

משוואת חום לא-הומוגנית עם תנאי שפה לא-הומוגניים

$$\underline{u_t = \alpha^2 u_{xx} + bu + cu_x + g(x,t)}, (\alpha \neq 0, t \geq 0, 0 \leq x \leq l), \quad u(x,0) = f(x)$$

$u(0,t) = \varphi_1(t), \quad u(l,t) = \varphi_2(t)$	$v(0,t) = v(l,t) = 0$
$u_x(0,t) = \varphi_1(t), \quad u(l,t) = \varphi_2(t)$	$v_x(0,t) = v(l,t) = 0$
$u(0,t) = \varphi_1(t), \quad u_x(l,t) = \varphi_2(t)$	$v(0,t) = v_x(l,t) = 0$

$$\boxed{u(x,t) = v(x,t) + A(t) + xB(t)}$$

$u_x(0,t) = \varphi_1(t), \quad u_x(l,t) = \varphi_2(t)$	$v_x(0,t) = v_x(l,t) = 0$
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משוואת חום לא-הומוגנית ותנאי שפה הומוגניים

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$(\alpha > 0, t \geq 0, 0 \leq x \leq l), \quad u(x,0) = f(x)$	
$u_t = \alpha^2 u_{xx} + bu + p(x,t) \quad b$ $u(x,t) = v(x,t) e^{bt}$ $b - const$	$u_t = \alpha^2 u_{xx} + cu_x + bu + p(x,t) \quad .a$ $u(x,t) = v(x,t) e^{Dx+St}$ $c,b,D,S - consts$
↙	↖
$\boxed{v_t = \alpha^2 v_{xx} + g(x,t)}$	

$$\underline{u_t = \alpha^2 u_{xx} + g(x,t)}, (\alpha \neq 0, t > 0, 0 < x < l), \quad u(x,0) = f(x) . II$$

$u(0,t) = 0$	$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{n\pi x}{l}$
$u_x(0,t) = 0$	$u(x,t) = \sum_{n=0}^{\infty} u_n(t) \cos \frac{(2n+1)\pi x}{2l}$
$u(l,t) = 0$	
$u_x(l,t) = 0$	

$u(0,t) = 0$	$u(x,t) = \sum_{n=0}^{\infty} u_n(t) \sin \frac{(2n+1)\pi x}{2l}$
$u_x(0,t) = 0$	$u(x,t) = \frac{u_0(t)}{2} + \sum_{n=1}^{\infty} u_n(t) \cos \frac{n\pi x}{l}$
$u_x(l,t) = 0$	

דוגמה 1:

$$u_t = u_{xx} + 3 + t \cos 2x, u_x(0, t) = u_x(\pi, t) = 0, t > 0, u(x, 0) = \begin{cases} \pi, 0 < x < \pi/2 \\ 0, \pi/2 < x < \pi \end{cases}$$

פתרון

$$u_x(0, t) = u_x(\pi, t) = 0 \Rightarrow u(x, t) = \frac{u_0(t)}{2} + \sum_{n=1}^{\infty} u_n(t) \cos nx$$

$$u_0(0) = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} u(x, 0) dx = \frac{2}{\pi} \int_0^{\pi} \pi dx = \pi$$

$$u_n(0) = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi} \pi \cos nx dx = \frac{2}{n} \sin \frac{n\pi}{2}$$

$$g(x, t) = \frac{g_0(t)}{2} + \sum_{n=1}^{\infty} g_n(t) \cos nx = 3 + t \cos 2x \Rightarrow \begin{cases} g_0(t) = 6 \\ g_n(t) = \begin{cases} t & , n = 2 \\ 0 & , n \neq 2 \end{cases} \end{cases}$$

$$\left. \begin{array}{l} u_t = u_{xx} + g(x, t) \\ u(x, t) = u_0(t)/2 + \sum_{n=1}^{\infty} u_n(t) \cos nx \\ g(x, t) = g_0(t)/2 + \sum_{n=1}^{\infty} g_n(t) \cos nx \end{array} \right\} \Rightarrow \begin{array}{l} \frac{u'_0(t)}{2} + \sum_{n=1}^{\infty} u'_n(t) \cos nx = - \sum_{n=1}^{\infty} n^2 u_n(t) \cos nx + \\ + \frac{g_0(t)}{2} + \sum_{n=1}^{\infty} g_n(t) \cos nx \end{array} \Rightarrow$$

$$\Rightarrow u'_0(t) = g_0(t), u'_n(t) + n^2 u_n(t) = g_n(t)$$

$$\{u'_0(t) = 6, u_0(0) = \pi\} \Rightarrow u_0(t) = 6t + \pi$$

$$u'_n(t) + n^2 u_n(t) = \begin{cases} t & , n = 2 \\ 0 & , n \neq 2 \end{cases}, u_n(0) = \frac{2}{n} \sin \frac{n\pi}{2} \Rightarrow$$

$$u'_2(t) + 4u_2(t) = t, u_2(0) = 0 \Rightarrow u_2(t) = 0.0625e^{-4t} + 0.25t - 0.0625$$

$$n \neq 2, u'_n(t) + n^2 u_n(t) = 0, u_n(0) = \frac{2}{n} \sin \frac{n\pi}{2} \Rightarrow u_n(t) = \frac{2}{n} \sin \frac{n\pi}{2} e^{-n^2 t}, n \neq 2$$

$$u(x, t) = 3t + 0.5\pi + \sum_{n=1}^{\infty} u_n(t) \cos nx, u_n(t) = \begin{cases} 0.0625(e^{-4t} + 4t - 1) & , n = 2 \\ u_n(t) = \frac{2}{n} \sin \frac{n\pi}{2} e^{-n^2 t} & n \neq 2 \end{cases}$$

$$u(x, t) = 3t + 0.5\pi + 0.0625(e^{-4t} + 4t - 1) \cos 2x + 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} e^{-(2k+1)^2 t} \cos(2k+1)x$$

