Wireless Capacity with Oblivious Power in General Metrics

Magnús M. Halldórsson, Pradipta Mitra

19/12/12
Outline

1. Introduction
   - Definition of the Problem


3. Algorithm

4. Range Assignment Problem
How much communication can occur simultaneously in any given wireless network? This basic question addressed here of the capacity of a wireless network has attracted tremendous attention, led by the highly-cited work on non-constructive average-case analysis by Gupta and Kumar. Less is known about algorithmic results on arbitrary instances.
$s_v$ (sender) and $r_v$ (receiver) are points in the metric space. Link $\ell_v$ represents a directed communication request from a sender $s_v$ to a receiver $r_v$. A power $P_v$ is associated with each link $l_v$. We will use the $SINR$ model to represent the interference between two links in the metric space.
Given a set of $n$ links we need to find the maximum number of links that can communicate simultaneously.
Definitions

Given is a set \( L = \{\ell_1, \ell_2, \ldots, \ell_n\} \) of links, where each link \( \ell_v \) represents a communication request from a sender \( s_v \) to a receiver \( r_v \).

- \( d(x, y) \) - distance between \( x \) and \( y \).
- \( d_{vw} = d(s_v, r_w) \) - distance from link \( \ell_v \) to link \( \ell_w \).

Let \( P_v \) denote the power assigned to link \( \ell_v \).
Assumptions

We will work with the path loss radio propagation model for the reception of signals. Let $\alpha$ be the path loss exponent, then the power $\mathcal{P}$ is $\mathcal{P}/d(x, y)^\alpha$, ($\alpha > 0$).
Assumptions

We worked in the SINR model of interference, in which a node $r_v$ successfully receives a message from a sender $s_v$ if and only if the following condition holds:

$$\frac{P_v/\ell_v^\alpha}{\sum_{\ell_w \in S \setminus \{\ell_v\}} P_w/\ell_w^\alpha + N} \geq \beta,$$  \hspace{2cm} (1)

Where $N$ is a constant noise, and $\beta \geq 1$ denotes the minimum SINR, and $S$ is the set of concurrently scheduled links in the same slot.
We say that $S$ is \textit{SINR-feasible} (or simply \textit{feasible}) if (1) is satisfied for each link in $S$. We want to find the maximum \textit{SINR-feasible} set of links.
The affectance $a^P_w(v)$ of link $\ell_v$ caused by another link $\ell_w$, with a given power assignment $P$, is the interference of $\ell_w$ on $\ell_v$ relative to the power received, or

$$a^P_w(v) = c_v \frac{P_w / d^\alpha_{wv}}{P_v / \ell^\alpha_v} = c_v \frac{P_w}{P_v} \cdot \left( \frac{\ell_v}{d_{wv}} \right)^\alpha,$$

where $c_v = \beta / (1 - \beta N \ell^\alpha_v / P_v)$ is a constant depending only on the length and power of the link $\ell_v$.

Let $a^P_v(S) = \sum_{w \in S} a^P_w(w)$ and $a^P_S(v) = \sum_{w \in S} a^P_w(v)$. Using this notation, Eqn. 1 can be rewritten as $a^P_S(v) \leq 1$. 
Definitions for power assignments

Definition

A power assignment $P$ is length-monotone if $P_v \geq P_w$ whenever $\ell_v \geq \ell_w$.

Definition

A power assignment $P$ is sub-linear if $\frac{P_v}{\ell_v^\alpha} \leq \frac{P_w}{\ell_w^\alpha}$ whenever $\ell_v \geq \ell_w$. 
We will see now the algorithm to the $P$-Capacity problem for fixed power assignments.

Algorithm $C(L,P)$:

Suppose the links $L = \{\ell_1, \ell_2, \ldots, \ell_n\}$ are in non-decreasing order of length.

