity. By [6, Lemma 5.5], K_T and K_{T_1} are equal as sets. Let (A,C) and (A_1,C_1) be the transformations arising out of the factorizations $T^{-1}U$ and $T_1^{-1}U$ as given by formula (2.23) and (2.25). As the constant term of $T_1^{-1}T$ is the identity it follows that for $f\in K_T=K_{T_1}$

$$Cf=(T^{-1}f)_{-1}=(T_1^{-1}TT^{-1}f)_{-1}=(T_1^{-1}f)_{-1}=C_1f$$

of $C=C_1$.

To complete the proof it suffices to show the existence of maps X: $K_{T_1}\to K_T$ and H: $F^p[\lambda]\to K_T$ such that

$$XA_1-AX=HC. \tag{3.3}$$

We will prove (3.3) for the map X given by $Xf=f$. Thus, using Lemma 3.1 it suffices to show that $\ker(A_1-A)\supset\ker C$. To this end let $f\in\ker C=\{f\in K_T|(T^{-1}f)_{-1}=0\}$. Computing S_Tf we find

$$S_Tf=\pi_T\lambda f=T\pi_-T^{-1}\lambda f=T\cdot T^{-1}\lambda f=\lambda f$$

as by our assumption $\lambda T^{-1}f$ is strictly proper. As the same is true for S_{T_1} it follows that $(S_T-S_{T_1})f=0$ for every $f\in\ker C$. This proves the theorem.

We pass into the characterization of (C,A)-invariant subspaces in polynomial terms. A subspace V of the state space X is called (C,A)-invariant if there exists a linear transformation H such that $(A+HC)V\subset V$. It has been shown in [11] that V is (C,A)-invariant if and only if $A(V\cap\ker C)\subset V$.

Theorem 3.3: Let (A,B,C) be the observable realization associated with the transfer function $G(\lambda)=T(\lambda)^{-1}U(\lambda)$. Then a subspace $V\subset K_T$ is a (C,A)-invariant subspaces if and only if

$$V=E_1K_{F_1} \tag{3.4}$$

where $T_1=E_1F_1$ is such that $T_1^{-1}T$ is a bicausal isomorphism.

We will give two proofs of the theorem.

Proof I: V is (C,A)-invariant if and only if it is invariant for $A_1=A+HC$. In the case of the pair (A,C) arising out of $G=T^{-1}U$ (A_1,C) will be associated, by Theorem 3.2, with $T_1^{-1}U$ where $T_1^{-1}T$ is a bicausal isomorphism. Thus, since K_T and K_{T_1} are equal as sets, V is an S_{T_1}-invariant subspace of K_{T_1}. Those are, by [6, Theorem 2.9], of the form $V=E_1K_{F_1}$ with $T_1=E_1F_1$.

Proof II: In this proof we use duality and the results of [6]. The subspace V of K_T is (C,A)-invariant if and only if $V^\perp\subset K_{\tilde T}$ is (A,C)-invariant, i.e., and $(S_{\tilde T},\pi_{\tilde T})$-invariant subspace. By [6, Theorem 4.2] there exists a $T_1\in F^{p\times p}[\lambda]$ such that TT_1^{-1} is a bicausal isomorphism and

$$V^\perp=\pi_{\tilde T}T_{\tilde T\tilde T_1^{-1}}(\tilde F_1K_{\tilde E_1})$$

where $T_1=E_1F_1$ (hence, also $\tilde T_1=\tilde F_1\tilde E_1$). By elementary properties of dual maps we have

$$T^*_{T\tilde T_1^{-1}}V=V_1\subset K_{T_1}$$

and $V_1^\perp=\tilde F_1K_{\tilde E_1}$. By Theorem 2.10 we have $V_1=E_1K_{F_1}$ and since

$$(\pi_{\tilde T}T_{\tilde T\tilde T_1^{-1}})^*K_T\to K_{T_1}$$

acts as the identity map, it follows that $V=E_1K_{F_1}$.

Corollary 3.4: If a (C,A)-invariant subspace of K_T of the form $E_1K_{F_1}$ contains $\mathsf{B}=$ Range $B=\{U\xi|\xi\in F^m\}$, then there exists a $U_1\in F^{p\times m}[\lambda]$ such that $U=E_1U_1$.

Proof: For each $\xi\in F^m$, $U\xi\in E_1K_{F_1}$ so $U\xi=E_1f_\xi$ from which the result follows.

Lemma 3.5: Let $V\subset K_T$ be a (C,A)-invariant subspace, having the representation $V=E_1K_{F_1}$ of Theorem 3.3. Then $f\in K_T$ is in V if $f=E_1g$ for some $g\in F^p[\lambda]$.

Proof: If $f\in E_1F_{K_1}$, then clearly $f=E_1g$ for some $g\in K_{F_1}\subset F^p[\lambda]$. Suppose conversely that $f\in K_T$ and $f=E_1g$. Since $f\in K_T$, and as K_T and K_{T_1} are equal, by [6, Lemma 5.5], as sets we have $f\in K_{T_1}$. Hence, $f=T_1h=E_1h$ for some $h\in\lambda^{-1}F^p[[\lambda^{-1}]]$. From $E_1F_1h=E_1g$ and the nonsingularity of E_1 it follows that $g=F_1h$ or $g\in K_{F_1}$ and the proof is complete.

Theorem 3.3 can be slightly generalized to yield a clear characterization of (C,A)-invariant subspaces of K_T. The result is the counterpart of [6, Theorem 4.6].

Theorem 3.6: A subspace $V\subset K_T$ is a (C,A)-invariant subspace if and only if $V=K_T\cap M$ for some submodule $M\subset F^p[\lambda]$.

