CHAPTER 5

Section 5.1, Page 237

2. Use the ratio test:
$$\lim_{n\to\infty} \frac{|(n+1)x^{n+1}/2^{n+1}|}{|nx^n/2^n|} = \lim_{n\to\infty} \frac{n+1}{n}\frac{1}{2}|x| = \frac{|x|}{2}.$$
Therefore the series converges absolutely for $|x| < 2$. For $x = 2$ and $x = -2$ the n^{th} term does not approach zero as $n \to \infty$ so the series diverge. Hence the radius of convergence is $\rho = 2$.

5. Use the ratio test:
$$\lim_{n\to\infty} \frac{|(2x+1)^{n+1}/(n+1)^2|}{|(2x+1)^n/n^2|} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2}|2x+1| = |2x+1|.$$
Therefore the series converges absolutely for $|2x+1| < 1$, or $|x+1/2| < 1/2$. At $x = 0$ and $x = -1$ the series also converge absolutely. However, for $|x+1/2| > 1/2$ the series diverges by the ratio test. The radius of convergence is $\rho = 1/2$.

9. For this problem $f(x) = \sin x$. Hence $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x, \ldots$. Then $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1, \ldots$. The even terms in the series will vanish and the odd terms will alternate in sign. We obtain $\sin x = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}/(2n+1)!$. From the ratio test it follows that $\rho = \infty$.

12. For this problem $f(x) = x^2$. Hence $f'(x) = 2x$, $f''(x) = 2$, and $f^{(n)}(x) = 0$ for $n > 2$. Then $f(-1) = 1$, $f'(-1) = -2$, $f''(-1) = 2$ and $x^2 = 1 - 2(x+1) + 2(x+1)^2/2! = 1 - 2(x+1) + (x+1)^2$. Since the series terminates after a finite number of terms, it converges for all x. Thus $\rho = \infty$.

13. For this problem $f(x) = \ln x$. Hence $f'(x) = 1/x$, $f''(x) = -1/x^2$, $f'''(x) = 1\cdot 2/x^3, \ldots$, and $f^{(n)}(x) = (-1)^{n+1}(n-1)!/x^n$. Then $f(1) = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 1\cdot 2, \ldots$, $f^{(n)}(1) = (-1)^{n+1}(n-1)!$ The Taylor series is $\ln x = (x-1) - (x-1)^2/2 + (x-1)^3/3 - \ldots = \sum_{n=1}^{\infty} (-1)^{n+1}(x-1)^n/n$. It follows from the ratio test that

the series converges absolutely for $|x-1| < 1$. However, the series diverges at $x = 0$ so $\rho = 1$.

18. Writing the individual terms of y, we have
$y = a_0 + a_1x + a_2x^2 + \ldots + a_ax^n + \ldots$, so
$y' = a_1 + 2a_2x + 3a_3x^2 + \ldots + (n+1)a_{n+1}x^n + \ldots$, and
$y'' = 2a_2 + 3\cdot 2a_3x + 4\cdot 3a_4x^2 + \ldots + (n+2)(n+1)a_{n+2}x^n + \ldots$.
If $y'' = y$, we then equate coefficients of like powers of x to obtain $2a_2 = a_0$, $3\cdot 2a_3 = a_1$, $4\cdot 3a_4 = a_2$, $\ldots$ $(n+2)(n+1)a_{n+2} = a_n$, which yields the desired result for $n = 0,1,2,3\ldots$.

19. Set $m = n-1$ on the right hand side of the equation. Then $n = m+1$ and when $n = 1$, $m = 0$. Thus the right hand side becomes $\sum_{m=0}^{\infty} a_m(x-1)^{m+1}$, which is the same as the left hand side when m is replaced by n.

