

משוואות לינאריות (המשך)

משוואת אוילר מסדר 2  $(a_1, a_0 \in \mathbf{R}) \quad x^2 y'' + a_1 x y' + a_0 y = 0$

דרך 1 אם אם  $x > 0$  ההצבה  $y = x^\lambda$  (אם  $x < 0$  אז ההצבה  $y = (-x)^\lambda$ ) במשוואת אוילר נותנת

$$\lambda^2 + (a_1 - 1)\lambda + a_0 = 0$$

$\lambda_2, \lambda_1$  פתרונות של משוואה זאת. אזי

1)  $\lambda_1, \lambda_2 \in \mathbf{R}, \lambda_1 \neq \lambda_2 \Rightarrow y = C_1 |x|^{\lambda_1} + C_2 |x|^{\lambda_2}$

2)  $\lambda_1, \lambda_2 \in \mathbf{R}, \lambda_1 = \lambda_2 = \lambda \Rightarrow y = |x|^\lambda (C_1 + C_2 \ln x)$

3)  $\alpha, \beta \in \mathbf{R}, \lambda_{1,2} = \alpha \pm \beta i \Rightarrow y = |x|^\alpha (C_1 \cos(\beta \ln |x|) + C_2 \sin(\beta \ln |x|))$

$(a_1, a_0, a, b \in \mathbf{R}) \quad \begin{cases} (ax+b)^2 y'' + a_1(ax+b)y' + a_0 y = 0 \\ y = (ax+b)^\lambda \end{cases} \Rightarrow \lambda^2 + (a_1 - 1)\lambda + a_0 = 0$

דרך 2 ההצבה  $x = e^t$

$$x = e^t \Rightarrow t = \ln x \Rightarrow y = y(t(x)) \Rightarrow y'_x = y'_t \cdot t'_x \Rightarrow y'_x = y'_t \cdot \frac{1}{x}, \quad y''_{xx} = y''_{tt} \cdot \frac{1}{x^2} - y'_t \cdot \frac{1}{x^2}$$

$$\{x^2 y'' + a_1 x y' + a_0 y = 0, \quad y'_x = y'_t / x, \quad y''_{xx} = (y''_{tt} - y'_t) / x^2\} \Rightarrow y''_{tt} + (a_1 - 1) y'_t + a_0 y = 0$$

משוואת אוילר מסדר n

$$(ax+b)^n y^{(n)} + a_{n-1} (ax+b)^{n-1} y^{(n-1)} + \dots + a_1 (ax+b) y' + a_0 y = 0$$

$$a_{n-1}, a_{n-2}, \dots, a_0 \in \mathbf{R},$$

ההצבה  $y = (ax+b)^\lambda$   
דוגמאות

1.  $2(x+5)^2 y'' + 3(x+5) y' - y = 0, \quad \left. \begin{matrix} 2(x+5)^2 y'' + 3(x+5) y' - y = 0 \\ y = (x+5)^\lambda \end{matrix} \right\} \Rightarrow$

$$2\lambda(\lambda-1) + 3\lambda - 1 = 0 \Rightarrow 2\lambda^2 + \lambda - 1 = 0 \Rightarrow \lambda_1 = \frac{1}{2}, \lambda_2 = -1 \Rightarrow y = C_1 |x+5|^{0.5} + C_2 (x+5)^{-1}$$

2.  $x^3 y''' + x y' - y = 0$

$$\left. \begin{matrix} x^3 y''' + x y' - y = 0 \\ y = x^\lambda \end{matrix} \right\} \Rightarrow \lambda(\lambda-1)(\lambda-2) + \lambda - 1 = 0 \Rightarrow (\lambda-1)(\lambda^2 - 2\lambda + 1) = 0 \Rightarrow$$

$$\lambda_1 = \lambda_2 = \lambda_3 = 1 \Rightarrow y = C_1 x + C_2 x \ln |x| + C_3 x \ln^2 |x|$$