Proof: The "only if" part follows from Theorem 3.3 and Lemma 3.5. To prove the "if" part assume $V=K_T\cap M$ where M is any submodule of $F^p[\lambda]$. We show that V is (C,A)-invariant. Let $f\in V\cap\ker C$, then $(T^{-1}f)_{-1}=0$ which implies that $S_Tf=\pi_T\lambda f=\lambda f$. But $S_Tf\in K_T$ and $S_Tf=\lambda f\in M$. Thus, $S_Tf\in K_T\cap M=V$ which proves the theorem.

Next we characterize the left factors $E_1\in F^{p\times p}[\lambda]$ that can be right multiplied to yield a polynomial $T_1=E_1F_1$ for which $T_1^{-1}T$ is a bicausal isomorphism. This is the dual result to [6, Theorem 4.4].

Theorem 3.7: Let $T,E_1\in F^{p\times p}[\lambda]$ be nonsingular. Then there exists $F_1\in F^{p\times p}[\lambda]$ such that

i) $T_1=E_1F_1$
ii) *$T_1^{-1}T$ is a bicausal isomorphism*

if and only if all the right Wiener–Hopf factorization indexes at infinity of $E_1^{-1}T$ are nonnegative.

Proof: The proof is as of [6, Theorem 4.4] or follows from that theorem by duality.

Theorem 3.8: Let $G(\lambda)=T(\lambda)^{-1}U(\lambda)$ be a strictly proper $p\times m$ rational function of full row rank and assume the factorization is left coprime. Let (A,B,C) be the realization associated with this factorization in the state space K_T. Let $E_\rho\in F^{p\times p}[\lambda]$ be such that $E_\rho F^p[\lambda]=UF^m[\lambda]$, i.e.,

$$U=E_\rho U_\rho \tag{3.5}$$

and U_ρ is right unimodular (right invertible element of $F^{p\times m}[\lambda]$). Then $V\subset K_T$ is a (C,A)-invariant subspace that contains $\mathsf{B}=$ Range B if and only if

$$V=E_\sigma K_{F_\sigma} \tag{3.6}$$

where $T_\sigma=E_\sigma F_{\sigma'} T_\sigma^{-1}T$ is a bicausal isomorphism and

$$E_\rho=E_\sigma\cdot H \tag{3.7}$$

for some $H\in F^{p\times p}[\lambda]$.

Proof: If $V\subset K_T$ has the representation (3.6) with $T_\sigma=E_\sigma F_\sigma' T_\sigma^{-1}T$ a bicausal isomorphism and (3.7) holds, then V is (C,A)-invariant by Theorem 3.3. By Lemma 3.4 $V=\{f\in K_T \mid f=E_\rho g,\ g\in F^p[\lambda]\}$. Now

$$\mathsf{B}=\{U(\lambda)\xi \mid \xi\in F^m\}=\{E_\rho(\lambda)U_\rho(\lambda)\xi) \mid \xi\in F^m\}$$
$$=\{E_\sigma(HU_\rho(\lambda)\xi) \mid \xi\in F^m\}\subset V.$$

To prove the converse we show first that there exists $F_\rho\in F^{p\times p}[\lambda]$ such that $T_\rho=E_\rho E_\rho$ and $T_\rho^{-1}T$ is a bicausal isomorphism.

To this end we show that all the right Wiener–Hopf factorization indexes at infinity of $T^{-1}E_\rho$ are nonpositive. $T^{-1}U$ and $T^{-1}E_\rho$ have the same right factorization indexes at infinity. To see this let $\begin{pmatrix} U_\rho \\ U_\tau \end{pmatrix}$ be any completion of U_ρ to a unimodular matrix in $F^{m\times m}[\lambda]$ and let $T^{-1}E_\rho=\Omega\Delta W$ be a right Wiener–Hopf factorization. Thus, Ω is a bicausal isomorphism, W is unimodular, and $\Delta(\lambda)=\operatorname{diag}(\lambda^{\alpha_1},\cdots,\lambda^{\alpha_p})$. Now $T^{-1}U=T^{-1}E_\rho U_\rho=\Omega\Delta W U_\rho=\Omega(\Delta 0)\begin{pmatrix} WU_\rho \\ U_\tau \end{pmatrix}$. $T^{-1}U$, being strictly proper, all its right factorization indexes α_i are nonpositive [7]. The existence of F_ρ follows from Theorem 3.6.

We proceed to show that the inclusion relation

$$E_\rho K_{F_\rho}\supset E_\rho K_{U_\rho} \tag{3.8}$$

holds. In fact, since $T_\rho=E_\rho F_\rho=T\Gamma$ where Γ is a bicausal isomorphism, it follows that $T_\rho^{-1}U=\Gamma^{-1}T^{-1}U=\Gamma^{-1}F_\rho^{-1}E_\rho E_\rho U_\rho=\Gamma^{-1}F_\rho^{-1}U_\rho$ or $F_\rho^{-1}U_\rho$ is strictly proper. This implies

$$K_{F_\rho}\supset K_{U_\rho} \tag{3.9}$$

and hence (3.8) follows too. We already saw at the beginning of the proof that $E_\rho K_{F\rho}\supset\mathsf{B}$.

Let now $V\subset K_T$ be (C,A)-invariant and assume $V\supset\mathsf{B}$. By Theorem 3.3 $V=E_\alpha K_{F_\alpha}$. Now $F^p[\lambda]\supset K_{F_\alpha}\supset E_\alpha^{-1}\mathsf{B}=\{E_\alpha^{-1}U\xi \mid \xi\in F^m\}$.

