I. <u>משוואת לפלס</u> $u_{xx}+u_{yy}=0$ <u>במלבן</u> $0\le x\le a,\quad 0\le y\le b$ <u>עם תנאי שפה</u>

1.

$u(0,y)=0$ $u(a,y)=0$	$\left\{\sin\frac{n\pi x}{a}\right\}$	$u(x,y)=\sum_{n=1}^{\infty}\left(a_n e^{\frac{n\pi y}{a}}+b_n e^{-\frac{n\pi y}{a}}\right)\sin\frac{n\pi x}{a}$
$u_x(0,y)=0$ $u(a,y)=0$	$\left\{\cos\frac{(2n+1)\pi x}{2a}\right\}$	$u(x,y)=\sum_{n=0}^{\infty}\left(a_n e^{\frac{(2n+1)\pi y}{2a}}+b_n e^{-\frac{(2n+1)\pi y}{2a}}\right)\cos\frac{(2n+1)\pi x}{2a}$
$u(0,y)=0$ $u_x(a,y)=0$	$\left\{\sin\frac{(2n+1)\pi x}{2a}\right\}$	$u(x,y)=\sum_{n=0}^{\infty}\left(a_n e^{\frac{(2n+1)\pi y}{2a}}+b_n e^{-\frac{(2n+1)\pi y}{2a}}\right)\sin\frac{(2n+1)\pi x}{2a}$
$u_x(0,y)=0$ $u_x(a,y)=0$	$\left\{1,\cos\frac{n\pi x}{a}\right\}$	$u(x,y)=\frac{p+qy}{2}+\sum_{n=1}^{\infty}\left(a_n e^{\frac{n\pi y}{a}}+b_n e^{-\frac{n\pi y}{a}}\right)\cos\frac{n\pi x}{a}$

2.

$u(x,0)=0$ $u(x,b)=0$	$\left\{\sin\frac{n\pi y}{b}\right\}$	$u(x,y)=\sum_{n=1}^{\infty}\left(a_n e^{\frac{n\pi x}{b}}+b_n e^{-\frac{n\pi x}{b}}\right)\sin\frac{n\pi y}{b}$
$u_y(x,0)=0$ $u(x,b)=0$	$\left\{\cos\frac{(2n+1)\pi y}{2b}\right\}$	$u(x,y)=\sum_{n=0}^{\infty}\left(a_n e^{\frac{(2n+1)\pi x}{2b}}+b_n e^{-\frac{(2n+1)\pi x}{2b}}\right)\cos\frac{(2n+1)\pi y}{2b}$
$u(x,0)=0$ $u_y(x,b)=0$	$\left\{\sin\frac{(2n+1)\pi y}{2b}\right\}$	$u(x,y)=\sum_{n=0}^{\infty}\left(a_n e^{\frac{(2n+1)\pi x}{2b}}+b_n e^{-\frac{(2n+1)\pi x}{2b}}\right)\sin\frac{(2n+1)\pi y}{2b}$
$u_y(x,0)=0$ $u_y(x,b)=0$	$\left\{1,\cos\frac{n\pi y}{b}\right\}$	$u(x,y)=\frac{p+qx}{2}+\sum_{n=1}^{\infty}\left(a_n e^{\frac{n\pi x}{b}}+b_n e^{-\frac{n\pi x}{b}}\right)\cos\frac{n\pi y}{b}$

II. <u>משוואת לפלס</u> $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0$ <u>בקואורדינטות קוטביות</u>

$$u(r,\theta)=\frac{a_0+c_0\ln r}{2}+\sum_{n=1}^{\infty}\left(a_n r^n+b_n r^{-n}\right)\cos n\theta+\sum_{n=1}^{\infty}\left(c_n r^n+d_n r^{-n}\right)\sin n\theta$$

פתור משוואת לפלס $u_{xx}+u_{yy}=0$ במלבן $0\le x\le a,\quad 0\le y\le b$ עם תנאי שפה

1) $\left(0<x<2,\,0<y<4\right),\ u_x(0,y)=u(2,y)=u(x,0)=0,\ \ u(x,4)=\begin{cases}1, & 0\le x\le 1\\ 2-x, & 1<x\le 2\end{cases}$

2) $\left(0<x<\pi,\,0<y<\pi\right),\ u(x,0)=u(x,\pi)=0,\ u(0,y)=0,\ \ u(\pi,y)=\sin y$

