Intorduction to Topology

Assignment no. 5- Solution

1. Our definition of a basic open set in $\prod_{\alpha\in I} X_\alpha$ is $U = \prod_{\alpha\in I} U_\alpha$ with U_α open, and a finite support $F \subseteq I$ such that $\forall \alpha_i \in F : U_{\alpha_i} \subsetneq X_{\alpha_i}$. eqivalently: $U = \{\eta \in \prod_{\alpha\in I} X_\alpha : \forall \alpha_i \in F \quad \eta(\alpha_i) \in U_{\alpha_i}\}$. U's complement is defined: $D = \prod_{\alpha\in I} X_\alpha \setminus U = \{\eta \in \prod_{\alpha\in I} X_\alpha : \exists \alpha_i \in F \; s.t. \; \eta(\alpha_i) \in X_{\alpha_i} \setminus U_{\alpha_i} = D_{\alpha_i}\}$. The proof that the set $\mathcal{D}$, of closed sets defined as such, is a closed base, is by definitions, and is left to the reader. Denote this topology as $\mathcal{T}$.

   Suppose $\mathcal{T}'$ is a topology on $\prod_{\alpha\in I} X_\alpha$ by which all projections are continuous. Let $\alpha \in I$ be arbitrary, and let $D_\alpha \subseteq X_\alpha$ be closed. $P_\alpha^{-1}(D) = \{\eta \in \prod_{\alpha\in I} X_\alpha : P_\alpha(\eta) \in D_\alpha\} = \{\eta \in \prod_{\alpha\in I} X_\alpha : \eta(\alpha) \in D_\alpha\}$ which is closed by $\mathcal{T}$, thus $\mathcal{T} \subseteq \mathcal{T}'$. Thus $\mathcal{T}$ is minimal.

2. In proving that any projection P_α is open it is nessacery to show that for any open set $U \subseteq \prod_{\alpha\in I} X_\alpha$ there exist $\{U_j\}_{j\in J}$ basic sets, such that $\bigcup_{j\in J} U_j = U$, and to apply a property of the projection function by which $P_\alpha(\bigcup_{j\in J} U_j) = \bigcup_{j\in J} P_\alpha(U_j)$. In the case of closed set, any closed set is an intersection of closed basic sets, and this property does not hold for intersection.

   As a counter-example, let $X_1 = X_2 = \{1,2\}$ with the discrete topology, and $X = X_1 \times X_2$, let $U_1 = \{1\} \times \{1,2\}$ and $U_1 = \{2\} \times \{1,2\}$. $U_1 \cap U_2 = \emptyset$, and so $P_2(U_1 \cap U_2) = P_2(\emptyset) = \emptyset$. On the other hand $P_2(U_2) = P_2(U_1) = \{1,2\}$ and $\bigcap_{i=1}^{2} P_2(U_i) = \{1,2\}$.

3. (a) Connectedness is preserved for any product space $\prod_{\alpha\in I} X_\alpha$. Let $\{X_\alpha\}_{\alpha\in I}$ be connected (with I finite or infinite), and assume by contradiction that $\prod_{\alpha\in I} X_\alpha$ is not connected. Let $\emptyset \neq U_1, U_2 \subseteq \prod_{\alpha\in I} X_\alpha$ be open such that $U_1 \sqcup U_2 = \prod_{\alpha\in I} X_\alpha$, and $\gamma \in I$ arbitrary. Let $P_\gamma : \prod_{\alpha\in I} X_\alpha \to X_\gamma$ be the projection on X_γ. P_γ is open, thus $V_1 = P_\gamma(U_1)$ and $V_2 = P_\gamma(U_2)$ are non-empty, disjoint open sets, such that $V_1 \sqcup V_2 = X_\gamma$ in contradiction.

   (b) Let $\{X_\alpha\}_{\alpha\in I}$ be discrete. If I is finite, then for any $(x_1, \ldots, x_n) \in \prod_{\alpha\in I} X_\alpha$: $\{(x_1, \ldots, x_n)\} = \prod_{\alpha\in I} \{x_\alpha\}$ which is an open basic set by definition.

   For $|I| \geq \aleph_0$ it is not the case that $\prod_{\alpha\in I} X_\alpha$ is always discrete. Suppose $X_\alpha = \{0,1\}$, with the discrete topology, for any $\alpha \in I$, and $I = \mathbb{N}$. Suppose $\prod_{\alpha\in I} X_\alpha$ is discrete, thus $= \{\bar{0}\} = \{0\}^{\mathbb{N}}$ in particular, is open. Hence, there exists a basic set $\bar{0} \in U \subseteq \{\bar{0}\}$, but this is a condratiction, since for any basic set $U \in \{0,1\}^{\mathbb{N}}$ there is finite support $F \subseteq \mathbb{N}$ such that $\forall i \notin F : P_i(U) = \{0,1\} \nsubseteq \{0\}$

   (c) Any product of Hausdorff spaces is Hausdorff. Let $\{X_\alpha\}_{\alpha\in I}$ be Hausdorff, and let $x \neq y \in \prod_{\alpha\in I} X_\alpha$ be arbitrary. $\exists \beta \in I$ such that $x(\beta) \neq y(\beta)$, thus there exist $U_x, U_y \subseteq X_\beta$ open neighbourhoods of x and y, respectively, such that $U_x \cap U_y = \emptyset$. P_β is continuous, so

$V_x = P_\beta^{-1}(U_x)$, $V_y = P_\beta^{-1}(U_y)$ are open such that $V_x \cap V_y = \emptyset$ and so- $\prod_{\alpha\in I} X_\alpha$ is Hausdorff.

