Geometric Optimization
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homework assignment no. 2

1. Let $S = \{s_1, \ldots, s_n\}$ be a set of n axis-parallel squares, and put $U = s_1 \cup s_2 \cup \cdots \cup s_n$. The *combinatorial complexity* of U is the number of vertices on U's boundary, where a vertex is either a corner of a square or an intersection point between two edges of two different squares.

   (a) Prove that the combinatorial complexity of U is $O(n)$.

   (b) Given a set P of n points in an axis-parallel rectangle R of area 1, compute the edge length of a largest empty square that is contained in R. A square is empty if there are no points of P in its interior. In your solution, you may use the following result: One can compute the area of the union of a set of n axis-parallel rectangles in time $O(n \log n)$.

2. Let P and Q be two sets of points, such that the points of P lie on the line $y = 0$ and the points of Q lie on the line $y = 1$. Assume that $|P| = |Q| = n$, and let k be an integer between 1 and n^2. Find the k-th smallest distance between a point in P and a point in Q in near linear time. (Use the technique of Frederickson and Johnson.)

3. An $m \times n$ matrix is *Monge* if for any two rows $1 \leq i < k \leq m$ and any two columns $1 \leq j < l \leq n$ we have

$$A[i,j] + A[k,l] \leq A[i,l] + A[k,j]\,.$$

   (a) Let $f_A(i)$ be the index of the column containing the leftmost minimum element of row i in matrix A. Prove that $f_A(1) \leq f_A(2) \leq \cdots \leq f_A(m)$ for any $m \times n$ Monge matrix A.

   (b) Describe an $O(m + n \log m)$ algorithm for computing the values $f_A(1), f_A(2), \ldots, f_A(m)$.

4. **Facts:** (i) The combinatorial complexity of the union of n disks in the plane is $O(n)$. (ii) Given n disks in the plane, one can preprocess them in expected time $O(n \log n)$, such that given a query point q, one can determine whether q lies in the union of the disks in expected time $O(\log n)$.

   Given a set P of n points in the plane, describe an algorithm for computing the distance between the closest pair of points in P in expected time $O(n \log^2 n)$. You may use the facts above in your solution.

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