It follows that $F^p[\lambda]\supset E_\alpha^{-1}E_\rho F^p[\lambda]$ and so $H=E_\alpha^{-1}E_\rho\in F^{p\times p}[\lambda]$ or (3.7) follows.

We point out that another proof of this theorem can be obtained from [6, Theorem 5.3] by duality considerations. The details are simple and omitted.

Corollary 3.9: Under the assumptions of Theorem 3.6 *the minimal (C,A)-invariant subspace containing* B, *denoted by* $\mathsf{V}_*(\mathsf{B})$, *given by*

$$\mathsf{V}_*(\mathsf{B})=E_\rho K_{F_\rho}. \tag{3.10}$$

IV. On the Maximal Reachability Subspace in Ker C

Let G be a $p\times m$ strictly proper transfer function and let

$$G(\lambda)=T(\lambda)^{-1}U(\lambda) \tag{4.1}$$

be a left coprime factorization of G. With this factorization is associated a state space realization in K_T as described in Section II.

That there is a direct relation between (A,B)-invariant subspaces in $\ker C$ and nonsingular right factors of the numerator polynomial matrix in a coprime factorization of the transfer function has been established by Emre in [14]. In [6], however, specific representations have been obtained.

It has been shown in [6] that relative to this realization of G, every (A,B)-invariant subspace V of K_T which is included in $\ker C$ is of the form

$$V=U_0K_{E_0} \tag{4.2}$$

where

$$U=U_0E_0 \tag{4.3}$$

is a factorization of U with E_0 nonsingular, and every such subspace has such a representation. On the other hand, it was also shown in [6] that subspaces of the form

$$V=E_1K_{U_1} \tag{4.4}$$

where

$$U=E_1U_1 \tag{4.5}$$

is a factorization of U, with $E_1\in F^{p\times p}[\lambda]$ nonsingular, is also an (A,B)-invariant subspace contained in $\ker C$, but not all such subspaces have a representation of the second kind. One naturally looks for an intrinsic characterization of the second class of subspaces and it may not come as a surprise that the problem has to do with reachability subspaces.

For the analysis that follows we will assume that the transfer function G, as a matrix over the field of rational functions, has full row rank. Thus, in a left coprime factorization (2.1) the numerator matrix $U\in F^{p\times m}[\lambda]$ has full row rank over $F[\lambda]$. This assumption is not really necessary and with some obvious modifications the theorems and proofs can be adapted to the general case. Thus, since the factors in a left coprime factorization are determined only up to a common left unimodular factor, this factor can be chosen so that U is of the form

$$U(\lambda)=\begin{pmatrix} U'(\lambda) \\ 0 \end{pmatrix}$$

with U' of full row rank. The main results characterizing $R^*(\ker C)$ the maximal reachability subspace in $\ker C$, closely resembles the work of Khargonekar and Emre [9] but the final form seems to be more satisfactory.

As in the previous section we let

$$U=E_\rho U_\rho \tag{4.6}$$

with U_ρ right unimodular. This is possible by [6, Theorem 3.7].

Theorem 4.1: Let $G=T^{-1}U$ be strictly proper, the factorization left coprime and U assumed of full row rank with (4.6) holding and U_ρ right unimodular. Then we have

$$\mathsf{R}^*(\ker C)=E_\rho K_{U_\rho}. \tag{4.7}$$

Proof: Let $R=E_\rho K_{U_\rho}$. Then we know from [6, Theorem 5.6] that R is an (A, B)-invariant subspace included in $\ker C$. Next we show that $K_U\cap \mathsf{B}\subset R$. In fact, if $f\in K_U\cap \mathsf{B}$ and taking into account that $\mathsf{B}=\{U\xi|\xi\in F^m\}$ and that $K_U=\{f\in P^p[\lambda]|\, f=Uh,\ h\in\lambda^{-1}F^m[[\lambda]]\}$, it follows that $f=Uh=U\xi$. So $E_\rho U_\rho h=E_\rho U_\rho\xi$ and as E_ρ is nonsingular $U_\rho h=U_\rho\xi$ or $U_\rho h\in K_{U_\rho}$. So $f=E_\rho U_\rho h\in E_\rho K_{U_\rho}=R$. This implies that $\mathsf{R}^*(\ker C)\subset R$.

To prove the converse it suffices to show that R is a reachability subspace. Since $R=E_\rho KD_{U_\rho}$ and U_ρ in right unimodular, every element of R has a representation, not necessarily unique, of the form $f(\lambda)=U(\lambda)g(\lambda)$ with $g(\lambda)=\gamma_0+\gamma_1\lambda+\cdots+\gamma_s\lambda^s\in F^m[\lambda]$. Let $L=\{\xi\in F^m|\exists h\in\lambda^{-1}F^m[[\lambda^{-1}]],\ U\xi=Uh\}$. We prove first two lemmas.

Lemma 4.2: If $f(\lambda)(\gamma_0+\cdots+\gamma_s\lambda^s)\in K_U$, then $\gamma_s\in L$.

Proof: If $f\in K_U$, then $f=Uh$ for some $h\in\lambda^{-1}F^m[[\lambda^{-1}]]$. Let

$$h(\lambda)=\frac{h_{-1}}{\lambda}+\frac{h_{-2}}{\lambda^2}+\cdots$$

then

$$U(\lambda)\left(\gamma_s\lambda^s+\cdots+\lambda_0-\frac{h_{-1}}{\lambda}-\cdots\right)=0.$$

Therefore,

$$U(\lambda)\gamma_s=U(\lambda)\left(-\frac{\gamma_{s-1}}{\lambda}-\cdots-\frac{-\gamma_0}{\lambda^s}+\frac{h-1}{\lambda^{s+1}}+\cdots\right)$$

and $\gamma_S\in L$.

