Investigate the character of the critical points using the sufficiency c tions. Find the second-order partial derivatives of the function u:

$$\frac{\partial^2 u}{\partial x^2} = -2y; \quad \frac{\partial^2 u}{\partial x\, \partial y} = a - 2x - 2y; \quad \frac{\partial^2 u}{\partial y^2} = -2x.$$

At the point $M_1(0, 0)$ we have $A = \frac{\partial^2 u}{\partial x^2} = 0$; $B = \frac{\partial^2 u}{\partial x\, \partial y} = a$, $C = \frac{\partial^2 u}{\partial y^2}$ $AC - B^2 = -a^2 < 0$. Hence, at the point M_1 there is neither a maximum a minimum. At the point $M_2(0, a)$ we have $A = \frac{\partial^2 u}{\partial x^2} = -2a$; $B = \frac{\partial^2 u}{\partial x\, \partial y} =$ $C = \frac{\partial^2 u}{\partial y^2} = 0$;

$$AC - B^2 = -a^2 < 0.$$

Which means that at the point M_2 there is neither a maximum nor a mi mum. At the point $M_3(a, 0)$ we have $A = 0$, $B = -a$, $C = -2a$:

$$AC - B^2 = -a^2 < 0.$$

At M_3 too there is neither a maximum nor a minimum. At the po $M_4\left(\frac{a}{3}, \frac{a}{3}\right)$ we have $A = -\frac{2a}{3}$; $B = -\frac{a}{3}$; $C = -\frac{2a}{3}$;

$$AC - B^2 = \frac{4a^2}{9} - \frac{a^2}{9} > 0; \; A < 0.$$

Hence, at M_4 we have a maximum.

SEC. 18. MAXIMUM AND MINIMUM OF A FUNCTION OF SEVERAL VARIABLES RELATED BY GIVEN EQUATIONS (CONDITIONAL MAXIMA AND MINIMA)

In many maximum and minimum problems, one has to find th extrema of a function of several variables that are not indepen dent, but are related to one another by side conditions (for example, they must satisfy given equations).

By way of illustration let us consider the following problem Using a piece of tin $2a$ in area it is required to build a close box in the form of a parallelepiped of maximum volume.

Denote the length, width and height of the box by x, y, and The problem reduces to finding the maximum of the function

$$v = xyz$$

provided that $2xy + 2xz + 2yz = 2a$. The problem here deals wi a **conditional extremum**: the variables x, y, z **are restricted by** the **condition** that $2xy + 2xz + 2yz = 2a$. In this section we shall con sider methods of solving such problems.

Let us first consider the question of the conditional extremum of a function of two variables if these variables are restricted by a **single** condition.

Let it be required to find the maxima and minima of the function

$$u = f(x, y) \tag{1}$$

with the proviso that x and y are connected by the equation

$$\varphi(x, y) = 0. \tag{2}$$

Given condition (2), of the two variables x and y there will be only **one** which **is independent** (for instance, x) since y is determined from (2) as a function of x. If we solved equation (2) for y and put into (1) the expression found in place of y, we would obtain a function of **one** variable, x, and would reduce the problem to one that would involve finding the maximum and minimum of a function of one independent variable, x.

But the problem may be solved without solving equation (2) for x or y. For those values of x at which the function u can have a maximum or minimum, the derivative of u with respect to x should vanish.

From (1) we find $\frac{du}{dx}$, remembering that y is a function of x:

$$\frac{du}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}.$$

Hence, at the points of the extremum

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0. \tag{3}$$

From equation (2) we find

$$\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{dy}{dx} = 0. \tag{4}$$

This equality is satisfied for all x and y that satisfy equation (2) (see Sec. 11, Ch. VIII).

Multiplying the terms of (4) by an (as yet) undetermined coefficient λ and adding them to the corresponding terms of (3), we have

$$\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}\right) + \lambda\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{dy}{dx}\right) = 0$$

or

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \varphi}{\partial x}\right) + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \varphi}{\partial y}\right)\frac{dy}{dx} = 0. \tag{5}$$

The latter equality is fulfilled at all extremum points. Choose λ such that for the values of x and y which correspond to the extremum of the function u, the second parentheses in (5) should vanish:

$$\frac{\partial f}{\partial y}+\lambda\frac{\partial \varphi}{\partial y}=0.$$

But then, for these values of x and y, from (5) we have

$$\frac{\partial f}{\partial x}+\lambda\frac{\partial \varphi}{\partial x}=0.$$

It thus turns out that at the extremum points three equatio[ns] (with three unknowns x, y, λ) are satisfied:

$$\left.\begin{aligned}\frac{\partial f}{\partial x}+\lambda\frac{\partial \varphi}{\partial x}&=0,\\ \frac{\partial f}{\partial y}+\lambda\frac{\partial \varphi}{\partial y}&=0,\\ \varphi(x,\ y)&=0.\end{aligned}\right\} \quad (6)$$

From these equations determine x, y, and λ; the latter only play[s] an auxiliary role and will not be needed any more.

