

משוואות הגלים על מיתר אינסופי

$$u_{tt} = \alpha^2 u_{xx}, \quad (\alpha \neq 0, t > 0, -\infty < x < +\infty), \quad u(x,0) = f(x), \quad u_t(x,0) = p(x)$$

$$u(x,t) = \frac{1}{2} [f(x-\alpha t) + f(x+\alpha t)] + \frac{1}{2\alpha} \int_{x-\alpha t}^{x+\alpha t} p(z) dz$$

משוואות הגלים על מיתר סופי – שיטת הפרצת המשתנים

I. **משוואות לא-הומוגניות עם תנאי שפה הומוגניים**

$$u_{tt} = \alpha^2 u_{xx} + \beta u + g(x,t), \quad (\alpha, \beta \in \mathbb{R})$$

$$(\alpha \neq 0, t > 0, 0 < x < l), \quad u(x,0) = f(x), \quad u_t(x,0) = p(x)$$

| | |
|----------------|--|
| $u(0,t) = 0$ | $u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{n\pi x}{l}$ |
| $u(l,t) = 0$ | $u(x,t) = \sum_{n=0}^{\infty} u_n(t) \cos \frac{(2n+1)\pi x}{2l}$ |
| $u_x(0,t) = 0$ | $u(x,t) = \sum_{n=0}^{\infty} u_n(t) \sin \frac{(2n+1)\pi x}{2l}$ |
| $u(l,t) = 0$ | $u(x,t) = \frac{u_0(t)}{2} + \sum_{n=1}^{\infty} u_n(t) \cos \frac{n\pi x}{l}$ |
| $u_x(0,t) = 0$ | |
| $u_x(l,t) = 0$ | |

II. **משוואות לא-הומוגניות**

$$u_{tt} = \alpha^2 u_{xx} + \beta u + g(x,t), \quad (\alpha \neq 0, t > 0, 0 < x < l), \quad u(x,0) = f(x), \quad u_t(x,0) = p(x)$$

עם תנאי שפה לא-הומוגניים

: **דוגמאות**

1. מצאו את ערך הפתרון $u(x,t)$ של הבעיה

$$u_{tt} = 9u_{xx}, \quad t > 0, \quad -\infty < x < \infty, \quad u(x,0) = f(x) = \begin{cases} 1 - |x|, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}, \quad u_t(x,0) = p(x) = \begin{cases} 1, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

$x = 1, t = 1/3$ בנקודת

: **פתרון**

$$u_{tt} = 9u_{xx} \Rightarrow \alpha = 3, \quad u(x,t) = \frac{1}{2} [f(x-3t) + f(x+3t)] + \frac{1}{2 \cdot 3} \int_{x-3t}^{x+3t} p(z) dz \Rightarrow$$

$$u\left(1, \frac{1}{3}\right) = \frac{1}{2} [f(0) + f(2)] + \frac{1}{6} \int_0^2 p(z) dz = \frac{1}{2}(1+0) + \frac{1}{6} \int_0^1 1 dz = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

2. פתרו את המשוואה הדיפרנציאלית החלקית הבאה

$$u_{tt} = 2u_{xx} - 3u + 3t - x, \quad t > 0, \quad 0 < x < \pi, \quad \begin{cases} u_x(0,t) = 0 \\ u(\pi,t) = t \end{cases}, \quad \begin{cases} u(x,0) = 0 \\ u_t(x,0) = 1 \end{cases}$$

פתרונות:

א)

$$\begin{cases} u_x(0,t) = 0 \\ u(\pi,t) = t \end{cases} \Rightarrow \begin{cases} u(x,t) = v(x,t) + t \\ v_x(0,t) = 0 \\ v(\pi,t) = 0 \end{cases}$$

$$\begin{cases} u(x,t) = v(x,t) + t \\ u_{tt} = 2u_{xx} - 3u + 3t - x \end{cases} \Rightarrow v_{tt} = 2v_{xx} - 3v - x$$

$$\begin{cases} u(x,t) = v(x,t) + t \\ u(x,0) = 0 \\ u_t(x,0) = 1 \end{cases} \Rightarrow \begin{cases} v(x,0) = 0 \\ v_t(x,0) = 0 \end{cases}$$

$$\left. \begin{array}{l} v_{tt} = 2v_{xx} - 3v - x \\ \begin{cases} v_x(0,t) = 0 \\ v(\pi,t) = 0 \end{cases}, \quad \begin{cases} v(x,0) = 0 \\ v_t(x,0) = 0 \end{cases} \end{array} \right\} \Rightarrow \begin{cases} v_{tt} = 2v_{xx} - 3v - x \\ \begin{cases} v_x(0,t) = 0 \\ v(\pi,t) = 0 \end{cases}, \quad \begin{cases} v(x,0) = 0 \\ v_t(x,0) = 0 \end{cases} \end{cases}$$