Lemma 4.3: Let $K\colon K_T\to F^m$ be such that $(S_T+BK)K_U\subset K_U$, then given $\gamma\in L$, $(S_T+BK)^S U\gamma=U(\gamma_0+\cdots+\gamma_S\lambda^S)$ with $\gamma_s=\gamma$.

Proof: By induction. For $s=1$ since $U_\gamma\in K_U$ we have $S_TU\gamma=\lambda U\gamma=U(\lambda\gamma)$. Also $BKU\gamma=U\gamma_0$ so $(S_T+BK)U\gamma=U(\gamma_0+\lambda\gamma_1)$ with $\gamma_1=\gamma$. Assume the result holds for $s-1$. Then

$$(S_T+BK)^sU\gamma=(S_T+BK)U(\gamma_0'+\cdots+\gamma_{s-1}'\lambda^{s-1})$$

with $\gamma_{s-1}'=\gamma$. Again $U(\gamma_0'+\cdots+\gamma_{s-1}'\lambda^{s-1})\in K_U$ and so

$$\begin{aligned}S_TU(\gamma_0'+\cdots+\gamma_{s-1}'\lambda^{s-1})&=\lambda U(\gamma_0'+\cdots+\gamma_{s-1}'\lambda^{s-1})\\&=U(\gamma_0'\lambda+\cdots+\gamma_{s-1}'\lambda^s)\end{aligned}$$

whereas $BKU(\gamma_0'+\cdots+\gamma_{s-1}'\lambda^{s-1})=U\gamma_0$ for some $\gamma_0\in F^m$. This proves the lemma.

We complete the proof of Theorem 4.1 by induction. Choose $K\colon K_T\to F^m$ so that $(S_T+BK)(K_U)\subset K_U$. We will show that if $f\in R$, then $f=\Sigma_j(S_T+BK)^jB\beta_j$ with $\beta_j\in L$.

If $f=U(\lambda)\xi\in R$ then, since $R\subset K_U$, $\xi\in L$ and we are done. Suppose we prove every $f\in R$ of the form $f(\lambda)=U(\lambda)(\gamma_0+\cdots+\gamma_{s-1}\lambda^{s-1})$ has such a representation. Let $f(\lambda)=U(\lambda)(\gamma_0+\cdots+\gamma_s\lambda^s)\in R$. By Lemma 4.2 $\gamma_s\in L$ and by Lemma 4.3 $(S_T+BK)^sU\gamma_s=U(\lambda)(\beta_0+\cdots+\beta_{s-1}\lambda^{s-1}+\gamma_s\lambda^s)$. Hence, $f-(S_T+BK)^sU\gamma_s=U(\lambda)(\gamma_0'+\cdots+\gamma_{s-1}'\lambda^{s-1})$ and we are done by the induction hypothesis.

Given a (A, B)-invariant subspace $V\subset K_T$ we let

$$F(V)=\{K\colon K_T\to F^m|(A+BK)V\subset V\}.$$

The following theorem will turn out to be a generalization of [1, Corollary 5.1].

Theorem 4.4: Let $K\colon K_T\to F^m$ be such that $K\in F(K_U)$. Then $K\in F(E_\alpha K_{U_\alpha})$ for every factorization

$$U=E_\alpha U_\alpha \tag{4.8}$$

with E_α nonsingular.

Proof: Given $f\in K_U$ we have $f=Uh$ for some $h\in\lambda^{-1}F^m[[\lambda^{-1}]]$. Thus, $T^{-1}f=T^{-1}Uh$ is the product of two strictly proper functions, hence $\lambda T^{-1}f=T^{-1}(\lambda f)$ is also proper. This implies that for $f\in K_U$

$$(S_Tf)(\lambda)=\pi_T\lambda f=\lambda f(\lambda). \tag{4.9}$$

Therefore, for $f\in K_U$ we have

$$(S_T+BK)f=\lambda f(\lambda)+U(\lambda)\xi_f$$

where $\xi_f=Kf\in F^m$ and depends linearly on f. If we assume the factorization (4.8) and that $f\in E_\alpha K_{U_\alpha}$, then $f=E_\alpha g$ with $g\in K_{U_\alpha}$ and

$$\begin{aligned}(S_T+BK)f&=\lambda f(\lambda)+U(\lambda)\xi_f\\&=E_\alpha(\lambda)\lambda g(\lambda)+E_\alpha(\lambda)U_\alpha(\lambda)\xi f\\&=E_\alpha(\lambda)\{\lambda g(\lambda)+U_\alpha(\lambda)\xi f\}.\end{aligned}$$

By Lemma 3.5 $(S_T+BK)f\in E_\alpha K_{U_\alpha}$ or $K\in F(E_\alpha K_{U_\alpha})$.

A special case is the following.

Corollary 4.5: $K\in F(\mathsf{V}^*(\ker C))$ implies $K\in F(\mathsf{R}^*(\ker C))$.

While Corollary 4.5 and later Corollary 4.8 do not present new results, it seems that the proof of Theorem 4.4, by analyzing the zero structure, sheds more light on this whole circle of results which may have been absent from the original proofs.

Given $K\in F(K_U)$ the K_U has a naturally induced $F[\lambda]$-module structure, namely, the one induced by the operator S_T+BK and $E_\rho K_{U_\rho}\simeq\mathsf{R}^*(\ker C)$ is a submodule. The next theorem identifies the quotient module structure.