3.  $x^2 y'' + 3x y' + 5y = 0$

$$\left. \begin{matrix} x^2 y'' + 3x y' + 5y = 0 \\ y = x^\lambda \end{matrix} \right\} \Rightarrow \lambda^2 + 2\lambda + 5 = 0 \Rightarrow \lambda_{1,2} = -1 \pm 2i \Rightarrow y = \frac{C_1 \cos(2 \ln |x|) + C_2 \sin(2 \ln |x|)}{x}$$

4.  $x^3 y''' - 3x^2 y'' + 6x y' - 6y = x^{3.5}$   $y = y_h + y_p$

a) 
$$\left. \begin{aligned} x^3 y''' - 3x^2 y'' + 6x y' - 6y = 0 \\ y = x^\lambda \end{aligned} \right\} \Rightarrow \lambda(\lambda-1)(\lambda-2) - 3\lambda(\lambda-1) + 6\lambda - 6 = 0 \Rightarrow$$

$(\lambda-1)[\lambda(\lambda-2) - 3\lambda + 6] = 0 \Rightarrow (\lambda-1)(\lambda^2 - 5\lambda + 6) = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3 \Rightarrow$

$y_1 = x, y_2 = x^2, y_3 = x^2$

b)  $y_p = M(x)x + N(x)x^2 + K(x)x^3$

$$\begin{cases} M'x + N'x^2 + K'x^3 = 0 \\ M' + 2N'x + 3K'x^2 = 0 \Rightarrow M' = 0.5x^{1.5}, N' = -x^{0.5}, K' = 0.5x^{-0.5} \Rightarrow \\ 2N' + 6K'x = x^{0.5} \end{cases}$$

$M = 0.2x^{2.5}, N = -\frac{2}{3}x^{1.5}, K = x^{0.5} \Rightarrow y_p = \frac{1}{5}x^{2.5}x - \frac{2}{3}x^{1.5}x^2 + x^{0.5}x^3 = \frac{8}{15}x^{3.5}$

$y = C_1x + C_2x^2 + C_3x^3 + \frac{8}{15}x^{3.5}$

תרגילים

I. פתור את המשוואות

1)  $x^2 y'' - 4x y' + 6y = 0,$

2)  $x^2 y''' = 2y',$

3)  $x^2 y'' - 3x y' + 5y = 0,$

4)  $x^3 y'''' - 6x y'' = 0,$

5)  $x^4 y'''' + 6x^3 y''' + 6x^2 y'' - 2y = 0,$

6)  $(x-2)^2 y'' - 3(x-2) y' + 4y = 0,$

7)  $(2x+3)^2 y'' + 2(2x+3) y' - 4y = 0,$

8)  $x^2 y'' + x y' + 4y = 10x,$

9)  $x^3 y'' - 2xy = 6 \ln x$

18)  $x^3 y''' - 3x^2 y'' + 6x y' - 6y = \frac{x^4}{\sqrt{x+1}}$

10)  $x^2 y'' - 6y = 5x^3 + 8x^2,$

11)  $x^2 y'' - x y' + y = 2x,$

12)  $x^2 y'' - 2x y' + 2y + x - 2x^3 = 0,$

13)  $x^2 y'' - x y' - 3y = 0$

14)  $x^3 y'' + x y' - y = 0$

15)  $x^2 y'' - x y' + y = \frac{\ln x}{x} + \frac{x}{\ln x}$

16)  $(x-2)^2 y'' - 3(x-2) y' + 4y = x,$

17)  $x^2 y'' - 2y = \sin(\ln x)$

## חזרה II

פתור משוואות :