3) $\left(0<x<4,\,0<y<2\right),\ u(x,0)=u(x,2)=u(4,y)=0,\ \ u(0,y)=y(2-y)$

4) $\left(0<x<4,\,0<y<2\right),\ \ u(x,2)=u(4,y)=0,\ \ u(0,y)=y(2-y),\ \ u(x,0)=3\sin\dfrac{\pi x}{4}$

5) $\left(0<x<2,\,0<y<4\right),\ u(x,0)=u(x,4)=u(2,y)=0,\ \ u(0,y)=\begin{cases}-y, & 0\le y\le 2\\ 0, & 2<y\le 4\end{cases}$

6) $\left(0<x<1,\,0<y<2\right),\ u_y(x,0)=u_y(x,2)=u(0,y)=0,\ u_x(1,y)=1-2\cos\pi y$

7) $\left(0<x<\pi,\,0<y<1\right),\ u_x(0,y)=u_x(\pi,y)=0,\ u(x,0)=2\cos x,\ \ u(x,1)=\cos^2 x-\sin^2 x$

8) $\left(0<x<\pi,\,0<y<\pi\right),\ u(0,y)=u(\pi,y)=u(x,0)=0,\quad u(x,\pi)=\sin 3x$

9) $\left(0<x<a,\,0<y<b\right),\ u(0,y)=u_x(a,y)=0,\ u(x,0)=\sin\dfrac{\pi x}{2a},\ u(x,b)=3\sin\dfrac{3\pi x}{2a}$

10) $\left(0<x<1,\,0<y<\pi\right),\ u_y(x,0)=0, u_y(x,\pi)=-\pi\sin 2\pi x,\ u(0,y)=0\ ,\ u(1,y)=y$

II . פתור משוואת לפלס $\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial\theta^2}=0$ עם תנאי שפה בתחום נתון:

11) $0\le r\le 3,\quad u(3,\theta)=\begin{cases}0.1\theta, & 0\le\theta\le\pi\\ 0.2\pi-0.1\theta, & \pi<\theta<2\pi\end{cases}$

12) $0\le r\le a,\quad u(a,\theta)=4a+3a\sqrt{a}\cos\theta-5a^3\sin 3\theta$

13) $1\le r\le 2,\quad u(1,\theta)=0,\quad u(2,\theta)=\theta,\quad 0\le\theta<2\pi$

14) $r\ge 3, u(3,\theta)=2\pi\theta-\theta^2,\quad 0\le\theta<2\pi$

15) $0<\alpha\le r\le\beta,\quad u(\alpha,\theta)=1-\sin\theta+3\cos 2\theta,\quad u(\beta,\theta)=0,\quad 0\le\theta<2\pi$

III פתור משוואת לפלס בשיטת הפרדת משתנים

16) $u_{xx}+u_{yy}-2u=0\quad(0\le x\le\pi,\quad 0\le y\le\pi)$

$u(x,0)=u(x,\pi)=0,\quad u(0,y)=\sin y,\quad u(\pi,y)=3\sin 2y$

17) $u_{xx}+u_{yy}-5u=0\quad(0\le x\le\pi,\quad 0\le y\le\pi)$

$u(x,\pi)=u_x(0,y)=0,\quad u_x(\pi,y)=0,\quad u(x,0)=\begin{cases}1, & 0<x<0.5\pi\\ 0, & 0.5\pi<x<\pi\end{cases}$

תשובות:

1) $$u(x,y)=\frac{16}{\pi^2}\sum_{n=0}^{\infty}\frac{\cos\frac{(2n+1)\pi}{4}}{(2n+1)^2\left(e^{(2n+1)\pi}-e^{-(2n+1)\pi}\right)}\left(e^{\frac{(2n+1)\pi y}{4}}-e^{-\frac{(2n+1)\pi y}{4}}\right)\cos\frac{(2n+1)\pi x}{4}$$

2) $$u(x,y)=\frac{1}{e^{\pi}-e^{-\pi}}\left(e^{x}-e^{-x}\right)\sin y$$

3) $$u(x,y)=\frac{16}{\pi^3}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3(1-e^{4n\pi})}\left(e^{n\pi x/2}-e^{4n\pi-\frac{n\pi x}{2}}\right)\sin\frac{n\pi y}{2}$$

