Mayer Goldberg Numbers

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Abstract

In this paper we analyze the so-called *Mayer Goldberg numbers* which contain all possible finite information encoded in given bases.

1 Introduction

Consider the following binary number:

$$\mathrm{mg}(2) = 0.10001101100000101001110010111011100000001\ldots_2$$

What's special about it? Taking a closer look, we discover its underlying structure:

$$\mathrm{mg}(2) = \underbrace{\underline{0.1}}_{\text{chunk 1}}\ \underbrace{\underline{00}\ \underline{01}\ \underline{10}\ \underline{11}}_{\text{chunk 2}}\ \underbrace{\underline{000}\ \underline{001}\ \underline{010}\ \underline{011}\ \underline{100}\ \underline{101}\ \underline{110}\ \underline{111}}_{\text{chunk 3}}\ \ldots_2 \tag{1.1}$$

This number, when encoded in binary, is a concatenation of all words over $\{0,1\}^+$ in increasing order of their value in base-2. Thus, it "contains" within itself binary encodings of all possible deterministic structures.

2 Computation

In order to represent $\mathrm{mg}(2)$ algebraically, we shall compute it via summation. For this purpose the following formula will prove helpful:

$$\sum_{k=1}^{m} kx^k = \frac{(mx - m - 1)x^{m+1} + x}{(x-1)^2} \tag{2.1}$$

It immediately follows from derivation of $\sum_{k=0}^{m} x^k$ by x.

Let us count the digits in $\mathrm{mg}(2)$ from left to right, starting with 0. Then, the index of the leftmost digit in chunk $n \geqslant 1$ is:

$$\mathrm{offs}(n) = \sum_{k=1}^{n-1} k2^k = (n-2)2^n + 2 \tag{2.2}$$

Now $\mathrm{mg}(2)$ can be rewritten as

$$\begin{aligned}\mathrm{mg}(2) = {} & 2^{-\operatorname{offs}(1)} \cdot \underline{0.1} \\ & + 2^{-\operatorname{offs}(2)} \cdot \underline{0.0}\ \underline{01}\ \underline{10}\ \underline{11} \\ & + 2^{-\operatorname{offs}(3)} \cdot \underline{0.00}\ \underline{001}\ \underline{010}\ \underline{011}\ \underline{100}\ \underline{101}\ \underline{110}\ \underline{111} \\ & \dots \\ = {} & \sum_{n=1}^{\infty} \frac{\operatorname{chunk}(n)}{2^{\operatorname{offs}(n)}} \\ = {} & \frac{1}{4} \sum_{n=1}^{\infty} \frac{\operatorname{chunk}(n)}{2^{(n-2)2^n}}\end{aligned} \tag{2.3}$$

Where

$$\begin{aligned}\operatorname{chunk}(n) &= \frac{1}{2^{n-1}} \sum_{k=1}^{2^n-1} k 2^{-nk} \\ &= \frac{1}{2^{n-1}} \sum_{k=1}^{2^n-1} k (2^{-n})^k \\ &= \frac{(2^{-n})^{2^n}\left((2^n-1)2^{-n}-2^n\right)+2^{-n}}{(2^{-n}-1)^2 2^{n-1}} \qquad \text{by (2.1)} \\ &= 2 \cdot \frac{2^{-n}-(2^n+2^{-n}-1)2^{-n2^n}}{2^n+2^{-n}-2}\end{aligned} \tag{2.4}$$

Finally resulting in

$$\boxed{\mathrm{mg}(2) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{2^{-n}-(2^n+2^{-n}-1)2^{-n2^n}}{(2^n+2^{-n}-2)2^{(n-2)2^n}}} \tag{2.5}$$

First 51 digits of $\mathrm{mg}(2)$ in decimal representation:

$$\boxed{\mathrm{mg}(2) = 0.55277423455897304746839735242380246111017997632137\dots}$$

3 Generalization

Mayer Goldberg number for base-2 can be easily generalized for other bases:

$$\boxed{\mathrm{mg}(\beta) = \beta^{1-\frac{\beta}{(\beta-1)^2}} \sum_{n=1}^{\infty} \frac{\beta^{-n}-(\beta^n+\beta^{-n}-1)\beta^{-n\beta^n}}{(\beta^n+\beta^{-n}-2)\beta^{\frac{((n-1)\beta-n)\beta^n}{(\beta-1)^2}}}} \tag{3.1}$$

For example, the number for base-4, corresponding to all possible information encoded on a 4-hole punch-tape is:

$$\boxed{\mathrm{mg}(4) = 0.42194444444444444444353739145534438965151883420879\dots}$$

Note that $\mathrm{mg}(3) > \mathrm{mg}(2)$.

4 Further Exploration

Here we shall ask how many such numbers are — how many numbers, when represented in base β, possess the property that they "contain" all words in $\{0, \ldots, \beta - 1\}^+$.

Easy to see, for each base $\beta \geqslant 2$ there are $\aleph_1$ such numbers. This can be shown by transposing words in the chunks of $\mathrm{mg}(\beta)$, for example.

However, in my opinion it is very unlikely that most of the numbers maintain this property.[1] I don't even think that this property is shared over bases in general, except for special cases like $\beta_1 = k\beta_2$.

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[1] Like they do with probability of 1, for example, in respect to equal representation of all digits in any base.