(d) A finite product of metric spaces is metrizable: we shall prove a stronger case in which $|I| = \aleph_0$(the finite case can be derived from this by denoting $X_i = \{1\}$ for any $i > n$ for some $n \in \mathbb{N}$). WLOG, assume $I = \mathbb{N}$ and let $\{(X_i, d_i)\}_{i\in\mathbb{N}}$ be metric spaces. By a claim, proved in class, it can be assumed that $Im(d_i) \subseteq [0,1]$. Define $d : (\prod_{i\in\mathbb{N}} X_i)^2 \to \mathbb{R}$ by:

$$(x,y) \overset{d}{\mapsto} \sum_{i=1}^{\infty} \frac{1}{2^i} d_i(x(i), y(i))$$

Proving that d is a metric is left to the reader. Also note that $d(x,y)$ is deinfed for any $x, y \in \prod_{i\in\mathbb{N}} X_i$. We shall now show that the topology induced by d is the product topology. It suffices to show two-directional refinement on basic sets.

- Let $U \subseteq \prod_{i\in\mathbb{N}} X_i$ be open in product topology, and $(x_1, \ldots, x_n, \ldots) = x \in U$. Let $i \in \mathbb{N}$ be such that $\forall j > i :\ P_j(U) = X_j$. Denote $\epsilon = \min_{j\le i}\{\sup\{\delta :\ B_{d_j}(x_j, \delta) \subseteq P_j(U)\}\}$. Thus for every $j \in \mathbb{N} :\ P_j(B_d(x, \frac{\epsilon}{2^{i+1}})) \subseteq U_j$, and thus $x \in B_d(x, \frac{\epsilon}{2^{i+1}}) \subseteq U$. By proving that the set of open ball covers $\prod_{i\in\mathbb{N}} X_i$ (left to the reader) it can be easily shown that the metric topology is a base for product topology.
- Let $x \in \prod_{i\in\mathbb{N}} X_i$ and $\epsilon > 0$ be arbitrary. Let i_0 be such that $\frac{1}{2^{i_0-1}} \sup\{d_j(a,b) : a, b \in X_j\} < \epsilon$. For $j \le i_0$, define $B_{d_j}(x_j, \frac{\epsilon}{2}) = U_j \subseteq X_j$, then $\sum_{j=1}^{i_0} \frac{d(x_j, y_j)}{2^j} < \frac{\epsilon}{2}$ by selection of i_0, it holds that $U_1 \times \ldots \times U_{i_0} \times \prod_{i_0+1\le j\in\mathbb{N}} X_j \subseteq B(x, \epsilon)$, and so the product toplogy refines the metric topology.

For $|I| > \aleph_0$ we've seen in a previous assignment that $\mathbb{R}^{\mathbb{R}}$ is not metrizable, although $\mathbb{R}$ is metric.

(e) Path-connectedness is preserved for any product of path-connected spaces. Let $\{X_\alpha\}_{\alpha\in I}$ be path-connected spaces, and let $x, y \in \prod_{\alpha\in I} X_\alpha$ be arbitrary. for any $\alpha \in I$ there exists a path $\varphi_\alpha : [0,1] \to X_\alpha$ such that $\varphi_\alpha(0) = x(\alpha)$, $\varphi_\alpha(1) = y(\alpha)$. Denote $\psi : [0,1] \to \prod_{\alpha\in I} X_{\alpha\in I}$ by $\psi(t)(\alpha) = \varphi_\alpha(t)$ for any $t \in [0,1]\,,\ \alpha \in I$. Clearly $\psi(0) = x$ and $\psi(1) = y$. It is left to show that ψ is contiuous. Let $U \subseteq \prod_{\alpha\in I} X_\alpha$ be open, with a finite support $F \subseteq I$.

$$\begin{aligned}
\psi^{-1}(U) &= \{t \in [0,1] :\ \psi(t) \in U\} \\
&= \{t \in [0,1] :\ \forall\alpha \in I \psi(t)(\alpha) \in U_\alpha\} \\
&= \{t \in [0,1] :\ \forall\alpha \in F \varphi_\alpha(t) \in U_\alpha\} \\
&= \bigcap_{\alpha\in F} \{t \in [0,1] :\ \varphi_\alpha(t) \in U_\alpha\} = \bigcap_{\alpha\in F} \varphi_\alpha^{-1}(U_\alpha)
\end{aligned}$$

which is a finite intesection of open sets.