Let $\gamma = 1/2$

$S \leftarrow \emptyset$

for $v \leftarrow 1$ to $n$ do

if $a^P_S(v) + a^P_v(S) < \gamma$

then add $\ell_v$ to $S$

Output $X = \{\ell_v \in S : a^P_S(v) \leq 1\}$
Example

![Diagram of range assignment problem](image)

- \(s_1\) to \(r_1\)
- \(s_3\) to \(r_4\)
- \(s_2\) to \(r_3\)
- \(s_4\) to \(r_2\)

**Introduction**

Model and Preliminaries.

**Algorithm**

Range Assignment Problem
For the following input:
$s_1(0, 10), s_2(15, 0), s_3(5, 16), s_4(52, 0)$.
$r_1(0, 0), r_2(39, 32), r_3(17, 0), r_4(40, 16)$.

Distances:

<table>
<thead>
<tr>
<th>R</th>
<th>S</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>10</td>
<td>45</td>
<td>20</td>
<td>40</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>15</td>
<td>40</td>
<td>2</td>
<td>30</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>17</td>
<td>38</td>
<td>20</td>
<td>35</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>52</td>
<td>35</td>
<td>35</td>
<td>20</td>
</tr>
</tbody>
</table>
For the following input:
\( s_1(0, 10), s_2(15, 0), s_3(5, 16), s_4(52, 0). \)
\( r_1(0, 0), r_2(39, 32), r_3(17, 0), r_4(40, 16). \)

We assign for every link \( v \), \( P_v = l^3_v \)
For the following input:
\(s_1(0, 10), s_2(15, 0), s_3(5, 16), s_4(52, 0)\).
\(r_1(0, 0), r_2(39, 32), r_3(17, 0), r_4(40, 16)\).
We calculate the affectance:
\[
 a_w^p(v) = c_v \frac{P_w/d_w^\alpha}{P_v/\ell_v^\alpha} = c_v \frac{P_w}{P_v} \cdot \left( \frac{\ell_v}{d_w} \right)^\alpha, 
\]
where \(c_v = \beta / (1 - \beta N \ell_v^\alpha / P_v)\)
With \(\beta = 1, N = 4, \alpha = 2\).
Affectance:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.67</td>
<td>0.01</td>
<td>0.16</td>
<td>0.04</td>
</tr>
<tr>
<td>2</td>
<td>47.41</td>
<td>1.11</td>
<td>1000.0</td>
<td>4.44</td>
</tr>
<tr>
<td>3</td>
<td>4.61</td>
<td>0.15</td>
<td>1.25</td>
<td>0.41</td>
</tr>
<tr>
<td>4</td>
<td>0.42</td>
<td>0.18</td>
<td>0.41</td>
<td>1.25</td>
</tr>
</tbody>
</table>
Example

\[ s_1 \rightarrow r_1 \]
\[ s_3 \rightarrow r_4 \]
\[ s_2 \rightarrow r_3 \]
\[ s_4 \rightarrow r_2 \]
Example

Model and Preliminaries.

Algorithm

Range Assignment Problem

Example

\[ s_1 \rightarrow r_1 \]
\[ s_3 \rightarrow r_4 \]
\[ s_4 \rightarrow r_2 \]
\[ s_2 \rightarrow r_3 \]
Example

- $s_1$
- $s_3$
- $r_4$
- $r_2$
- $r_1$
- $s_2$
- $r_3$
- $s_4$
Example

```
s1
r1

s3

r4

s2

r3

s4
```
Example
Example

\[ r_1 \rightarrow s_1 \rightarrow r_4 \rightarrow s_3 \rightarrow r_2 \rightarrow s_4 \rightarrow r_3 \rightarrow s_2 \rightarrow r_1 \]
Example

Introduction
Model and Preliminaries.
Algorithm
Range Assignment Problem
Example

Range Assignment Problem

\[ s_1 \rightarrow r_1 \quad s_3 \rightarrow r_2 \quad s_2 \rightarrow r_3 \quad s_4 \rightarrow r_4 \]
Main Theorem

Theorem

Algorithm $C$ approximates $\mathcal{P}$-Capacity within a constant factor for any length-monotone, sub-linear $\mathcal{P}$.

