Geometric Optimization
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homework assignment no. 1

1. In the L_∞ *2-center problem* we are given a set P of n points in the plane, and the goal is to cover P with two (axis-parallel) squares of smallest (equal) size.

    a. What can one say about the side length of the squares in an optimal solution?

    b. Give an efficient algorithm for solving the corresponding decision problem.

    c. Describe a solution to the L_∞ 2-center problem that is based on parametric search.

2. In the *minimum output rate problem* we are given consecutive time intervals $T_1, \ldots, T_n$, input rates $I_1, \ldots, I_n$ (where I_j is the input rate during the j-th time interval), and a buffer size B, and we have to find the minimum output rate R^* required to assure that the buffer does not overflow.

    Let $a_i = T_i I_i$ denote the total amount of data received during the i-th interval, for $i = 1, \ldots, n$, and define a set H of $\binom{n}{2} + n$ lines of negative slope:

    $$l_{i,j} = (a_i + \cdots + a_j) - (T_i + \cdots + T_j)x, \quad 1 \le i \le j \le n \ .$$

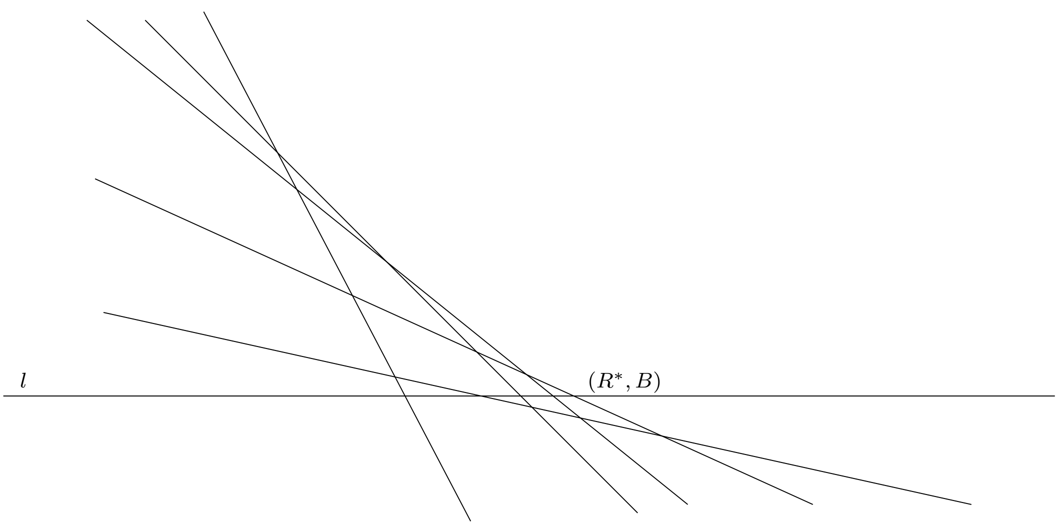

Figure 1: Question 2.

a. Prove that the desired rate R^* is equal to the x-coordinate of the intersection point of the upper envelope of H and the horizontal line l at height B. That is, it is equal to the x-coordinate of the rightmost intersection point of the lines of H with l.

b. Describe an $O(n \log n)$ randomized algorithm for computing R^*.

3. Devise an algorithm, based on randomized halving, for selecting the kth smallest element in a set S of n distinct real numbers. Show that with high probability the algorithm performs only $O(n)$ comparisons.
Hint: Randomly pick (with replacements) $n^{3/4}$ elements of S. Let $X_i = 1$ if the ith random sample is not greater than the kth smallest element in S, and 0 otherwise. Then $X = \sum_{i=1,n^{3/4}} X_i$ is the number of samples that are not greater than the kth smallest element. Notice that $\mu_X = kn^{-1/4}$ and that $\sigma_X \leq n^{3/8}/2$. Apply Chebyshev's inequality in your analysis, see below.

Chebyshev's Inequality: $Pr[|X - \mu_X| \geq t\sigma_X] \leq 1/t^2$.

Submission: November 2, 2010