23. Multiplying each term of the first series by x yields
$x\sum_{n=1}^{\infty} na_nx^{n-1} = \sum_{n=1}^{\infty} na_nx^n = \sum_{n=0}^{\infty} na_nx^n$, where the last equality can be verified by writing out the first few terms. Changing the index from k to n ($n=k$) in the second series then yields
$$\sum_{n=0}^{\infty} na_nx^n + \sum_{n=0}^{\infty} a_nx^n = \sum_{n=0}^{\infty}(n+1)a_nx^n.$$

25. $\sum_{m=2}^{\infty} m(m-1)a_mx^{m-2} + x\sum_{k=1}^{\infty} ka_kx^{k-1} =$
$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{k=1}^{\infty} ka_kx^k =$
$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + na_n]x^n$. In the first case we have let $n = m - 2$ in the first summation and multiplied each term of the second summation by x. In the second case we have let $n = k$ and noted that for $n = 0$, $na_n = 0$.

28. If we shift the index of summation in the first sum by

letting m = n−1, we have

$\sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{m=0}^{\infty} (m+1)a_{m+1}x^m$. Substituting this into the given equation and letting m = n again, we obtain:

$\sum_{n=0}^{\infty} (n+1)a_{n+1} x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$, or

$\sum_{n=0}^{\infty} [(n+1)a_{n+1} + 2a_n]x^n = 0$.

Hence $a_{n+1} = -2a_n/(n+1)$ for n = 0,1,2,3,... . Thus $a_1 = -2a_0$, $a_2 = -2a_1/2 = 2^2a_0/2$, $a_3 = -2a_2/3 = -2^3a_0/2\cdot3 = -2^3a_0/3!\ldots$ and $a_n = (-1)^n2^na_0/n!$. Notice that for n = 0 this formula reduces to a_0 so we can write

$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (-1)^n2^n\, a_0x^n/n! = a_0\sum_{n=0}^{\infty} (-2x)^n/n! = a_0e^{-2x}$.

Section 5.2, Page 247

2. $y = \sum_{n=0}^{\infty} a_n x^n$; $y' = \sum_{n=1}^{\infty} na_n x^{n-1}$ and since we must multiply y' by x in the D.E. we do not shift the index; and

$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$. Substituting in the D.E., we obtain

$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$. In order to have the starting point the same in all three summations, we let n = 0 in the first and third terms to obtain the following

$$(2\cdot1\; a_2 - a_0)x^0 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} - (n+1)a_n]x^n = 0.$$

Thus $a_{n+2} = a_n/(n+2)$ for n = 1,2,3,... . Note that the recurrence relation is also correct for n = 0. We show how to calculate the odd a's:

$a_3 = a_1/3$, $a_5 = a_3/5 = a_1/5\cdot3$, $a_7 = a_5/7 = a_1/7\cdot5\cdot3, \ldots$. Now notice that $a_3 = 2a_1/(2\cdot3) = 2a_1/3!$, that $a_5 = 2\cdot4a_1/(2\cdot3\cdot4\cdot5) = 2^2\cdot2a_1/5!$, and that $a_7 = 2\cdot4\cdot6a_1/(2\cdot3\cdot4\cdot5\cdot6\cdot7) = 2^3\cdot3!\ a_1/7!$. Likewise $a_9 = a_7/9 = 2^3\cdot3!\ a_1/(7!)9 = 2^3\cdot3!\ 8a_1/9! = 2^4\cdot4!\ a_1/9!$. Continuing we have $a_{2m+1} = 2^m\cdot m!\ a_1/(2m+1)!$. In the same way we find that the even a's are given by $a_{2m} = a_0/2^m\, m!$. Thus

$$y = a_0\sum_{m=0}^{\infty}\frac{x^{2m}}{2^m m!} + a_1\sum_{m=0}^{\infty}\frac{2^m m!\ x^{2m+1}}{(2m+1)!}.$$

3. $y = \sum_{n=0}^{\infty} a_n(x-1)^n$; $y' = \sum_{n=1}^{\infty} na_n(x-1)^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^n$, and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n.$$

Substituting in the D.E. and setting $x = 1 + (x-1)$ we obtain

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n - \sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^n - \sum_{n=1}^{\infty} na_n(x-1)^n - \sum_{n=0}^{\infty} a_n(x-1)^n = 0,$$

where the third term comes from:

$$-(x-1)y' = \sum_{n=1}^{\infty}(n+1)a_{n+1}(x-1)^{n+1} = -\sum_{n=1}^{\infty} na_n(x-1)^n.$$