תרגילים

מצא פתרון פרטי עבור משוואת החום הנתונה המקיים את התנאים המצורפים

- 1) $u_t = 4u_{xx}$, $u(0,t) = 0$, $u(10,t) = 100$, $t > 0$, $u(x,0) = \begin{cases} 20x, & 0 \leq x < 5 \\ 100, & 5 \leq x \leq 10 \end{cases}$
- 2) $u_t = u_{xx} - x$, $u(0,t) = 0$, $u_x(1,t) = 0$, $t > 0$, $u(x,0) = \sin(0.5\pi x)$, $0 \leq x \leq 1$
- 3) $u_t = u_{xx} + 1 - |1 - 2x|$, $u(0,t) = u(1,t) = 0$, $t > 0$, $u(x,0) = 0$, $0 \leq x \leq 1$
- 4) $u_t = u_{xx} + t$, $u(0,t) = u_x(\pi,t) = 0$, $t > 0$, $u(x,0) = x^2 - 2\pi x$, $0 < x < \pi$
- 5) $u_t = u_{xx} - 2u_x + u + te^x \sin x$, $u(0,t) = u(\pi,t) = 0$, $t > 0$, $u(x,0) = 3e^x \sin 5x$, $0 < x < \pi$
- 6) $u_t = 9u_{xx} + 18u_x + 10u$, $u(0,t) = 0$, $u(\pi,t) = e^{-\pi}$, $t > 0$, $u(x,0) = \frac{x}{\pi}e^{-x}$, $0 < x < \pi$
- 7) $u_t = 2u_{xx} - 3u + \cos x$, $u_x(0,t) = u_x(\pi,t) = 0$, $t > 0$, $u(x,0) = \cos 2x$, $0 < x < \pi$
- 8) $u_t = 2u_{xx} - u + x(t+1)$, $u(0,t) = 1$, $u_x(\pi,t) = t$, $t > 0$, $u(x,0) = 1 + 4 \sin 1.5x$, $0 < x < \pi$

תשובות:

- 1) $u(x,t) = 10x + \frac{400}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin(0.5n\pi)}{n^2} e^{-0.04n^2\pi^2t} \sin \frac{n\pi x}{10}$
- 2) $u(x,t) = e^{-0.25\pi^2t} \sin(0.5\pi x) + \frac{32}{\pi^4} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^4} \left(1 - e^{-0.25(2n-1)^2\pi^2t}\right) \sin \frac{(2n-1)\pi x}{2}$
- 3) $u(x,t) = \frac{8}{\pi^4} \sum_{n=1}^{\infty} \frac{\sin(0.5n\pi)}{n^4} (1 - e^{-n^2\pi^2t}) \sin n\pi x =$
 $= \frac{8}{\pi^4} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^4} \left(1 - e^{-(2k-1)^2\pi^2t}\right) \sin(2k-1)\pi x$
- 4) $u(x,t) = \frac{16}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \left[2 \left(\frac{2}{(2n-1)^2} - 1 \right) e^{-\frac{(2n-1)^2}{4}t} - \frac{4}{(2n-1)^2} + t \right] \sin \frac{(2n-1)x}{2}$
- 5) $u(x,t) = e^x \left((e^{-t} + t - 1) \sin x + 3e^{-25t} \sin 5x \right)$
- 6) $u(x,t) = \frac{e^{-x}}{\pi} \left(x + 2e^t \sum_{n=1}^{\infty} \frac{(-1)^n}{n(9n^2-1)} (e^{-9n^2t} - e^{-t}) \sin nx \right)$
- 7) $u(x,t) = 0.2(1 - e^{-5t}) \cos x + e^{-11t} \cos 2x$
- 8) $\begin{cases} u(x,t) = 1 + xt + e^{-t} \sum_{n=1}^{\infty} \left[\left(w_n(0) - \frac{\gamma_n}{\alpha_n + 1} \right) e^{-\alpha_n t} + \frac{\gamma_n}{\alpha_n + 1} e^t \right] \sin \frac{(2n-1)x}{2} \\ w_n(0) = \begin{cases} 4, & n = 2 \\ 0, & n \neq 2 \end{cases}, \quad \gamma_n = -\frac{4}{\pi(2n-1)}, \quad \alpha_n = \frac{(2n-1)^2}{2} \end{cases}$
 $u(x,t) = 1 + xt + 4e^{-5.5t} \sin 1.5x + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{e^{-(\alpha_n+1)t} - 1}{(2n-1)(\alpha_n + 1)} \sin \frac{(2n-1)x}{2}, \quad \alpha_n = \frac{(2n-1)^2}{2}$