Definition: A simple polygon $P$ is star-shaped if there exists a point $c \in P$, such that for any point $p \in P$ the line segment $cp$ is contained in $P$. The point $c$ is called a center point of $P$.

1. Let $P$ be a star-shaped polygon, and let $c$ be a center point of $P$. Show that, given a query point $q$, one can determine in $O(\log n)$ time whether $q$ lies in $P$. Assume that $P$ is given as an array of its $n$ vertices in sorted order along the boundary.

2. Give an expected linear-time algorithm to decide whether a simple polygon is star-shaped.

3. On $n$ parallel railway tracks $n$ trains are moving with constant speeds $v_1, \ldots, v_n$. At time $t = 0$ the trains are at positions $k_1, \ldots, k_n$. Give an $O(n \log n)$ algorithm that detects all trains that at some moment in time are leading. (Hint: use the algorithm for computing the intersection of half-planes.)

4. Let $P$ be a set of $n$ points in the plane. Describe an $O(n \log n)$-time algorithm that finds for each point $p \in P$ the point in $P$ that is closest to $p$.

5. The Gabriel graph of a set $\mathcal{P}$ of points in the plane consists of all edges $pq$, $p, q \in \mathcal{P}$, such that the circle with diameter $pq$ does not contain any point of $\mathcal{P}$ in its interior.

   (a) Prove that the Delaunay triangulation of $\mathcal{P}$ contains the Gabriel graph of $\mathcal{P}$.
   
   (b) Prove that $pq$ is an edge of the Gabriel graph if and only if $pq$ intersects the Voronoi edge between Vor ($p$) and Vor ($q$).

   (c) Show that the Gabriel graph can be computed in $O(n \log n)$ time.

6. Let $H = \{h_1, \ldots, h_n\}$ be a set of $n$ lines in the plane. Assuming that there are no two lines in $H$ that are parallel to each other, and there are no three lines in $H$ that share a point, prove that the number of faces that are formed by the lines in $H$ is

$$\frac{n^2}{2} + \frac{n}{2} + 1 .$$

7. Let $R$ be a set of $n$ points in the plane, and let $B$ be a set of $n$ blue points in the plane. A line $l$ is a separator for $R$ and $B$ if all points of $R$ lie on one side of $l$ and all points of $B$ lie on the other side of $l$. Describe an algorithm for deciding in expected $O(n)$ time whether there exists a separator for $R$ and $B$.

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