Letting $n = 0$ in the first, second, and the fourth sums, we obtain

$$(2\cdot1\cdot a_2 - 1\cdot a_1 - a_0)(x-1)^0 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} - (n+1)a_{n+1} - (n+1)a_n](x-1)^n = 0.$$

Thus $(n+2)a_{n+2} - a_{n+1} - a_n = 0$ for $n = 0,1,2,\ldots$. This recurrance relation can be used to solve for a_2 in terms of a_0 and a_1, then for a_3 in terms of a_0 and a_1, etc. In many cases it is easier to first take $a_0 = 0$ and generate one solution and then take $a_1 = 0$ and generate the second linearly independent solution. Thus, choosing $a_0 = 0$ we find that $a_2 = a_1/2$, $a_3 = (a_2+a_1)/3 = a_1/2$, $a_4 = (a_3+a_2)/4 = a_1/4$, $a_5 = (a_4+a_3)/5 = 3a_1/20, \ldots$. This yields the solution $y_2(x) = a_1[(x-1) + (x-1)^2/2 + (x-1)^3/2 + (x-1)^4/4 + 3(x-1)^5/20 + \ldots]$. The second independent solution may be obtained by choosing $a_1 = 0$. Then $a_2 = a_0/2$, $a_3 = (a_2+a_1)/3 = a_0/6$, $a_4 = (a_3+a_2)/4 = a_0/6$, $a_5 = (a_4+a_3)/5 = a_0/15, \ldots$. This yields the solution $y_1(x) = a_0[1+(x-1)^2/2+(x-1)^3/6+(x-1)^4/6+(x-1)^5/15+\ldots]$.

5. $y = \sum_{n=0}^{\infty} a_n x^n$; $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$; and $y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$.
Substituting in the D.E. and shifting the index in both summations for y'' gives

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} (n+1)n\, a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n =$$

$$(2\cdot1\cdot a_2 + a_0)x^0 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + a_n]x^n = 0.$$

Thus $a_2 = -a_0/2$ and $a_{n+2} = na_{n+1}/(n+2) - a_n/(n+2)(n+1)$, $n = 1,2,\ldots$. Choosing $a_0 = 0$ yields $a_2 = 0$, $a_3 = -a_1/6$, $a_4 = 2a_3/4 = -a_1/12, \ldots$ which gives one solution as $y_2(x) = a_1(x - x^3/6 - x^4/12 + \ldots)$. A second linearly independent solution is obtained by choosing $a_1 = 0$. Then $a_2 = -a_0/2$, $a_3 = a_2/3 = -a_0/6$, $a_4 = 2a_3/4 - a_2/12 = -a_0/24, \ldots$ which gives $y_1(x) = a_0(1 - x^2/2 - x^3/6 - x^4/24 + \ldots)$.

8. If $y = \sum_{n=1}^{\infty} a_n(x-1)^n$ then

$$xy = [1+(x-1)]y = \sum_{n=1}^{\infty} a_n(x-1)^n + \sum_{n=1}^{\infty} a_n(x-1)^{n+1},$$

$$y' = \sum_{n=1}^{\infty} na_n(x-1)^{n-1}, \text{ and}$$

$xy'' = [1+(x-1)]y''$

$$= \sum_{n=1}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=1}^{\infty} n(n-1)a_n(x-1)^{n-1}.$$

14. You will need to rewrite x+1 as 3 + (x-2) in order to multiply x+1 times y' as a power series about $x_0 = 2$.

16a. From Problem 6 we have

$$y(x) = c_1(1 - x^2 + \frac{1}{6}x^4 + \ldots) + c_2(x - \frac{1}{4}x^3 + \frac{7}{160}x^5 + \ldots).$$

Now $y(0) = c_1 = -1$ and $y'(0) = c_2 = 3$ and thus

$$y(x) = -1+x^2-\frac{1}{6}x^4+3x-\frac{3}{4}x^3 = -1+3x+x^2-\frac{3}{4}x^3-\frac{1}{6}x^4 + \ldots .$$

16c. By plotting $f = -1 + 3x + x^2 - 3x^3/4$ and $g = f - x^4/6$ between -1 and 1 it appears that f is a reasonable approximation for $|x| < 0.7$.