- |   |   |  |
|---|---|--|
| 1. $xy' - y = xe^{y/x}$   | 2. $t s' = s + t^2 \sin^2 t$                            | 3. $(1 - x^2)t' - xt = xt^2, t(0) = 0.5$ |
| 4. $2x \cos^2 y dx + (8\sqrt[3]{y} - x^2 \sin 2y) dy = 0, y(1) = 0$ | 5. $\cos y dx + (\sin y + e^x) dy = 0$                  | 7. $(x + y + 1) dx = (2x + 2y - 1) dy$   |
| 6. $2xy dx - (3y + x^2) dy = 0$                                     | 9. $r' - r \tan \varphi + r^2 \cos \varphi = 0$         | 12. $y'' \tan y = 2(y')^2$               |
| 8. $(x + ye^{-1/y}) y' = y^2$                                       | 10. $y''' = x^{-3}$                                     | 11. $xy'' = y'(\ln y' - \ln x)$          |
| 13. $2y(y')^3 + y'' = 0, y(0) = 0, y'(0) = -3$                      | 14. $x^2 y'' - xy' + y = 0$                             | 17. $y''' + 9y' = 0$                     |
| 15. $y'' - 16y = 0$   | 16. $y'' + 16y = 0$                                     | 19. $y^{IV} - 4y''' + 13y'' = 0$         |
| 18. $y^V + 16y' = 0$  | 20. $y'' + 4y' + 4y = 3xe^{-2x}$                        | 21. $y'' - 4y' = 4x - 5 + 10e^x \cos x$  |
| 22. $x'' + x = \cos t + \cos 2t$                                    | 23. $y'' - 2y' + 10y = 74 \sin 3x, y(0) = 6, y'(0) = 3$ | 24. $y'' + y = \cos^{-3} x$              |
| 25. $x^2 y'' - 3x y' + 4y = \ln x$                                  | 26. $x^2 y'' + x y' + 4y = \sin(\ln x)$                 |  |
| 27. $y'' + \frac{\mu}{x^2} y = 0$                                   | a) $\mu > 0.25$ b) $\mu = 0.25$ c) $\mu < 0.25$         |  |
| 28. $y''' + y' = \frac{\sin x}{\cos^2 x}$                           | 29. $x^3 y''' - x y' - 3y = x^2$                        |  |

תשובות :

I

- |   |   |
|---|---|
| 1) $y = C_1 x^2 + C_2 x^3$  | 11) $y = C_1 x + C_2 x \ln  x  + x \ln^2  x $                       |
| 2) $y = C_1 + C_2 \ln  x  + C_3 x^3$  | 12) $y = C_1 x + C_2 x^2 + x \ln  x  + x^3$                         |
| 3) $y = x^2 (C_1 \cos(\ln  x ) + C_2 \sin(\ln  x ))$  | 13) $y = C_1 x^{-1} + C_2 x^3$                                      |
| 4) $y = C_1 + C_2 \ln  x  + C_3 x + C_4 x^5$  | 14) $y = x(C_1 + C_2 \ln  x  + C_3 \ln^2  x )$                      |
| 5) $y = C_1  x ^{\sqrt{2}} + C_2  x ^{-\sqrt{2}} + C_3 \cos(\ln  x ) + C_4 \sin(\ln  x )$                                       |   |
| 6) $y = C_1 (x-2)^2 + C_2 (x-2)^2 \ln  x-2 $  |   |
| 7) $y = C_1 (2x+3) + C_2 (2x+3)^{-1}$   | 15) $y = x[C_1 + (C_2 + \ln  \ln x ) \ln x] + \frac{1 + \ln x}{4x}$ |
| 8) $y = C_1 \cos(2 \ln  x ) + C_2 \sin(2 \ln  x ) + 2x$   |   |
| 9) $y = C_1 x^2 + \frac{1}{x} \left( C_2 - \frac{2}{3} \ln x - \ln^2 x \right)$   | 16) $y = (x-2)^2 (C_1 + C_2 \ln  x-2 ) + x + 1.5$                   |
| 10) $y = C_1 x^{-2} + C_2 x^3 + x^3 \ln  x  - 2x^2$   | 17) $y = C_1 x^{-1} + C_2 x^2 + 0.1 \cos(\ln x) - 0.3 \sin(\ln x)$  |
| 18) $y = C_1 x + C_2 x^2 + C_3 x^3 + \frac{8}{15} x^{3.5} - \frac{5}{3} x^{2.5} + x^{1.5} - (x - 2x^2 + x^3) \ln(\sqrt{x} + 1)$ |   |

