תרגילים נוספים למשוואת החום

1) $u_t = 4u_{xx} - u + t,\ u(0,t) = t,\ u(\pi,t) = t+3,\ t>0,\ u(x,0) = 3,\ 0<x<\pi$

2) $u_t = u_{xx} - 2u + x(1+2t) + 5\sin 2t \cos x\, e^{-2t}$

$u_x(0,t) = t,\ u(\pi/2,t) = \pi t/2,\ t>0,\ u(x,0) = \begin{cases} 1, & 0 \le x \le \pi/4 \\ 0, & \pi/4 < x < \pi/2 \end{cases}$

3) $u_t = 9u_{xx} - u + (2\sin \pi x + 9\pi^2 t \sin 3\pi x + 2)e^{-t}$

$u(0,t) = (2t+1)e^{-t},\ u_x(1/2,t) = 0,\ t>0,\ u(x,0) = x+1,\ 0 \le x \le 1/2$

4) $u_t = \pi u_{xx} - 3u - 2\pi + 3x^2 + e^{-3t} \sin \pi x$

$u_x(0,t) = e^{-3t},\ u_x(\pi,t) = 2\pi + e^{-3t},\ t>0,\ u(x,0) = x^2 + 2x,\ 0<x<\pi$

5) $u_t = 4u_{xx} - 3u + 12x + 2e^{-t} + te^{-3t} \sin 2\pi x$

$u(0,t) = e^{-t},\ u(1,t) = e^{-t} + 4,\ t>0,\ u(x,0) = 1 + 4x - 5\sin 3\pi x,\ 0<x<1$

6) $u_t = 9u_{xx} - u + 1,\ u_x(0,t) = 0,\ u(\pi,t) = t,\ t>0,\ u(x,0) = \cos(3x/2),\ 0<x<\pi$

7) $u_t = 3u_{xx} - 2u + \gamma(2x - 2 + \cos 2x),\ \gamma = const$

$u_x(0,t) = u_x(\pi,t) = \gamma,\ t>0,\ u(x,0) = \gamma(x + 2\cos^2 x),\ 0<x<\pi$

תשובות:

1) $$u(x,t) = t + 3x/\pi + \sum_{n=1}^{\infty}\left[\frac{\beta_n}{4n^2+1} + \left(\frac{6}{n\pi} - \frac{\beta_n}{4n^2+1}\right)e^{-(4n^2+1)t}\right]\sin nx,\ \beta_n = 2\frac{4(-1)^n - 1}{n\pi}$$

2) $$u(x,t) = tx + \left(2(1+\sqrt{2}/\pi)e^{-3t} + (\sin 2t - 2\cos 2t)e^{-2t}\right)\cos x +$$
$$+\frac{4}{\pi}\sum_{n=2}^{\infty}\frac{1}{2n-1}\sin\frac{(2n-1)\pi}{4}e^{-(2n-1)^2 t}\cos(2n-1)x$$

3) $$u(x,t) = e^{-t}\left[2t+1+\frac{1}{9\pi^2}\left((34e^{-9\pi^2 t} + 2)\sin \pi x + \left(\left(\frac{1}{81} - 4\right)e^{-81\pi^2 t} + \frac{81\pi^2 t - 1}{81}\right)\sin 3\pi x\right)\right] +$$
$$+e^{-t}\frac{4}{\pi^2}\sum_{n=3}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}e^{-9(2n-1)^2\pi^2 t}\sin(2n-1)\pi x$$

4) $u(x,t)=x^2+xe^{-3t}+e^{-3t}\left[\frac{2t}{\pi}+\frac{1}{2}-\frac{4}{\pi^2}e^{-\pi^3 t}\cos\pi x+\sum_{n=2}^{\infty}w_n\cos n\pi x\right]$

$$w_n=\left(\gamma_n-\frac{\beta_n}{n^2\pi^3}\right)e^{-n^2\pi^3 t}+\frac{\beta_n}{n^2\pi^3},\ n=2,3,\ldots$$

$$\left.\begin{array}{l}\beta_n=-2\dfrac{(-1)^n+1}{(n^2-1)}=\begin{cases}\dfrac{-4}{(n^2-1)}, & n=2k\\ 0 & ,\ n=2k+1\end{cases}\\ \gamma_n=2\dfrac{(-1)^n-1}{n^2\pi^2}=\begin{cases}\dfrac{-4}{n^2\pi^2} & ,\ n=2k+1\\ 0 & ,\ n=2k\end{cases}\end{array}\right\}\Rightarrow w_n=\begin{cases}\dfrac{\beta_n}{n^2\pi^3}\left(1-e^{-n^2\pi^3 t}\right), & n=2k\\ \gamma_n e^{-n^2\pi^3 t} & ,\ n=2k+1\end{cases}$$

$$w_n=\frac{-4}{\pi^2}\begin{cases}\dfrac{1}{n^2(n^2-1)\pi}\left(1-e^{-n^2\pi^3 t}\right) & ,\ n=2k\\ \dfrac{1}{n^2}e^{-n^2\pi^3 t} & ,\ n=2k+1\end{cases},\quad k=1,2,\ldots$$

5) $u(x,t)=4x+e^{-t}+e^{-3t}\dfrac{e^{-16\pi^2 t}+16\pi^2 t-1}{256\pi^4}\sin 2\pi x-5e^{-(36\pi^2+3)t}\sin 3\pi x$

6) $u(x,t)=t+\sum_{n=1}^{\infty}\left[C_n e^{-\gamma_n t}+\frac{\beta_n}{\gamma_n}\left(t-\frac{1}{\gamma_n}\right)\right]\cos\frac{2n-1}{2}x$

$$\gamma_n=1+\frac{9(2n-1)^2}{4}\quad,\quad \beta_n=\frac{4(-1)^n}{\pi(2n-1)}\quad,\quad C_n=\frac{\beta_n}{\gamma_n^2}+\begin{cases}1, & n=2\\ 0, & n\neq 2\end{cases}$$

7) $u(x,t)=\gamma\left(x+(2e^{-2t}-1)+\frac{1}{14}(13e^{-14t}+1)\cos 2x\right)$