19. The D.E. transforms into $u''(t) + t^2u'(t) + (t^2+2t)u(t) = 0$.

Assuming that $u(t) = \sum_{n=0}^{\infty} a_nt^n$, we have $u'(t) = \sum_{n=1}^{\infty} na_nt^{n-1}$ and

$u''(t) = \sum_{n=2}^{\infty} n(n-1)a_nt^{n-2}$. Substituting in the D.E. and shifting indices yields

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}t^n + \sum_{n=2}^{\infty} (n-1)a_{n-1}t^n + \sum_{n=2}^{\infty} a_{n-2}t^n + \sum_{n=1}^{\infty} 2a_{n-1}t^n = 0,$$

$$2\cdot1\cdot a_2t^0 + (3\cdot2\cdot a_3 + 2\cdot a_0)t^1 + \sum_{n=2}^{\infty}[(n+2)(n+1)a_{n+2} + (n+1)a_{n-1} + a_{n-2}]t^n = 0.$$

It follows that $a_2 = 0$, $a_3 = -a_0/3$ and $a_{n+2} = -a_{n-1}/(n+2) - a_{n-2}/[(n+2)(n+1)]$, $n = 2,3,4\ldots$. We obtain one solution by choosing $a_1 = 0$. Then $a_4 = -a_0/12$, $a_5 = -a_2/5 - a_1/20 = 0$, $a_6 = -a_3/6 - a_2/30 = a_0/18, \ldots$. Thus one solution is $u_1(t) = a_0(1 - t^3/3 - t^4/12 + t^6/18 + \ldots)$ so

$$y_1(x) = u_1(x-1) = a_0[1 - (x-1)^3/3 - (x-1)^4/12 + (x-1)^6/18 + \ldots].$$

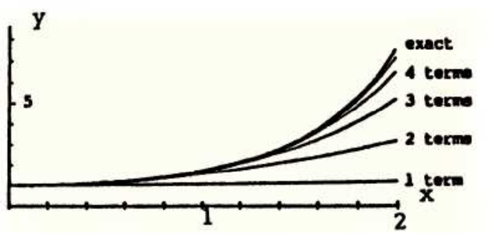
We obtain a second solution by choosing $a_0 = 0$. Then $a_4 = -a_1/4$, $a_5 = -a_2/5 - a_1/20 = -a_1/20$, $a_6 = -a_3/6 - a_2/30 = 0$, $a_7 = -a_4/7 - a_3/42 = a_1/28, \ldots$. Thus a second linearly independent solution is $u_2(t) = a_1[t - t^4/4 - t^5/20 + t^7/28 + \ldots]$ or

$$y_2(x) = u_2(x-1)$$
$$= a_1[(x-1) - (x-1)^4/4 - (x-1)^5/20 + (x-1)^7/28 + \ldots].$$

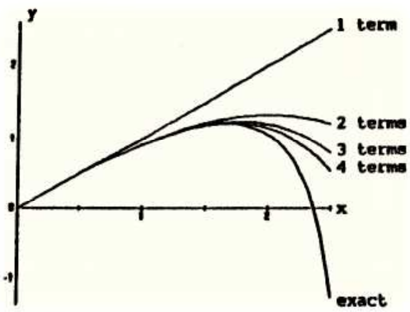
The Taylor series for $x^2 - 1$ about $x = 1$ may be obtained by writing $x = (x-1) + 1$ so $x^2 = (x-1)^2 + 2(x-1) + 1$ and $x^2 - 1 = (x-1)^2 + 2(x-1)$. The D.E. now appears as $y'' + (x-1)^2y' + [(x-1)^2 + 2(x-1)]y = 0$ which is identical to the transformed equation with $t = x - 1$.

