1. Let $S = \{s_1, \ldots, s_n\}$ be a set of $n$ axis-parallel squares, and put $U = s_1 \cup s_2 \cup \cdots \cup s_n$. The combinatorial complexity of $U$ is the number of vertices on $U$’s boundary, where a vertex is either a corner of a square or an intersection point between two edges of two different squares.

(a) Prove that the combinatorial complexity of $U$ is $O(n)$.

(b) Given a set $P$ of $n$ points in an axis-parallel rectangle $R$ of area $1$, compute the edge length of a largest empty square that is contained in $R$. A square is empty if there are no points of $P$ in its interior. In your solution, you may use the following result: One can compute the area of the union of a set of $n$ axis-parallel rectangles in time $O(n \log n)$.

2. Let $P$ and $Q$ be two sets of points, such that the points of $P$ lie on the line $y = 0$ and the points of $Q$ lie on the line $y = 1$. Assume that $|P| = |Q| = n$, and let $k$ be an integer between 1 and $n^2$. Find the $k$-th smallest distance between a point in $P$ and a point in $Q$ in near linear time. (Use the technique of Frederickson and Johnson.)

3. An $m \times n$ matrix is Monge if for any two rows $1 \leq i < k \leq m$ and any two columns $1 \leq j < l \leq n$ we have


(a) Let $f_A(i)$ be the index of the column containing the leftmost minimum element of row $i$ in matrix $A$. Prove that $f_A(1) \leq f_A(2) \leq \cdots \leq f_A(m)$ for any $m \times n$ Monge matrix $A$.

(b) Describe an $O(m + n \log m)$ algorithm for computing the values $f_A(1), f_A(2), \ldots, f_A(m)$.
4. **Facts:** (i) The combinatorial complexity of the union of $n$ disks in the plane is $O(n)$. (ii) Given $n$ disks in the plane, one can preprocess them in expected time $O(n \log n)$, such that given a query point $q$, one can determine whether $q$ lies in the union of the disks in expected time $O(\log n)$.

Given a set $P$ of $n$ points in the plane, describe an algorithm for computing the distance between the closest pair of points in $P$ in expected time $O(n \log^2 n)$. You may use the facts above in your solution.

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