ב)

$$\begin{cases} v_{tt} = 2v_{xx} - 3v - x \\ \begin{cases} v_x(0,t) = 0 \\ v(\pi,t) = 0 \end{cases}, \quad \begin{cases} v(x,0) = 0 \\ v_t(x,0) = 0 \end{cases} \end{cases}$$

$$\begin{cases} l = \pi \\ v_x(0,t) = 0 \\ v(\pi,t) = 0 \end{cases} \Rightarrow \begin{cases} v(x,t) = \sum_{n=1}^{\infty} v_n(t) \cos \frac{(2n-1)x}{2} \\ g(x,t) = \sum_{n=1}^{\infty} g_n(t) \cos \frac{(2n-1)x}{2} \end{cases}, \quad v_{tt} = 2v_{xx} - 3v + g(x,t)$$

$$\Rightarrow v_n''(t) + \left(\frac{(2n-1)^2}{2} + 3 \right) v_n(t) = g_n(t)$$

$$g(x,t) = -x = \sum_{n=1}^{\infty} g_n(t) \cos \frac{(2n-1)x}{2} \Rightarrow g_n(t) = -\frac{2}{\pi} \int_0^{\pi} x \cos \frac{(2n-1)x}{2} dx = \frac{4}{2n-1} \left[\frac{2}{\pi(2n-1)} + (-1)^n \right]$$

$$v(x,0) = 0 = \sum_{n=1}^{\infty} v_n(0) \cos \frac{(2n-1)x}{2} \Rightarrow v_n(0) = 0$$

$$v_t(x,0) = 0 = \sum_{n=1}^{\infty} v_n'(0) \cos \frac{(2n-1)x}{2} \Rightarrow v_n'(0) = 0$$

$$\lambda) \quad v_n''(t) + \left(\frac{(2n-1)^2}{2} + 3 \right) v_n(t) = \frac{4}{2n-1} \left[\frac{2}{\pi(2n-1)} + (-1)^n \right], \quad v_n(0) = 0, \quad v_n'(0) = 0$$

ט)

$$\begin{cases} y'' + \lambda^2 y = \delta, \\ \lambda \neq 0 \end{cases}, \quad y(0) = 0, \quad y'(0) = 0 \Rightarrow \begin{cases} y = C_1 \cos \lambda t + C_2 \sin \lambda t + \frac{\delta}{\lambda^2} \\ y(0) = 0, \quad y'(0) = 0 \end{cases} \Rightarrow y = \frac{\delta}{\lambda^2} (1 - \cos \lambda t)$$

$$\tau) \quad \lambda^2 = \frac{(2n-1)^2 + 6}{2}, \quad \delta = \frac{4}{2n-1} \left[\frac{2}{\pi(2n-1)} + (-1)^n \right]$$

תשובה

$$\begin{cases} v_n(t) = \frac{8}{2n-1} \left[\frac{2}{\pi(2n-1)} - (-1)^n \right] \cdot \left[1 - \cos \left(t \frac{\sqrt{(2n-1)^2 + 6}}{\sqrt{2}} \right) \right] \\ u(x,t) = t + \sum_{n=1}^{\infty} v_n(t) \cos \frac{(2n-1)x}{2} \quad (0 < x < \pi, \quad t > 0) \end{cases}$$

I . מצא פתרון פרטי עבור משוואת הגלים הנתונה המקיים את התנאים המצורפים

1) $u_{tt} = 9u_{xx}, \quad t > 0, \quad -\infty < x < \infty, \quad u(x,0) = \begin{cases} 2 - |x|, & |x| \leq 2 \\ 0, & |x| > 2 \end{cases}, \quad u_t(x,0) = 0$
 $u(2,1) = ?, \quad u(1,1/3) = ?$

2) $u_{tt} = 4u_{xx}, \quad t > 0, \quad -\infty < x < \infty, \quad u(x,0) = 0, \quad u_t(x,0) = \begin{cases} -0.02, & -1 < x < 0 \\ 0.02, & 0 < x < 1 \\ 0, & |x| > 1 \end{cases}$
 $u(1,0.3) = ?, \quad u(1,2) = ?$

3) $u_{tt} = 100u_{xx}, \quad t > 0, \quad -\infty < x < \infty, \quad u(x,0) = \frac{\sin x}{x}, \quad u_t(x,0) = 0$

II .