Theorem 4.6: We have the $F[\lambda]$-module isomorphism

$$K_U/E_\rho K_{U_\rho}=\simeq K_{E_\rho}. \tag{4.10}$$

Proof: Choose $K\in F(K_U)$ which implies that $K\in F(E_\rho K_{U_\rho})$ and $E_\rho K_{U_\rho}$ is a submodule of K_U. Define a map R: $K_U \to K_{E_\rho}$ by

$$Rf=\pi_{E_\rho}f \qquad \text{for } f\in K_U. \tag{4.11}$$

We will show that R is a module homorphism of K_U onto K_{E_ρ} with $\ker R=E_\rho K_{U_\rho}$.

Indeed, for $f\in K_U$ we have

$$\begin{aligned}R(S_T+BK)f&=R(\lambda f+U\xi_f)\\&=\pi_{E_\rho}(\lambda f+U\xi_f)=\pi_{E_\rho}\lambda f\\&=\pi_{E_\rho}\lambda\pi_{E_\rho}f=S_{E_\rho}Rf\end{aligned}$$

or

$$R(S_T+BK)+S_{E_\rho}R \tag{4.12}$$

which shows that R is a module homorphism. To show that R is surjective we note that $K_U+E_\rho F^p[\lambda]=K_U+UF^m[\lambda]$.

Now U is assumed to be of full row rank; hence there exists a rational Ω such that $U\Omega=I$. Given $g\in F^p[\lambda]$ we have $g=U\Omega g=Ug_+ + U_{g_-}$ with $g_+=\pi_+\Omega g$ and $g_-=\pi_-\Omega g$. It follows that $U_{g-}=g-Ug_+\in K_U$ and $Ug_+\in UF^m[\lambda]$. This implies $K_U+UF^m[\lambda]=F^p[\lambda]$ or

$$K_U+E_\rho F^p[\lambda]=F^p[\lambda]. \tag{4.13}$$

Since $\pi_{E_\rho}F^p[\lambda]=K_{E_\rho}$ the map R is clearly surjective. Finally, $f\in\ker R$ is and only if $f=E_\rho f'$ for some $f'\in F^p[\lambda]$. By Lemma 3.5 this implies the equality $\ker R=E_\rho K_U$. This completes the proof. The proof of the surjectivity of the map R is adapted from [9].

Theorem 4.6 gives some insight into the nature of the transmission zeros of the transfer function $G=T^{-1}U$. The transmission zeros are usually defined, with (A,B,C) a canonical realization of G, to be the eigenvalues of the map $\overline{A+BK}$: $V^*(\ker C)/R^*(\ker C)\to V^*(\ker C)/R^*(\ker C)$ induced by $A+BK$ where $K\in F(V^*(\ker C))$. Now by (4.12) the map $A+BK$ is isomorphic to S_{E_ρ} and hence the transmission zeros are just the zeros of $\det E_\rho$. Moreover, the invariant factors of $\overline{S_T+BK}$ coincide with the invariant factors of E_ρ and thus the use of the Smith–McMillan form can be avoided.

Corollary 4.7: A subspace $V\subset K_T$ is an (A,B)-invariant subspace contained in $\ker C$ and containing $\mathsf{R}^*(\ker C)=E_\rho K_{U_\rho}$ if and only if

$$V=E_\alpha K_{U_\alpha} \tag{4.14}$$

with

$$U=E_\alpha U_\alpha \tag{4.15}$$

and E_α nonsingular, and for some H

$$E_\rho=E_\alpha H. \tag{4.16}$$

Proof: Assume V is of the form (4.14) with (4.15) and (4.16) satisfied. Then

$$\mathsf{R}^*(\ker C)=E_\rho K_{U_\rho}=E_\alpha HK_{U_\rho}\subset E_\alpha K_{U_\alpha}=V$$

where $U_\alpha=HU_\rho$.

To prove the converse let V be (A,B)-invariant contained in $\ker C$ and containing $\mathsf{R}^*(\ker C)$. Since $V\subset K_U+\mathsf{V}^*(\ker C)$, V and K_U are compatible [6], [12] and hence there exists $K\in F(V)\cap F(K_U)$. By Theorem 4.4 $K\in F(E_\rho K_{U_\rho})$. Thus, we have the module inclusions $K_U\supset V\supset E_\rho K_{U_\rho}$. Let R: $K_U\to K_{E_\rho}$ be defined by (4.11). $R(V)=\pi_{E_\rho}(V)$ is a submodule of K_{E_ρ} and hence of the form $\pi_{E_\rho}(V)=E_\alpha K_H$ with $E_\rho=E_\alpha H$. Now $f\in K_U$ and $\pi_{E_\rho}f\in E_\alpha K_H$ if and only if $f=E_\alpha g+E_\rho p$ with $g\in K_H$ and $p\in F^p[\lambda]$. Thus, $f=E_\alpha(g+Hp)$ and by Lemma 3.4 $f\in E_\alpha K_{U_\alpha}$. Conversely, if $f\in E_\alpha K_{U_\alpha}$, then $f=E_\alpha g$ and

$$\pi_{E_\rho}f=E_\alpha H\pi_-H^{-1}E_\alpha^{-1}E_\alpha g=E_\alpha\pi_H g=E_\alpha g' \qquad E_\alpha K_H.$$

This implies $V=E_\alpha K_{U_\alpha}$ and the theorem is proved.

The following result has previously been obtained by Emre and Hautus [1].

Corollary 4.8: If $K\in F(V^*(\ker C))$, then $K\in F(V)$ for every V that is (A,B)-invariant, is contained in $\ker C$ *and contains* $\mathsf{R}^*(\ker C)$.

Proof: Follows from Corollaries 4.4 and 4.7.