From this conclusion it follows that equations (6) are necessa[ry] conditions of a conditional extremum; or equations (6) are satisfie[d] at the extremum points. But there will not be a conditional extr[e]mum for every x and y (and λ) that satisfy equations (6). A su[p]plementary investigation of the nature of the critical point is re[-]quired. In the solution of concrete problems it is sometimes po[s]sible to establish the character of the critical point from th[e] statement of the problem. It will be noted that the left-hand sid[es] of equations (6) are partial derivatives of the function

$$F(x,\ y,\ \lambda)=f(x,\ y)+\lambda\varphi(x,\ y) \quad (7)$$

with respect to the variables x, y and λ.

Thus, in order to find the values of x and y which satisfy con[-]dition (2), for which the function $u=f(x,\ y)$ can have a condi[-]tional maximum or a conditional minimum, one has to constr[uct] an auxiliary function (7), equate to zero its derivatives with re[-]spect to x, y, and λ, and from the three equations (6) th[us] obtained determine the sought-for x, y (and the auxiliary factor). The foregoing method can be extended to a study of the conditio[n]al extremum of a function of any number of variables.

Let it be required to find the maxima and minima of a functio[n] of n variables, $u=f(x_1,\ x_2\ldots,\ x_n)$ provided that the variabl[es]

*) For the sake of definiteness, we shall assume that at the critical poi[nt]

$$\frac{\partial \varphi}{\partial y}\neq 0.$$

$x_1, x_2, \ldots, x_n$ are connected by $m\,(m<n)$ equations:

$$\left.\begin{aligned} \varphi_1(x_1, x_2, \ldots, x_n) &= 0, \\ \varphi_2(x_1, x_2, \ldots, x_n) &= 0, \\ \cdots\cdots\cdots&\cdots \\ \varphi_m(x_1, x_2, \ldots, x_n) &= 0. \end{aligned}\right\} \tag{8}$$

In order to find the values of $x_1, x_2, \ldots, x_n$, for which there may be conditional maxima and minima, one has to form the function

$$F(x_1, x_2, \ldots, x_n, \lambda_1, \ldots, \lambda_m) = f(x_1, \ldots, x_n) + \lambda_1\varphi_1(x_1, \ldots, x_n) + \\ + \lambda_2\varphi_2(x_1, \ldots, x_n) + \ldots + \lambda_m\varphi_m(x_1, \ldots, x_n),$$

equate to zero its partial derivatives with respect to $x_1, x_2, \ldots, x_n$:

$$\left.\begin{aligned} \frac{\partial f}{\partial x_1} + \lambda_1\frac{\partial \varphi_1}{\partial x_1} + \ldots + \lambda_m\frac{\partial \varphi_m}{\partial x_1} &= 0, \\ \frac{\partial f}{\partial x_2} + \lambda_1\frac{\partial \varphi_1}{\partial x_2} + \ldots + \lambda_m\frac{\partial \varphi_m}{\partial x_2} &= 0, \\ \cdots\cdots\cdots\cdots&\cdots \\ \frac{\partial f}{\partial x_n} + \lambda_1\frac{\partial \varphi_1}{\partial x_n} + \ldots + \lambda_m\frac{\partial \varphi_m}{\partial x_n} &= 0 \end{aligned}\right\} \tag{9}$$

and from the $m+n$ equations (8) and (9) determine $x_1, x_2, \ldots, x_n$ and the auxiliary unknowns $\lambda_1, \ldots, \lambda_m$. Just as in the case of a function of two variables, we shall, in the general case, leave undecided the question of whether the function, for the values found, will have a maximum or minimum or will have neither. We will decide this matter on the basis of additional reasoning.

Example 1. Let us return to the problem formulated at the beginning of this section: to find the maximum of the function

$$v = xyz$$

provided that

$$xy + xz + yz - a = 0 \quad (x>0,\ y>0,\ z>0). \tag{10}$$

We form the auxiliary function

$$F(x y z \lambda) = F(x, y, \lambda) = xyz + \lambda(xy + xz + yz - a).$$

Find its partial derivatives and equate them to zero:

$$\left.\begin{aligned} yz + \lambda(y+z) &= 0, \\ xz + \lambda(x+z) &= 0, \\ xy + \lambda(x+y) &= 0. \end{aligned}\right\} \tag{11}$$

The problem reduces to solving a system of four equations (10) and (11) in four unknowns (x, y, z and λ). To solve this system, multiply the first of equations (11) by x, the second by y, the third by z, and add; taking (… into account we find that $\lambda=-\frac{3xyz}{2a}$. Putting this value of λ into equatio… (11) we get

$$yz\left[1-\frac{3x}{2a}(y+z)\right]=0,$$

$$xz\left[1-\frac{3y}{2a}(x+z)\right]=0,$$

$$xy\left[1-\frac{3z}{2a}(x+y)\right]=0.$$

Since it is evident from the statement of the problem that x, y, z are dif… rent from zero, we get from the latter equations

$$\frac{3x}{2a}(y+z)=1,\quad \frac{3y}{2a}(x+z)=1,\quad \frac{3z}{2a}(x+y)=1.$$

From the first two equations we find $x=y$, from the second and third equatio… $y=z$. But then from equation (10) we get $x=y=z=\sqrt{\frac{a}{3}}$. This is … only system of values of x, y, and z, for which there can be a maximum… minimum.