4) $$u(x,y)=\frac{3}{1-e^{\pi}}\left(e^{\frac{\pi y}{4}}-e^{\pi-\frac{\pi y}{4}}\right)\sin\frac{\pi x}{4}+\frac{16}{\pi^3}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3(1-e^{4n\pi})}\left(e^{\frac{n\pi x}{2}}-e^{4n\pi-\frac{n\pi x}{2}}\right)\sin\frac{n\pi y}{2}$$

5) $$u(x,y)=\sum_{n=1}^{\infty}\frac{\frac{4}{n\pi}\cos\frac{n\pi}{2}-\frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}}{1-e^{n\pi}}\left(e^{\frac{n\pi x}{4}}-e^{n\pi-\frac{n\pi x}{4}}\right)\sin\frac{n\pi y}{4}$$

6) $$u(x,y)=x-\frac{2}{\pi\,(e^{\pi}+e^{-\pi})}(e^{\pi x}-e^{-\pi x})\cos\pi y$$

7) $$u(x,y)=\frac{1}{e^{2}-e^{-2}}\left(e^{2y}-e^{-2y}\right)\cos 2x+\frac{2}{1-e^{2}}\left(e^{y}-e^{2-y}\right)\cos x$$

8) $$u(x,y)=\frac{1}{e^{3\pi}-e^{-3\pi}}\left(e^{3y}-e^{-3y}\right)\sin 3x$$

9) $$u(x,y)=\frac{3}{e^{3\pi b/(2a)}-e^{-3\pi b/(2a)}}\left(e^{3\pi y/(2a)}-e^{-3\pi y/(2a)}\right)\sin\frac{3\pi x}{2a}+$$
$$+\frac{1}{1-e^{\pi b/a}}\left(e^{\pi y/(2a)}-e^{-\pi y/(2a)}e^{\pi b/a}\right)\sin\frac{\pi x}{2a}$$

10) $$u(x,y)=\frac{-1}{2\left(e^{2\pi^2}-e^{-2\pi^2}\right)}\left(e^{2\pi y}+e^{-2\pi y}\right)\sin 2\pi x+$$
$$+\frac{\pi}{2}x+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^2\left(e^{n}-e^{-n}\right)}\left(e^{nx}-e^{-nx}\right)\cos ny$$

11) $u(r,\theta)=\frac{\pi}{20}+\frac{0.2}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{3}\right)^{n}\frac{(-1)^{n}-1}{n^{2}}\cos n\theta=\frac{\pi}{20}-\frac{0.4}{\pi}\sum_{k=1}^{\infty}\left(\frac{r}{3}\right)^{2k-1}\frac{1}{(2k-1)^{2}}\cos(2k-1)\theta$

12) $u(r,\theta)=4a+3\sqrt{a}\,r\cos\theta-5r^{3}\sin 3\theta$

13) $u(r,\theta)=\frac{\pi}{\ln 2}\ln r-2\sum_{n=1}^{\infty}\frac{1}{n(2^{n}-2^{-n})}(r^{n}-r^{-n})\sin n\theta$

14) $u(r,\theta)=\frac{2\pi^{2}}{3}-4\sum_{n=1}^{\infty}\frac{3^{n}}{n^{2}}r^{-n}\cos n\theta$

15) $u(r,\theta)=\frac{\ln r-\ln\beta}{\ln\alpha-\ln\beta}+\frac{3}{\alpha^{2}-\beta^{4}\alpha^{-2}}(r^{2}-\beta^{4}r^{-2})\cos 2\theta-\frac{1}{\alpha-\beta^{2}\alpha^{-1}}(r-\beta^{2}r^{-1})\sin\theta$

16) $u(x,y)=\frac{1}{1-e^{2\sqrt{3}\pi}}\left(e^{\sqrt{3}\,x}-e^{2\sqrt{3}\pi-\sqrt{3}x}\right)\sin y+\frac{1}{e^{\sqrt{6}\pi}-e^{-\sqrt{6}\pi}}\left(e^{\sqrt{6}\,x}-e^{-\sqrt{6}x}\right)\sin 2y$

17) $u(x,y)=\frac{\pi-y}{2\pi}+\sum_{n=1}^{\infty}a_{n}\left(e^{\sqrt{n^{2}+5}\,y}-e^{2\pi\sqrt{n^{2}+5}}e^{-\sqrt{n^{2}+5}\,y}\right)\cos nx$

$$a_{n}=\frac{2}{n\pi(1-e^{2\pi\sqrt{n^{2}+5}})}\sin\frac{n\pi}{2}$$