We denote by $\mathsf{V}_*(\mathsf{B})$ the minimal (C,A)-invariant subspace that contains B.

Corollary 4.9: The following inclusion holds:

$$\mathsf{R}^*(\ker C)\subset\mathsf{V}_*(\mathsf{B}). \tag{4.17}$$

Proof: Relation (3.8) obtained in the proof of Theorem 3.6 is equivalent to (4.17) where we use the identification of $\mathsf{R}^*(\ker C)$ and $\mathsf{V}_*(\mathsf{B})$ given by Theorem 4.1 and Corollary 3.9, respectively.

Actually, a more precise result holds as proved first, in a completely different way, by Morse [10].

Corollary 4.10: The following equality holds:

$$\mathsf{R}^*(\ker C)=\mathsf{V}_*(\mathsf{B})\cap\mathsf{V}^*(\ker C). \tag{4.18}$$

Proof: The inclusion $\mathsf{R}^*(\ker C)\subset\mathsf{V}^*(\ker C)$ together with (4.17) imply

$$\mathsf{R}^*(\ker C)\subset\mathsf{V}_*(\mathsf{B})\cap\mathsf{V}^*(\ker C). \tag{4.19}$$

Conversely, let $f\in\mathsf{V}_*(\mathsf{B})\cap\mathsf{V}^*(\ker C)$. Since $\mathsf{V}_*(\mathsf{B})=E_\rho K_{F_\rho}$ and $\mathsf{V}^*(\ker C)=K_U$ we have $f=E_\rho g$ for some $g\in F^p[\lambda]$ and also $f=Uh=E_\rho U_\rho h$ for some $h\in\lambda^{-1}F^m[[\lambda^{-1}]]$. This implies $g=U_\rho h$ or $g\in K_{U_\rho}$. Hence, $f\in E_\rho K_{U_\rho}=\mathsf{R}^*(\ker C)$ which completes the proof.

V. Illustrative Examples

To illustrate the preceding development we work out in detail some examples.

Example 1: Let

$$g(\lambda)=\frac{\lambda-1}{\lambda^3-\lambda^2-6}=\frac{\lambda-1}{\lambda(\lambda+2)(\lambda-3)}$$

and we write $t(\lambda)=\lambda^3-\lambda^2-6$ and $U(\lambda)=\lambda-1$. K_t is identified with the set of all polynomials of degree $\leqslant 2$ and S_t is multiplication by $\lambda(\operatorname{mod} t)$. Relative to $g=t^{-1}u$ and the choice of $\{1,\lambda,\lambda^2\}$ as a basis for K_t we obtain the realization

$$A=\begin{bmatrix}0&0&0\\1&0&6\\0&1&1\end{bmatrix},\quad B=\begin{bmatrix}-1\\1\\0\end{bmatrix},$$
$$\text{and}\quad C=(0\quad 0\quad 1).$$

A (C,A)-invariant subspace is of the form $E_1K_{F_1}$. Thus, let $t_1(\lambda)$ be any polynomial of degree 3, say $t_1(\lambda)=(\lambda+1)(\lambda-1)(\lambda+3)$; then $(\lambda+1)(\lambda-1)\;K_{\lambda+3}$ is an example of a one-dimensional (C,A)-invariant subspace, namely, the subspace spanned by λ^2-1 or in the state space representation span $\begin{bmatrix}-1\\0\\1\end{bmatrix}$.

$V=(\lambda+1)K(\lambda-1)(\lambda+3)$ is an example of a (C,A)-invariant subspace of dimension 2 spanned by $\{\lambda+1,\lambda^2+\lambda\}$ or in the state space representation span

$$\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}.$$

The minimal (C,A)-invariant subspace containing $\mathsf{B}=$ Range B is $\{(\lambda-1)(a+b\lambda)|\,a,b\in F\}$ and a basis is given by $\{\lambda-1,\lambda^2-\lambda\}$. Alternatively, in the state-space representation

$$\mathsf{V}_*(B)+\operatorname{span}\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix},\begin{bmatrix}0\\-1\\1\end{bmatrix}\right\}.$$

The maximal reachability subspace in $\ker C$ is $(\lambda-1)$ $K_1=\{0\}$. The maximal (A,B)-invariant subspace in $\ker C$ is $\mathsf{V}^*(\ker C)K_{\lambda-1}$ which is the set of constant polynomials. Since $S_t(1)=\lambda=(\lambda-1)+1$ with $(\lambda-1)\in\mathsf{B}$ and $1\in K_{\lambda-1}$, we can take $K:\to K_t$ F by $K(1)=1$, $K(\lambda)=K(\lambda^2)=0$. The matrix representation of K is $(1\quad 0\quad 0)$. Thus,

$$(A-BK)=\begin{bmatrix}1&0&0\\0&0&6\\0&1&2\end{bmatrix}$$

and the induced map $\overline{A-BK}$ in $\mathsf{V}^*(\ker C)/\mathsf{R}^*(\ker C)$ is the identity map with the corresponding invariant factor $\lambda-1$. This is in total agreement with the factorization $u(\lambda)=(\lambda-1)\cdot 1=e_\rho(\lambda)u_\rho(\lambda)$ with $e_\rho(\lambda)=\lambda-1$ and $u_\rho(\lambda)=1$.