22b. $y = a_0 + a_1x + a_2x^2 + \ldots$, $y^2 = a_0^2 + 2a_0a_1x + (2a_0a_2 + a_1^2)x^2 + \ldots$, $y' = a_1 + 2a_2x + 3a_3x^2 + \ldots$, and $(y')^2 = a_1^2 + 4a_1a_2x + (6a_1a_3 + 4a_2^{\ 2})x^2 + \ldots$. Substituting these into $(y')^2 = 1 - y^2$ and collecting coefficients of like powers of x yields $(a_1^2 + a_0^2 - 1) + (4a_1a_2 + 2a_0a_1)x + (6a_1a_3 + 4a_2^2 + 2a_0a_2 + a_1^2)x^2 + \ldots = 0$. As in the earlier problems, each coefficient must be zero. The I.C. $y(0) = 0$ requires that $a_0 = 0$, and thus $a_1^2 + a_0^2 - 1 = 0$ gives $a_1^2 = 1$. However, the D.E.indicates that y' is always positive, so $y'(0) = a_1 > 0$ implies $a_1 = 1$. Then $4a_1a_2 + 2a_0a_1 = 0$ implies that $a_2 = 0$; and $6a_1a_3 + 4a_2^2 + 2a_0a_2 + a_1^2 = 6a_1a_3 + a_1^2 = 0$ implies that $a_3 = -1/6$. Thus $y = x - x^3/3! + \ldots$.

23.



26.



26. We have $y(x) = a_0y_1 + a_1y_2$, where y_1 and y_2 are found in

Problem 10. Now $y(0) = a_0 = 0$ and $y'(0) = a_1 = 1$. Thus
$$y(x) = x - \frac{x^3}{12} - \frac{x^5}{240} - \frac{x^7}{2240}.$$

Section 5.3, Page 253

1. The D.E. can be solved for y'' to yield $y'' = -xy' - y$. If $y = \phi(x)$ is a solution, then $\phi''(x) = -x\phi'(x) - \phi(x)$ and thus setting $x = 0$ we obtain $\phi''(0) = -0 - 1 = -1$. Differentiating the equation for y'' yields $y''' = -xy'' - 2y'$ and hence setting $y = \phi(x)$ again yields $\phi'''(0) = -0 - 0 = 0$. In a similar fashion $y^{iv} = -xy''' - 3y''$ and thus $\phi^{iv}(0) = -0 - 3(-1) = 3$. The process can be continued to calculate higher derivatives of $\phi(x)$.

6. The zeros of $P(x) = x^2 - 2x - 3$ are $x = -1$ and $x = 3$. For $x_0 = 4$, $x_0 = -4$, and $x_0 = 0$ the distance to the nearest zero of $P(x)$ is 1,3, and 1, respectively. Thus a lower bound for the radius of convergence for series solutions in powers of $(x-4)$, $(x+4)$, and x is $\rho = 1$, $\rho = 3$, and $\rho = 1$, respectively.

9a. Since $P(x) = 1$ has no zeros, the radius of convergence for $x_0 = 0$ is $\rho = \infty$.

9f. Since $P(x) = x^2 + 2$ has zeros at $x = \pm\sqrt{2}\, i$, the lower bound for the radius of convergence of the series solution about $x_0 = 0$ is $\rho = \sqrt{2}$.

9h. Since $x_0 = 1$ and $P(x) = x$ has a zero at $x = 0$, $\rho = 1$.

10a. If we assume that $y = \sum_{n=2}^{\infty} a_n x^n$, then $y' = \sum_{n=2}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$. Substituting in the D.E., shifting indices of summation, and collecting coefficients of like powers of x yields the equation
$(2\cdot1\cdot a_2 + \alpha^2 a_0)x^0 + [3\cdot2\cdot a_3 + (\alpha^2-1)a_1]x^1$

$$+ \sum_{n=2}^{\infty}[(n+2)(n+1)a_{n+2} + (\alpha^2-n^2)a_n]x^n = 0.$$

Hence the recurrence relation is
$a_{n+2} = (n^2-\alpha^2)a_n/(n+2)(n+1)$, $n = 0, 1,2,\ldots$. For the first solution we choose $a_1 = 0$. We find that
$a_2 = -\alpha^2 a_0/2\cdot1$, $a_3 = 0$, $a_4 = (2^2-\alpha^2)a_2/4\cdot3 = -(2^2-\alpha^2)\alpha^2 a_0/4!$
$\ldots$, $a_{2m} = -[(2m-2)^2 - \alpha^2]\ldots(2^2-\alpha^2)\alpha^2 a_0/(2m)!$,
and $a_{2m+1} = 0$, so $y_1(x) = 1 - \dfrac{\alpha^2}{2!}x^2 - \dfrac{(2^2-\alpha^2)\alpha^2}{4!}x^4 - \ldots$
$-\dfrac{[(2m-2)^2-\alpha^2]\ldots(2^2-\alpha^2)\alpha^2}{(2m)!}x^{2m} - \ldots,$
where we have set $a_0 = 1$. For the second solution we take $a_0 = 0$ and $a_1 = 1$ in the recurrence relation to obtain the desired solution.

