Homework assignment no. 4

1. Let S be a set of n (axis-aligned) squares in the plane. We wish to find a piercing set for S of minimum cardinality. That is, we wish to find a set of points P, such that (i) each square $s \in S$ is pierced by at least one point in P (i.e., $s \cap P \neq \emptyset$), and (ii) P is as small as possible (i.e., $|P| \leq |Q|$, for any other piercing set for S).

   Alma suggested the following algorithm: Let Q be an initially empty set. As long as S is non-empty do the following: Let $s_{\min} \in S$ be the smallest square in S, add the 4 corners of $s_{\min}$ to Q and remove $s_{\min}$ and all the squares in S that intersect $s_{\min}$ from S.

   Prove that (i) The set Q returned by Alma's algorithm is a piercing set for S.
   (ii) $|Q| \leq 4|P|$, where P is a minimum-cardinality piercing set for S.

2. Let $S = \{s_1, \ldots, s_n\}$ be a set of n axis-parallel squares in the plane, and put $U = \cup_{i=1}^{n} s_i$. Prove that the combinatorial complexity of the boundary of U is $O(n)$.

3. **Definition:** (i) A simple polygon P is *star-shaped* if there exists a point $c \in P$, such that for any point $p \in P$ the line segment $\overline{cp}$ is contained in P.
   (ii) A simple polygon P is *monotone* if there exists a direction $\vec{d}$, $0 \leq \vec{d} < 180$, such that P is monotone with respect to $\vec{d}$. For example P is monotone with respect to $\vec{0}$ (or, P is x-monotone), if for any vertical line l, $l \cap P$ is either empty or connected.

   (a) Give an example of a polygon P that is star-shaped but not monotone.

   (b) Give an example of a polygon P that is monotone (and show a direction $\vec{d}$ with respect to which it is monotone) but not star-shaped.

   (c) Do examples for parts (a) and (b) exist if we require that P is rectilinear (i.e., all its edges are axis-parallel)?

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