The feasibility of the solution output by $C$ is evident by virtue of the last line of the algorithm.
Definition A \( \delta \)-signal set is one where the affectance of any link is at most \( 1/\delta \).

A set is SINR-feasible iff it is a 1-signal set. Let \( OPT_\delta^P = OPT_\delta^P(L) \) be the maximum number of links in a \( \delta \)-signal set given a power assignment \( P \).
Algebraic definitions

**Definition**
For a matrix $\mathbf{M}$ (vector $\mathbf{v}$), let $Se(\mathbf{M})$ ($Se(\mathbf{v})$) denote the sum of entries in the matrix (vector).

**Definition**
A matrix $\mathbf{M} = (m_{ij})$ is *q-approximately-symmetric* if, for any two indices $i$ and $j$, $m_{ij} \leq q \cdot m_{ji}$ for some $q \geq 0$. 
Claim

Let $\mathbf{M}$ be a $n \times n$ matrix that contains only non-negative entries. Assume $\text{Se}(\mathbf{M}) \leq \gamma n$ for some $\gamma \geq 0$. Then, for any $\lambda > 1$, there are at least $(1 - \frac{1}{\lambda})n$ rows (columns) $\mathbf{r}$ of $\mathbf{M}$ for which $\text{Se}(\mathbf{r}) \leq \gamma \cdot \lambda$.

Lemma

Let $n$ be a number and $\mathbf{p}$ be a positive $n$-dimensional vector. Let $\mathbf{M}$ be a non-negative $q$-approximately-symmetric $n$-by-$n$ matrix such that $\mathbf{M}\mathbf{p} \leq \mathbf{p}$. Then, $\text{Se}(\mathbf{M}) \leq (q + 1)n$. 
Proposition

Let $L$ be set of links and $P$ be a power assignment for $L$. Then, for any $\tau \leq \delta$, $\text{OPT}_\delta^P(L) \geq \frac{\text{OPT}_\tau^P(L)}{4\delta/\tau}$.

From: Oblivious Interference Scheduling [2009]

Lemma (3)

Let $\ell_u, \ell_v$ be links in a $q^\alpha$-signal set. Then, $d_{uv} \cdot d_{vu} \geq q^2 \cdot \ell_u \ell_v$. 
Lemma

Let $S$ and $X$ be the sets created by the algorithm $C$ on a given link set. Then, $|X| \geq |S|(1 - \gamma)$.

Proof.

Observe that by the algorithm construction, the sum $Se(S) = \sum_{v,w \in S} a_v(w) = \sum_{v \in S} a_S(v)$ is upper bounded by $\gamma|S|$. Then the Lemma follows from Claim 1 by plugging in $\lambda = \frac{1}{\gamma}$.
Now we will relate the size of the set $S$ to the optimal solution. We need another definition here:

**Definition**

For a set $S$ and link $\ell_v$, let $S_v^- = \{ \ell_w \in S : \ell_w \leq \ell_v \}$ and $S_v^+ = \{ \ell_w \in S : \ell_w \geq \ell_v \}$. 
Lemma (Red-Blue Lemma)

Let $R$ and $B$ be disjoint sets of links, referred to as red and blue links in a length-monotone, sub-linear power metric. If $|B| > 4|R|$ and $B$ is a $3^\alpha$-signal set, then there is a blue link $\ell_b$ such that $a_{R_b^-}(b) + a_b(R_b^-) \leq 3^\alpha(a_B(b) + a_b(B))$. 
**Lemma**

Let $R$ and $B$ be disjoint sets of links, referred to as red and blue links in a length-monotone power metric. If $|B| > 2|R|$ and $B$ is a $3^\alpha$-signal set, then there is a set $B' \subseteq B$ of size at least $|B| - 2|R|$ such that for all $\ell_b \in B'$, $a_{R_b}(b) \leq 3^\alpha a_B(b)$.