10b. If α is an even integer $2k$ then $(2m-2)^2 - \alpha^2 = (2m-2)^2 - 4k^2 = 0$. Thus when $m = k+1$ all terms in the series for $y_1(x)$ are zero after the x^{2k} term. A similar argument shows that if $\alpha = 2k+1$ then all terms in $y_2(x)$ are zero after the x^{2k+1}.

11. The Taylor series about $x = 0$ for $\sin x$ is
$\sin x = x - x^3/3! + x^5/5! - \ldots$. Assuming that
$y = \sum_{n=2}^{\infty} a_n x^n$ we find $y'' + (\sin x)y = 2a_2 + 6a_3x + 12a_4x^2$
$+ 20a_5x^3 + 30a_6x^4 + 42a_7x^5 + \ldots$
$+ (x-x^3/3!+x^5/5!-\ldots)(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots)$
$= 2a_2 + (6a_3+a_0)x + (12a_4+a_1)x^2 + (20a_5+a_2-a_0/6)x^3 +$
$(30a_6+ a_3-a_1/6)x^4 + (42a_7+a_4+a_0/5!)x^5 + \ldots = 0$. Hence
$a_2 = 0$, $a_3 = -a_0/6$, $a_4 = -a_1/12$, $a_5 = a_0/120$,
$a_6 = (a_1+a_0)/180$, $a_7 = -a_0/7! + a_1/504$, $\ldots$. We set $a_0 = 1$ and $a_1 = 0$ and obtain
$y_1(x) = (1 - x^3/6 + x^5/120 + x^6/180 + \ldots)$. Next we set $a_0 = 0$ and $a_1 = 1$ and obtain
$y_2(x) = (x - x^4/12 + x^6/180 + x^7/504 +\ldots)$. Since

p(x) = 1 and q(x) = sinx both have $\rho = \infty$, the solution in this case converges for all x, that is, $\rho = \infty$

18. We know that $e^x = 1 + x + x^2/2! + X^3/3! + \dots$, and therefore $e^{x^2} = 1 + x^2 + x^4/2! + x^6/3! + \dots$. Hence, if $y = \Sigma a_n x^n$, we have

$$a_1 + 2a_2 x + 3a_3 x^2 + \dots = (1+x^2 + x^4/2+\dots)(a_0+a_1x+a_2x^2+\dots)$$
$$= a_0 + a_1 x + (a_0+a_2)x^2 + \dots:$$

Thus, $a_1 = a_0$ $2a_2 = a_1$ and $3a_3 = a_0 + a_2$, which yield the desired solution.

20. Substituting $y = \sum_{n=2}^{\infty} a_n x^n$ into the D.E. we obtain $\sum_{n=2}^{\infty} n a_n x^{n-1} - \sum_{n=2}^{\infty} a_n x^n = x^2$. Shifting indices in the summation yields $\sum_{n=2}^{\infty} [(n+1)a_{n+1} - a_n]x^n = x^2$. Equating coefficients of both sides then gives: $a_1 - a_0 = 0$, $2a_2 - a_1 = 0$, $3a_3 - a_2 = 1$ and $(n+1)a_{n+1} = a_n$ for n = 3,4,... . Thus $a_1 = a_0$, $a_2 = a_1/2 = a_0/2$, $a_3 = 1/3 + a_2/3 = 1/3 + a_0/2\cdot3$, $a_4 = a_3/4 = 1/3\cdot4 + a_0/2\cdot3\cdot4$, ..., $a_n = a_{n-1}/n = 2/n! + a_0/n!$ and hence

$$y(x) = a_0(1 + x + \frac{x^2}{2!}+\dots+\frac{x^n}{n!}\dots) + 2(\frac{x^3}{3!}+\frac{x^4}{4!}+\dots+\frac{x^n}{n!}+\dots).$$

Using the power series for e^x, the first and second sums can be rewritten as $a_0 e^x + 2(e^x - 1 - x - x^2/2)$.