Example 2: We consider the full row rank transfer function

$$G(\lambda)=\begin{bmatrix}\dfrac{2\lambda^2}{(\lambda^2-4)(+1)^2}&\dfrac{2\lambda}{(\lambda^2-4)(+1)^2}&\dfrac{\lambda^3-4\lambda^2-\lambda}{(\lambda^2-4)(\lambda+1)^2}\\[2ex]\dfrac{\lambda+1}{(\lambda-1)^2}&\dfrac{\lambda+1}{(\lambda-1)^2}&\dfrac{-2(\lambda+1)}{(\lambda-1)^2}\end{bmatrix}.$$

Clearly, $G=T^{-1}U$ with

$$T(\lambda)=\begin{bmatrix}(\lambda^2-4)(\lambda+1)^2&0\\0&(\lambda-1)^2\end{bmatrix}$$

and

$$U(\lambda)=\begin{pmatrix}2\lambda^2&2\lambda&\lambda^3-4\lambda^2-\lambda\\\lambda+1&\lambda+1&-2(\lambda+1)\end{pmatrix}$$

and it is easily checked, say by computing the rank of $(T(\lambda)U(\lambda))$ at the points $\lambda=\pm1,\ \pm2$, that this is a left coprime factorization of G. Since

$$K_T=K_{(\lambda^2-4)(\lambda+1)^2}\oplus K(\lambda-1)^2$$

a state space realization can be written immediately as

$$A=\begin{bmatrix}0&0&0&4&0&0\\1&0&0&8&0&0\\0&1&0&3&0&0\\0&0&1&-2&0&0\\0&0&0&0&0&1\\0&6&0&0&1&-2\end{bmatrix}$$

$$B=\begin{bmatrix}0&0&0\\0&2&-1\\2&0&-4\\0&0&1\\1&1&-2\\1&1&-2\end{bmatrix}$$

and

$$C=\begin{pmatrix}0&0&0&1&0&0\\0&0&0&0&0&1\end{pmatrix}.$$

Since T happens to be row proper with row indexes 4 and 2 any row proper T_1 with the same row indexes would satisfy $T_1^{-1}T$ is a bicausal isomorphism. Thus, let

$$T_1(\lambda)=\begin{pmatrix}\lambda^3(\lambda-1)&1\\0&\lambda(\lambda+1)\end{pmatrix}.$$

T_1 admits a factorization $T_1=E_1F_1$ with

$$E_1(\lambda)=\begin{pmatrix}1&-(\lambda^3-2\lambda^2+2\lambda-2)\\\lambda(\lambda+1)&2\lambda\end{pmatrix}$$

and

$$F_1(\lambda)=\begin{pmatrix}2&1\\-(\lambda+1)&0\end{pmatrix}.$$

Since

$$K_{F_1}=\left\{\begin{pmatrix}0\\a\end{pmatrix}\middle| a\in F\right\}$$

we have

$$V=E_1K_{F_1}=\operatorname{span}\left\{\begin{pmatrix}1\\(\lambda+1)\end{pmatrix}\right\}.$$

So this is a one-dimensional (C, A)-invariant subspace. Of course, there are multitudes of (C, A)-invariant subspaces related to different factorizations of polynomial matrices T_1 constructed in the same way.

To compute K_U we notice that U in row proper with row indexes 3 and 1. So

$$K_U=K_{\lambda 3}\oplus K_\lambda=\left\{\begin{pmatrix}a+b\lambda+c\lambda^2\\d\end{pmatrix}\middle| a,b,e,d\in F\right\}.$$

Or in the state space representation

$$\mathsf{V}^*(\ker C)=\operatorname{span}\left\{\begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix}\right\}.$$

Next we pass to the factorization $U=E_\rho U_\rho$:

$$\begin{aligned}U(\lambda)&=\begin{pmatrix}2\lambda^2 & 2\lambda & \lambda^3-4\lambda^2-\lambda\\ \lambda+1 & \lambda+1 & -2(\lambda+1)\end{pmatrix}\\
&=\begin{pmatrix}\lambda & 0\\0 & \lambda+1\end{pmatrix}\begin{pmatrix}2\lambda & 2 & \lambda^2-4\lambda-1\\1 & 1 & -2\end{pmatrix}\\
&=\begin{pmatrix}\lambda & 0\\0 & \lambda+1\end{pmatrix}\begin{pmatrix}1 & 2\lambda\\0 & 1\end{pmatrix}\begin{pmatrix}0 & 2-2\lambda & \lambda^2-1\\1 & 1 & -2\end{pmatrix}\\
&=\begin{pmatrix}\lambda & 0\\0 & \lambda+1\end{pmatrix}\begin{pmatrix}1 & 2\lambda\\0 & 1\end{pmatrix}\begin{pmatrix}0 & \lambda-1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & 1 & -2\\0 & 2 & \lambda+1\end{pmatrix}\\
&=\begin{pmatrix}2\lambda^2 & \lambda(\lambda-1)\\ \lambda+1 & 0\end{pmatrix}\begin{pmatrix}1 & 1 & -2\\0 & 2 & \lambda+1\end{pmatrix}\end{aligned}$$

so

$$E_\rho(\lambda)=\begin{pmatrix}2\lambda^2 & \lambda(\lambda-1)\\ \lambda+1 & 0\end{pmatrix}$$

and

$$U_\rho(\lambda)=\begin{pmatrix}1 & 1 & -2\\0 & 2 & \lambda+1\end{pmatrix}.$$

Of course, E_ρ is determined only up to a right unimodular factor. Now U_ρ is row proper with row indexes 0 and 1 so

$$K_{U_\rho}=\{0\}\oplus K_\lambda=\left\{\begin{pmatrix}0\\a\end{pmatrix}\middle| a\in F\right\}$$

and $E_\rho K_{U_\rho}$ is spanned by

$$\begin{pmatrix}\lambda(\lambda-1)\\0\end{pmatrix}.$$

Hence $\mathsf{R}^*(\ker C)$ is one-dimensional. In terms of the state space realization $\mathsf{R}^*(\ker C)$ is spanned by $\tilde{X}$ where $X=(0\quad -1\quad 1\quad 0\quad 0\quad 0)$.