**Lemma**

Let $R$ and $B$ be disjoint sets of links, referred to as red and blue links in a sub-linear power metric. If $|B| > 2|R|$ and $B$ is a $3^\alpha$-signal set, then there is a set $B' \subseteq B$ of size at least $|B| - 2|R|$ such that for all $\ell_b \in B'$, $a_b(R_b^-) \leq 3^\alpha a_b(B)$.
Proof

We will show proof only for the first lemma, the other is very similar.
For each link $\ell_r \in R$, we will assign a set of “guards" $X_r \subseteq B$. For different $\ell_r$, the $X_r$’s will be of size at most 2. We remove all the guard sets from $S$ and it will satisfied the lemma.

Here’s how we choose the guards:

\[
B' \leftarrow B.
\]
For each link $\ell_r \in R$
\begin{align*}
&\text{add to } X_r \text{ the link } \ell_x \in B_r^+ \text{ with the sender nearest to } s_r \text{ (if one exists)} \\
&\text{add the link } \ell_y \in B_r^+ \text{ with the receiver nearest to } r_r \text{ (if one exists; possibly, } \ell_y = \ell_x) 
\end{align*}
Figure: Example configuration of guards and points. Guard $s_x$ (on left) is the nearest blue sender to red point $s_r$ (in center) among those of links larger than $\ell_r$. 

Proof (cont.)
Since $|B| > 2|R|$, $|B'| \geq |B| - 2|R| > 0$, by construction. Consider any link $\ell_b \in B'$. In what follows, we will show that the affectance of any link $l_r \in R_b^-$ on $\ell_b$ is comparable to the affectance of one of the guards of $l_r$ on $\ell_b$. Once we recall that the guards are blue, this implies that the overall affectance on $\ell_b$ from $R_b^-$ is not much worse than that from $B$. Since $\ell_b \notin X_r$, $X_r$ is non-empty and contains guards $\ell_x$ and $\ell_y$ (potentially $\ell_x = l_y$). Consider any $l_r \in R_b^-$. Let $d$ denote $d(s_x, s_r)$ and observe that since $\ell_b \notin X_r$, $d(s_b, s_r) \geq d$. 

Proof (cont.)
We claim that $d_{rb} \geq d/2$. Before proving the claim, let us see how this leads to the proof of the Lemma. If the claim is true, then by the triangular inequality,

$$d_{xb} = d(s_x, r_b) \leq d(s_x, s_r) + d(s_r, r_b) = d + d_{rb} \leq 3d_{rb}.$$ 

Since $\ell_x \geq \ell_r$ and the power metric is length-monotone, $P_x \geq P_r$. Thus,

$$\frac{a_x(b)}{a_r(b)} = \frac{P_x}{P_r} \left( \frac{d_{rb}}{d_{xb}} \right)^\alpha \geq 3^{-\alpha}.$$ 

Summing over all links in $B$,

$$a_B(b) = \sum_{\ell_r \in R_b} a_x(b) \geq 3^{-\alpha} \sum_{\ell_r \in R_b} a_r(b) = 3^{-\alpha} a_{R_b}(b).$$
To prove the claim that $d_{rb} \geq d/2$, let us suppose otherwise for contradiction. Then, by the triangular inequality,

$$\ell_b = d(s_b, r_b) \geq d(s_b, s_r) - d(r_b, s_r) > d - d/2 = d/2.$$  

Since $\ell_y$ was chosen into $X_r$, its receiver is at least as close to $s_r$ as $r_b$, that is $d(r_y, s_r) \leq d(r_b, s_r) < d/2$, and its sender is also at least far as $s_x$, or $d(s_y, s_r) \geq d$. So, $\ell_y \geq d(s_y, s_r) - d(s_r, r_y) > d/2$. 
Proof (cont.)

Now, \( d(r_y, r_b) \leq d(r_y, s_r) + d(s_r, r_b) < d \) and \( d_{yb} \cdot d_{by} \leq (\ell_y + d(r_y, r_b)) \cdot (\ell_b + d(r_y, r_b)) < (\ell_y + d) \cdot (\ell_b + d) < 9 \cdot \ell_y \ell_b \).