4) $u_{tt} = 4u_{xx}, \quad u(0,t) = u(1,t) = 0, \quad u(x,0) = x^2 - x, \quad u_t(x,0) = x - 1, \quad (0 < x < 1, \quad t > 0)$

5) $u_{tt} = u_{xx}, \quad u(0,t) = u_x(\pi,t) = 0, \quad u(x,0) = \frac{x}{\pi}, \quad u_t(x,0) = 0, \quad (0 < x < \pi, \quad t > 0)$

6) $u_{tt} = 4u_{xx}, \quad u(0,t) = u_x(1,t) = 0, \quad u(x,0) = 0, \quad u_t(x,0) = 0.2, \quad (0 \leq x \leq 1, \quad t > 0)$

7) $u_{tt} = u_{xx}, \quad u_x(0,t) = u_x(1,t) = 0, \quad u(x,0) = \frac{1}{2} - \frac{1}{5} \cos 3\pi x, \quad u_t(x,0) = 0.4 \cos 2\pi x, \quad (0 < x < 1, \quad t > 0)$

8) $u_{tt} = u_{xx} + \frac{\sin 3t}{10}, \quad u(0,t) = u(1,t) = 0, \quad u(x,0) = 0, \quad u_t(x,0) = 0, \quad (0 < x < 1, \quad t > 0)$

9) $u_{tt} = 9u_{xx} - \cos \frac{5x}{2}, \quad u_x(0,t) = u(\pi,t) = 0, \quad u(x,0) = u_t(x,0) = 0, \quad (0 < x < \pi, \quad t > 0)$

10) $u_{tt} = 4u_{xx}, \quad u(0,t) = 0, \quad u(\pi,t) = \sin t, \quad u(x,0) = 0, \quad u_t(x,0) = \frac{x}{\pi}, \quad (0 < x < \pi, \quad t > 0)$

11) $u_{tt} = 4u_{xx} - u + g(x,t), \quad (0 < x < 2, \quad t > 0), \quad u_x(0,t) = u_x(2,t) = 0,$

$$u(x,0) = 0, \quad u_t(x,0) = \cos \pi x, \quad g(x,t) = \begin{cases} 1, & 0 < x \leq 1 \\ 0, & 1 < x < 2 \end{cases}$$

III מצא פתרון דלמבר לביעות קושי הבאות :

1) $u_{tt} = u_{xx}, \quad u(x,0) = 0, \quad u_t(x,0) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases} \quad u(x,t) = ?$

2) $u_{tt} = u_{xx}, \quad u(x,0) = 0, \quad u_t(x,0) = \begin{cases} x, & |x| < 1 \\ 0, & |x| > 1 \end{cases} \quad u(x,t) = ?$

- 3) $u_{tt} = u_{xx}$, $u(x,0) = \begin{cases} -x^2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$, $u_t(x,0) = 0$ $u(x,t) = ?$
- 4) $u_{tt} = u_{xx}$, $u(x,0) = \begin{cases} 1, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$, $u_t(x,0) = \begin{cases} 1, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$ $u(x,t) = ?$
- 5) $u_{tt} = u_{xx}$, $u(x,0) = \begin{cases} 1, & 0 \leq x \leq 1 \text{ and } 2 \leq x \leq 3 \\ 0, & \text{אחרת} \end{cases}$, $u_t(x,0) = 0$ $u(x,t) = ?$
- 6) $u_{tt} - 4u_{xx} = 0$, $u(x,0) = x$, $u_t(x,0) = 3x^2$ $u(3,2) = ?$

תשובות:

1) $u(2,1) = 0.5$, $u\left(1, \frac{1}{3}\right) = 1$

2) $u(1,0.3) = 0.003$, $u(1,2) = 0$

3) $u(x,t) = \frac{1}{2} \left[\frac{\sin(x-10t)}{x-10t} + \frac{\sin(x+10t)}{x+10t} \right]$

4) $u(x,t) = \sum_{n=1}^{\infty} \left[\frac{4((-1)^n - 1)}{n^3 \pi^3} \cos 2n\pi t - \frac{1}{n^2 \pi^2} \sin 2n\pi t \right] \sin n\pi x$

5) $u(x,t) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \cos \frac{(2n-1)t}{2} \sin \frac{(2n-1)x}{2}$