II תשובות (חזרה)

$$1. 1 + e^{y/x} \ln |Cx| = 0 \quad 2. s = \frac{t^2}{2} - \frac{t}{4} \sin 2t + Ct \quad 3. t = \frac{1}{3\sqrt{|1-x^2|} - 1}$$

$$4. x^2 \cos^2 y + 6y \sqrt[3]{y} - 1 = 0 \quad 5. y = C + e^{-x} \cos y \quad 6. x^2 = y(C + 3 \ln |y|), y = 0$$

$$7. C + 2y = x + \ln |x + y|, y = -x \quad 8. C + \ln |y| = xe^{1/y}$$

$$9. r = \frac{1}{(\varphi + C) \cos \varphi}, r = 0 \quad 10. y = 0.5 \ln |x| + C_1 x^2 + C_2 x + C_3$$

$$11. y = \frac{1}{C_1} e^{C_1 x + 1} \left( x - \frac{1}{C_1} \right) + C_2, y = \frac{e}{2} x^2 + C$$

$$12. C_1 x + C_2 = -\cot y, \begin{cases} y = C \\ C \neq 0.5\pi + k\pi, k \in Z \end{cases} \quad 13. 3x = y^3 - y$$

$$14. y = x(C_1 + C_2 \ln |x|)$$

$$15. y = C_1 e^{4x} + C_2 e^{-4x} \quad 16. y = C_1 \cos 4x + C_2 \sin 4x$$

$$17. y = C_1 + C_2 \cos 3x + C_3 \sin 3x$$

$$18. y = C_1 + e^{\sqrt{2}x} (C_2 \cos(x\sqrt{2}) + C_3 \sin(x\sqrt{2})) + e^{-\sqrt{2}x} (C_4 \cos(x\sqrt{2}) + C_5 \sin(x\sqrt{2}))$$

$$19. y = C_1 + C_2 x + e^{2x} (C_3 \cos 3x + C_4 \sin 3x)$$

$$20. y = e^{-2x} (C_1 + C_2 x + 0.5x^3)$$

$$21. y = C_1 + C_2 e^{4x} - 0.5x^2 + x - e^x (2 \cos x + \sin x)$$

$$22. x = C_1 \cos t + C_2 \sin t + \frac{1}{2} t \sin t - \frac{1}{3} \cos 2t$$

$$23. y = e^x (\sin 3x - 6 \cos 3x) + 12 \cos 3x + 2 \sin 3x$$

$$24. y = C_1 \cos x + C_2 \sin x + \frac{1}{2 \cos x}$$

$$25. y = x^2 (C_1 + C_2 \ln x) + 0.25 \ln x + 0.25$$

$$26. y = C_1 \cos(2 \ln x) + C_2 \sin(2 \ln x) + \frac{1}{3} \sin(\ln x)$$

$$27. a) r_1 = 0.5 + \sqrt{0.25 - \mu} \quad r_2 = 0.5 - \sqrt{0.25 - \mu}, \quad y = C_1 |x|^{r_1} + C_2 |x|^{r_2}$$

$$b) r_{1,2} = 0.5, \quad y = |x|^{0.5} (C_1 + C_2 \ln |x|)$$

$$c) r_{1,2} = 0.5 \pm \sqrt{0.25 - \mu} = 0.5 \pm \tau i, \quad y = |x|^{0.5} (C_1 \cos(\tau \ln |x|) + C_2 \sin(\tau \ln |x|))$$

$$28. y_p = \frac{1}{\cos x} + \cos x \ln |\cos x| + \sin x (x - \tan x) = \cos x + \cos x \ln |\cos x| + x \sin x$$

$$y = \cos x (C_1 + \ln |\cos x|) + (C_2 + x) \sin x + C_3$$

$$29. y = C_1 x^3 + C_2 \cos(\ln |x|) + C_3 \sin(\ln |x|) - 0.2x^2$$