But since \( B \) is a \( 3^a \)-signal set, this is a contradiction by Lemma 3. Hence, any link \( \ell_r \) in \( R_b^- \) satisfies \( d(s_r, r_b) = d_{rb} \geq d/2 \).
Lemma

Let $S$ be the set found by the algorithm and $\tau = 3^{\alpha + 1}/2\gamma$. Then, $|S| \geq \text{OPT}_{2\tau}/10$.

Proof.

By Claim 1, there is a set $O \subseteq \text{OPT}_{2\tau}$ of size at least $\text{OPT}_{2\tau}/2$ such that for all $u \in O$, $a_u(O) \leq \frac{1}{\tau}$. By definition, $a_O(u) \leq \frac{1}{2\tau}$.

We claim that $|S| \geq |O|/5$. Suppose otherwise. Then, $|S| < |O \setminus S|/4$. Applying Lemma 6 with $B = O \setminus S$ and $R = S$, we find that there is a link $\ell_b$ in $O \setminus S$ that satisfies $a_{S_b^-}(b) + a_b(S_b^-) \leq 3^\alpha (a_B(b) + a_b(B)) \leq 3^\alpha (\frac{1}{\tau} + \frac{1}{2\tau}) = \gamma$. The operation of the algorithm is then such that the algorithm would have added the link $\ell_b$ to $S$, which is a contradiction.
Introduction
Model and Preliminaries.
Algorithm
Range Assignment Problem

The problem Definition

Given a set of points \( A \) in the three-dimensional space, a positive number \( q \).
We want to find a complete range assignment for \( A \) of total cost at most \( q \).

This problem is NP-hard, we will show an 2-approximation algorithm for the problem where \( q \) is the minimum cost available.
2-Approximation Algorithm

Given a set $V = \{x_1, x_2, \ldots, x_n\}$ of points in 3-space.

Algorithm \text{approximation}(V):

1. Construct an undirected weighted complete graph $G(V)$ with vertices $V$ and where the weight of the edge between $x_i$ and $x_j$ is $d(x_i, x_j)^2$ for all $i$ and $j$.
2. Find a minimum weight spanning tree $T$ of $G(V)$.
3. For $i = 1, \ldots, n$ assign the range of $x_i$ to be the maximum of $d(x_i, x_j)$ over $j$ such that $(x_i, x_j)$ is an edge in $T$. 
Algorithm Input
Construct full graph
Find weighted MST
Algorithm Result
Theorem

Let \( \text{OPT}(V) \) be the minimum cost of a complete range assignment for \( V \) and let \( \text{APP}(V) \) be the cost of the complete range assignment for \( V \) found by the algorithm, Then \( \text{APP}(V) < 2 \times \text{OPT}(V) \).

Proof.

Let \( \text{MST}(V) \) be the cost of the minimum weight spanning tree \( T \) of \( G(V) \), The theorem follows immediately from the following claims:
Claim 1

Claim

\[ \text{OPT}(V) > \text{MST}(V). \]

Proof.

Given an optimal assignment we can construct a spanning tree of \( G(V) \) by choosing any vertex, constructing a shortest path destination tree with the chosen vertex as the destination. Since each of the \( n - 1 \) vertices other than the root vertex must have a range assigned which establishes the edges of the destination tree, \( \text{OPT}(V) \) is greater than the weight of the resulting spanning tree which in turn is \( \geq \text{MST}(V) \).
Claim 2

\[ APP(V) < 2 \times MST(V) \]

Proof.

\[ APP(V) = \sum_{i=1}^{n} \max_{\{j|\{x_i, x_j\} \in T}\} d(x_i, x_j)^2 \leq \sum_{i=1}^{n} \sum_{\{j|\{x_i, x_j\} \in T\}} d(x_i, x_j)^2 = 2 \times MST(V) \]