6) $u(x,t) = \frac{0.8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \sin(2n-1)\pi t \sin \frac{(2n-1)\pi x}{2}$

7) $u(x,t) = \frac{1}{2} - \frac{1}{5} \cos 3\pi t \cos 3\pi x + \frac{1}{5\pi} \sin 2\pi t \cos 2\pi x$

8) $u(x,t) = \frac{1}{5\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n(n^2\pi^2 - 9)} \left(\sin 3t - \frac{3}{n\pi} \sin n\pi t \right) \sin n\pi x$

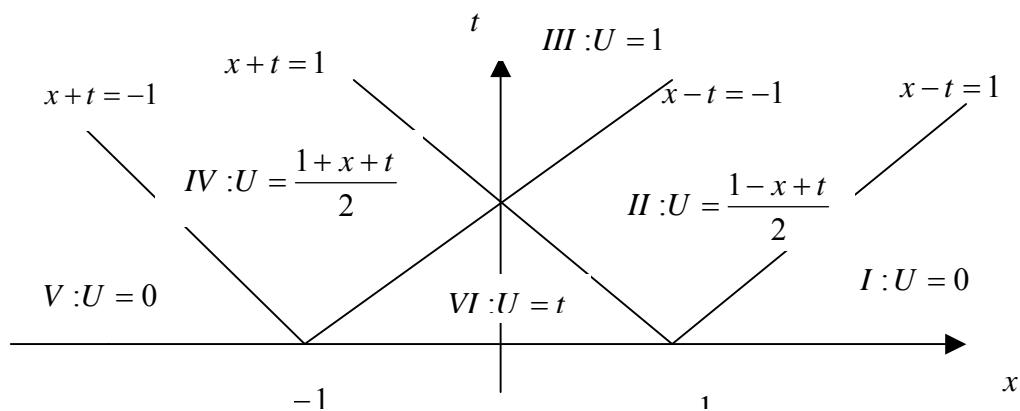
9) $u(x,t) = \frac{4}{225} \left(\cos \frac{15t}{2} - 1 \right) \cos \frac{5x}{2}$

10) $u(x,t) = \frac{x}{\pi} \sin t + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(4n^2 - 1)n} \left(\frac{1}{2n} \sin 2nt - \sin t \right) \sin nx$

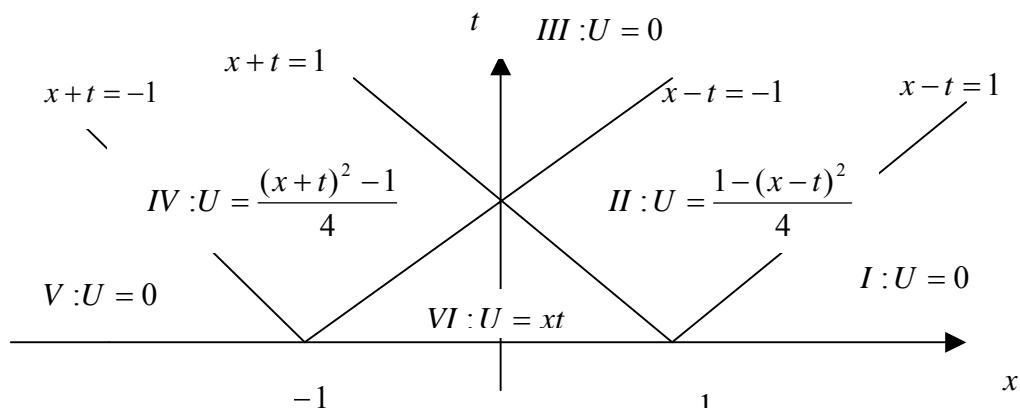
11) $u(x,t) = \frac{1 - \cos t}{2} + \sum_{n=1}^{\infty} u_n(t) \cos \frac{n\pi x}{2},$

$$u_n(t) = \begin{cases} \frac{g_n}{\alpha_n^2} (1 - \cos \alpha_n t), & n \neq 2 \\ \frac{1}{\sqrt{4\pi^2 + 1}} \sin \left(t \sqrt{4\pi^2 + 1} \right), & n = 2 \end{cases}, \quad \alpha_n = \sqrt{n^2\pi^2 + 1}, \quad g_n = \frac{2}{n\pi} \sin \frac{n\pi}{2}$$