22. Substituting $y = \sum_{n=2}^{\infty} a_n x^n$ into the Legendre equation, shifting indices, and collecting coefficients of like powers of x yields

$$[2\cdot1\cdot a_2 + \alpha(\alpha+1)a_0]x^0 + \{3\cdot2\cdot a_3 - [2\cdot1 - \alpha(\alpha+1)]a_1\}x^1 +$$
$$\sum_{n=2}^{\infty}\{(n+2)(n+1)a_{n+2} - [n(n+1) - \alpha(\alpha+1)]a_n\}x^n = 0.$$

Thus $a_2 = -\alpha(\alpha+1)a_0/2!$, $a_3 = [2\cdot1 - \alpha(\alpha+1)]a_1/3! = -(\alpha-1)(\alpha+2)a_1/3!$ and the recurrence relation is

$(n+2)(n+1)a_{n+2} = -[\alpha(\alpha+1) - n(n+1)]a_n = -(\alpha-n)(\alpha+n+1)a_n$, $n = 2,3,\ldots$. Setting $a_1 = 0$, $a_0 = 1$ yields a solution with $a_3 = a_5 = a_7 = \ldots = 0$ and

$a_4 = \alpha(\alpha-2)(\alpha+1)(\alpha+3)/4!,\ldots,\ a_{2m} = (-1)^m\alpha(\alpha-2)(\alpha-4)\ \ldots$ $(\alpha-2m+2)(\alpha+1)(\alpha+3)\ \ldots\ (\alpha+2m-1)/(2m)!,\ldots$. The second linearly independent solution is obtained by setting $a_0 = 0$ and $a_1 = 1$. The coefficients are $a_2 = a_4 = a_6 = \ldots = 0$ and $a_3 = -(\alpha-1)(\alpha+2)/3!$, $a_5 = -(\alpha-3)(\alpha+4)a_3/5\cdot 4 = (\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)/5!,\ldots$.

26. Using the chain rule we have:

$$\frac{dF(\phi)}{d\phi} = \frac{dF[\phi(x)]}{dx}\frac{dx}{d\phi} = -f'(x)\sin\phi(x) = -f'(x)\sqrt{1-x^2},$$

$$\frac{d^2F(\phi)}{d\phi^2} = \frac{d}{dx}[-f'(x)\sqrt{1-x^2}]\frac{dx}{d\phi} = (1-x^2)f''(x) - xf'(x),$$

which when substituted into the D.E. yields the desired result.

28. Since $[(1-x^2)y']' = (1-x^2)y'' - 2xy'$, the Legendre Equation, from Problem 22, can be written as shown. Thus, carrying out the steps indicated yields the two equations:

$$P_m[(1-x^2)P_n']' = -n(n+1)P_nP_m$$

$$P_n[(1-x^2)P_m']' = -m(m+1)P_nP_m.$$

As long as $n \neq m$ the second equation can be subtracted from the first and the result integrated from -1 to 1 to obtain

$$\int_{-1}^{1}\{P_m[(1-x^2)P_n']'-P_n[(1-x^2)P_m']'\}dx = [m(m+1)-n(n+1)]\int_{-1}^{1}P_nP_m dx$$

The left side may be integrated by parts to yield

$$[P_m(1-x^2)P_n' - P_n(1-x^2)P_m']_{-1}^{1} + \int_{-1}^{1}[P_m'(1-x^2)P_n' - P_n'(1-x^2)P_m']dx,$$

which is zero. Thus $\int_{-1}^{1}P_n(x)P_m(x)dx = 0$ for $n \neq m$.

Section 5.4, Page 259

1. Since the coefficients of y, y' and y'' have no common factors and since $P(x)$ vanishes only at $x = 0$ we conclude that $x = 0$ is a singular point. Writing the D.E. in the form $y'' + p(x)y' + q(x)y = 0$, we obtain $p(x) = (1-x)/x$ and $q(x) = 1$. Thus for the singular point we have