It can be proved that the solution obtained yields a maximum. Incidental… this is also evident from geometrical reasoning (the statement of the probl… indicates that the volume of the box cannot be big without bound; i… therefore natural to expect that for some definite values of the sides… volume will be a maximum).

Thus, for the volume of the box to be a maximum, the box must be… cube, an edge of which is equal to $\sqrt{\frac{a}{3}}$.

Example 2. Determine the maximum value of the nth root of a prod… of numbers $x_1, x_2, \ldots, x_n$ provided that their sum is equal to a given nu… ber a. Thus, the problem is stated as follows: it is required to find the ma… imum of the function $u=\sqrt[n]{x_1\ldots x_n}$ on the condition that

$$x_1+x_2+\ldots+x_n-a=0$$

$$(x_1>0,\ x_2>0,\ \ldots,\ x_n>0).$$

Form an auxiliary function:

$$F(x_1,\ \ldots,\ x_n,\ \lambda)=\sqrt[n]{x_1\ \ldots\ x_n}+\lambda(x_1+x_2+\ldots+x_n-a).$$

Find its partial derivatives:

$$F'_{x_1}=\frac{1}{n}\,\frac{x_2x_3\ldots x_n}{(x_1\ \ldots\ x_n)^{\frac{n-1}{n}}}+\lambda=\frac{1}{n}\,\frac{u}{x_1}+\lambda=0 \quad\text{or}\quad u=-n\lambda x_1,$$

$$F'_{x_2}=\frac{1}{n}\,\frac{u}{x_2}+\lambda=0 \quad\text{or}\quad u=-n\lambda x_2,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$F'_{x_n}=\frac{1}{n}\,\frac{u}{x_n}+\lambda=0 \quad\text{or}\quad u=-n\lambda x_n.$$

y y, the third by z, and add; taking
$\frac{3xyz}{2a}$. Putting this value of λ into equ

$$-\frac{3x}{2a}(y+z)\Big]=0,$$

$$-\frac{3y}{2a}(x+z)\Big]=0,$$

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ement of the problem that x, y, z are
latter equations

$$\frac{y}{a}(x+z)=1,\quad \frac{3z}{2a}(x+y)=1.$$

nd $x=y$, from the second and third eq
(10) we get $x=y=z=\sqrt{\frac{a}{3}}$. Thi
nd z, for which there can be a maxi
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ted as follows: it is required to find th
$\ldots x_n$ on the condition that

$$+\ldots+x_n-a=0$$

$$,\ x_2>0,\ \ldots,\ x_n>0).$$

$$\overline{x_1\ \ldots\ x_n}+\lambda(x_1+x_2+\ldots+x_n-a).$$

$$\frac{n}{-1}+\lambda=\frac{1}{n}\frac{u}{x_1}+\lambda=0 \text{ or } u=-n\lambda x_1,$$

$$\text{or } u=-n\lambda x_2,$$

. . . .

$$\text{or } u=-n\lambda x_n.$$

m the foregoing equations we find

$$x_1=x_2=\ldots=x_n,$$

from equation (12) we have

$$x_1=x_2=\ldots=x_n=\frac{a}{n}.$$

By the meaning of the problem these values yield a maximum of the ction $\sqrt[n]{x_1\ \ldots\ x_n}$ equal to $\frac{a}{n}$.

Thus, for any positive numbers x_1, x_2, $\ldots$, x_n connected by the relation- p $x_1+x_2+\ldots+x_n=a$, the inequality

$$\sqrt[n]{x_1\ \ldots\ x_n}\leqslant\frac{a}{n} \tag{13}$$

ulfilled (since it has already been proved that $\frac{a}{n}$ is the maximum of this ction). Now substituting into (13) the value of a obtained from (12), we get

$$\sqrt[n]{x_1x_2\ \ldots\ x_n}\leqslant\frac{x_1+\ldots+x_n}{n}. \tag{14}$$

is inequality holds for all positive numbers x_1, x_2, $\ldots$, x_n. The expression the left-hand side of (14) is called the *geometric mean* of these numbers. us, the geometric mean of several positive numbers is not greater than r arithmetic mean.

SEC. 19. SINGULAR POINTS OF A CURVE

The concept of a partial derivative is used in investigating ves.

Let a curve be given by the equation

$$F(x,\ y)=0.$$

The slope of the tangent to the curve is determined from the mula

$$\frac{dy}{dx}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

e Sec. 11, Ch. VIII).

If at a given point $M(x,\ y)$ of the curve under consideration, least one of the partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ does not vanish, n at this point either $\frac{dy}{dx}$ or $\frac{dx}{dy}$ is completely determined. The ve $F(x,\ y)=0$ has a very definite line tangent at this point. this case, the point $M(x,\ y)$ is called an *ordinary* point.