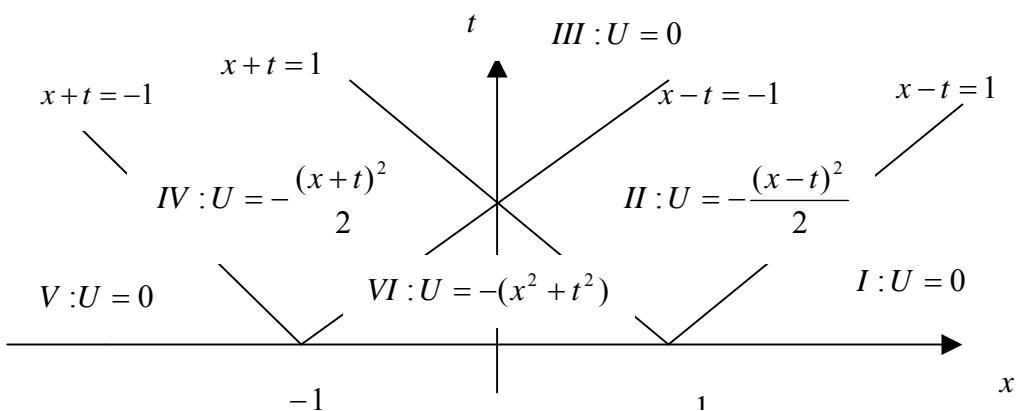
III.1



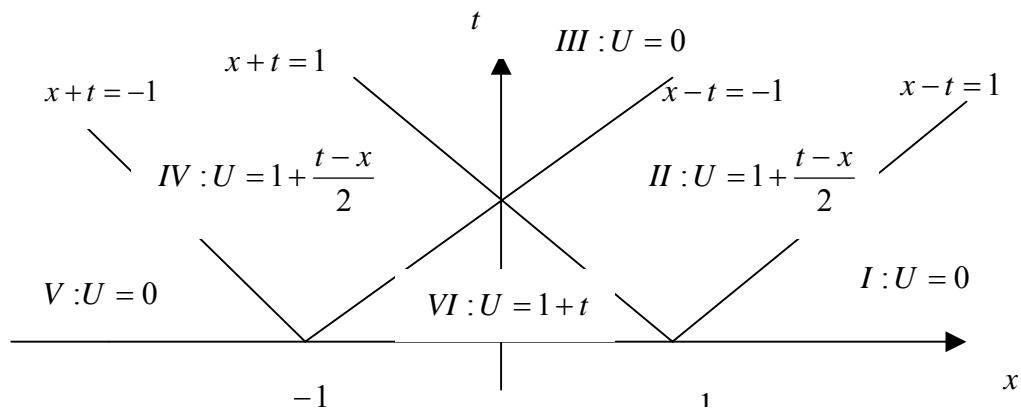
III.2



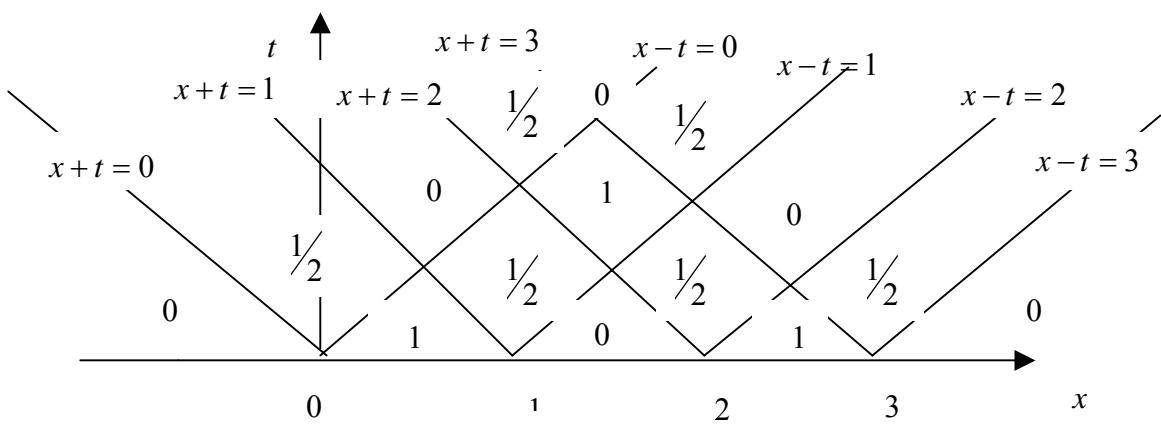
III.3



III.4



III.5



$$u(3,2)=59 \quad \text{III.6}$$