Next we want to compute the transmission polynomials. Of course, they are, by Theorem 4.6, the invariant factors of E_ρ which in our case is just $\lambda(\lambda^2-1)$. It is of interest however to compute the induced map $\overline{A+BK}$ in $\mathsf{V}^*(\ker C)/\mathsf{R}^*(\ker C)$.

For $f\in K_U$ we know $S_T f=\lambda f$. Let us choose a basis of K_U consisting of the vector polynomials

$$\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}\lambda\\0\end{pmatrix},\begin{pmatrix}\lambda^2\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}.$$

Clearly,

$$S_T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\lambda\\0\end{pmatrix}\in K_U,\quad S_T\begin{pmatrix}\lambda\\0\end{pmatrix}=\begin{pmatrix}\lambda^2\\0\end{pmatrix}\in K_U,$$

$$S_T\begin{pmatrix}\lambda^2\\0\end{pmatrix}=\begin{pmatrix}\lambda^3\\0\end{pmatrix}=U(\lambda)\begin{bmatrix}2\\0\\1\end{bmatrix}+\begin{pmatrix}\lambda\\0\end{pmatrix},$$

and finally

$$S_T\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\\lambda\end{pmatrix}=U(\lambda)\begin{bmatrix}0\\1\\0\end{bmatrix}-\begin{pmatrix}2\lambda\\1\end{pmatrix}.$$

If we define $K\colon K_U\to F^3$ by

$$K\begin{pmatrix}1\\0\end{pmatrix}=K\begin{pmatrix}\lambda\\0\end{pmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix},\quad K\begin{pmatrix}\lambda^2\\0\end{pmatrix}=\begin{bmatrix}2\\0\\1\end{bmatrix}$$

and

$$K\begin{pmatrix}0\\1\end{pmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix}$$

we obtain $(S_T-BK)K_U\subset K_U$. In terms of the basis

$$\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}\lambda\\0\end{pmatrix},\begin{pmatrix}\lambda^2\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$$

the map S_T-BK has the matrix representation

$$\begin{bmatrix}0 & 0 & 0 & 0\\1 & 0 & 1 & -2\\0 & 1 & 0 & 0\\0 & 0 & 0 & -1\end{bmatrix}.$$

This is not a convenient basis for computing the induced map. Thus, we change the basis to

$$\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}\lambda\\0\end{pmatrix},\begin{pmatrix}\lambda^2-\lambda\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}.$$

The matrix representation becomes

$$\begin{bmatrix}0 & 0 & 0 & 0\\1 & 1 & 0 & -2\\0 & 1 & -1 & 0\\0 & 0 & 0 & -1\end{bmatrix}$$

and the induced map has the matrix representation

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix}.$$

The invariant factors of

$$\begin{bmatrix} \lambda & 0 & 0 \\ -1 & \lambda-1 & 2 \\ 0 & 0 & \lambda+1 \end{bmatrix}$$

are clearly $\lambda(\lambda^2-1)$, 1, 1, in total agreement with the previous computation of the transmission polynomials.

Finally, it may be of interest to compute $V_*(B)$. We look for F_ρ and that with $T_\rho = E_\rho F_\rho$, $T_\rho^{-1}T = \Gamma$ is a bicausal isomorphism. Or, in other words, $E_\rho^{-1}T = F_\rho \Gamma$. Now

$$E_\rho^{-1}T = \begin{bmatrix} 0 & \dfrac{1}{\lambda+1} \\ \dfrac{1}{\lambda(\lambda-)} & \dfrac{2\lambda^2}{(-1)(+1)} \end{bmatrix}$$

$$\cdot \begin{bmatrix} (\lambda^2-4)(\lambda+1)^2 & 0 \\ 0 & (\lambda-1)^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \dfrac{(-1)^2}{+1} \\ \dfrac{(\lambda^2-4)(\lambda+1)^2}{(\lambda-1)} & \dfrac{2\lambda^2(\lambda-1)^2}{\lambda(\lambda-1)(\lambda+1)} \end{bmatrix}$$

$$= \frac{1}{\lambda(\lambda^2-1)} \begin{bmatrix} 0 & \lambda(\lambda-1)^3 \\ (\lambda^2-4)(\lambda+1)^3 & 2\lambda^2(\lambda-1)^2 \end{bmatrix}.$$

This can be factored as

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^2 \end{pmatrix} \Gamma$$

for some bicausal isomorphism Γ. So F_ρ can be taken as

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^2 \end{pmatrix}.$$

Hence, since

$$K_{F_\rho} = K_\lambda \oplus K_{\lambda^2}$$
$$= \left\{ \begin{pmatrix} a \\ b+c\lambda \end{pmatrix} \middle| a, b, c \in F \right\}$$
$$V^*(B) = E_\rho K_{F_\rho}$$
$$= \operatorname{span}\left\{ \begin{pmatrix} 2\lambda^2 \\ \lambda+1 \end{pmatrix}, \begin{pmatrix} \lambda^2-\lambda \\ 0 \end{pmatrix}, \begin{pmatrix} \lambda^2(\lambda-1) \\ 0 \end{pmatrix} \right\}$$

and we clearly have $V_*(B) \supset R^*(\ker C)$.

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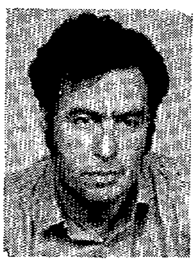

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