(f) The existence of a countable base is preserved for a finite product. Let $X_1, \ldots, X_n$ be spaces and let $B_1, \ldots, B_n$ be their respective countable bases. Denote $\mathcal{B} := \{\prod_{i=1}^{n} U_i : U_i \in B_i\}$. The proof that $\mathcal{B}$ is a base was given in class. Also $|\mathcal{B}| \leq \underbrace{\aleph_0 \times \ldots \times \aleph_0}_{n} = \aleph_0$.

For an infinite product it is not the case. Let $X_\alpha = \{0, 1\}$ with the discrete topology, for any $\alpha \in I$, and $I = \mathbb{R}$. Let $\mathcal{B}$ be a base for $\{0,1\}^I$, by product topology. By construction of the product topology, the set $A = \{\{0\} \times \prod_{\beta \neq \alpha \in I} X_\alpha : \beta \in I\}$, is a subset of $\mathcal{B}$, and $|A| = \aleph$.

4. (a) $A = \{f \in [0,1]^{\mathbb{R}} : f \text{ is contiuous}\}$:

- A is not open. Let $f \in A$, and let $U = \prod_{i \in \mathbb{R}} U_i$ be a basic open neighbourhood of f. Let $F \subseteq \mathbb{R}$ be U's finite support. Let

$$g(x) = \begin{cases} f(x) & if\ x \in F \\ 0 & if\ x \notin F \wedge f(x) \neq 0 \\ 1 & if\ x \notin F \wedge f(x) = 0 \end{cases}$$

be a function in $[0,1]^{\mathbb{R}}$. $g \in U$, g is not continuous, so $g \notin A$. Thus $U \cap ([0,1]^{\mathbb{R}} \setminus A) \neq \emptyset$ and so A isn't open.

- A is not closed. Let $f \in [0,1]^{\mathbb{R}} \setminus A$ be a non-continuous function, and let $U \subseteq [0,1]^{\mathbb{R}}$ be an open neighbourhood of f. Let $\{i_1, \ldots, i_n\} = F \subseteq \mathbb{R}$ be U's finite support. Let

$$g(x) = \begin{cases} f(i_1) & if\ x < i_1 \\ tf(i_j) + (1-t)f(i_{j+1}) & if\ x \in [i_j, i_{j+1}),\ t = \frac{i_{j+1}-x}{i_{j+1}-i_j},\ 1 \leq j < n \\ f(i_n) & if\ x \geq i_n \end{cases}$$

g is contiuous, and $g \in U$, so $U \cap A \neq \emptyset$ and so $X \setminus A$ is not open.

- A is path-connected, and thus connected. Let $f, g \in A$ be arbitrary, there exist a path $\varphi : [0,1] \to [0,1]^{\mathbb{R}}$ defined by $t \overset{\varphi}{\mapsto} tf + (1-t)g$. Clearly $\varphi([0,1]) \subseteq A$ (as a linear combination of contiuous functions), and $\varphi(0) = g$, $\varphi(1) = f$.

(b) $A = \{f \in [0,1]^{\mathbb{R}} : f(0) \neq f(100)\}$

- A is open. Let $f \in A$ be arbitrary. Denote $\delta := \left|\frac{f(100)-f(0)}{3}\right| > 0$, and let $U = \{[0,1]^{\mathbb{R}} : \eta(0) \in B(0,\delta) \wedge \eta(100) \in B(100,\delta)\}$. U is a basic open set (by definition). Also, $\forall \eta \in U$:

$$\begin{aligned} |\eta(100) - \eta(0)| &= |\eta(100) - f(100) + f(100) - f(0) + f(0) - \eta(0)| \\ &\geq |f(100) - f(0)| - |\eta(100) - f(100) + f(0) - \eta(0)| \\ &\geq |f(100) - f(0)| - |\eta(100) - f(100)| - |f(0) - \eta(0)| \\ &> 3\delta - \delta - \delta = \delta > 0 \end{aligned}$$

So $\eta(0) \neq \eta(100)$ and thus $\eta \in U$. So A is open.

- A is not closed. We've seen that any product of connected spaces is connected, so $[0,1]^{\mathbb{R}}$ is connected, and thus U is clopen iff $U = \emptyset$ or $U = [0,1]^{\mathbb{R}}$. A is neither, so A is not closed.
- A is not connected. Let $U = \{f \in [0,1]^{\mathbb{R}} : f(0) < f(100)\}$ and $V = \{f \in [0,1]^{\mathbb{R}} : f(0) > f(100)\}$. Clearly $U \cap V = \emptyset$ and $U \cup V = A$. The proff that U and V are open is similar to that of A being open, and is left to the reader.

(c) $A = \{f \in [0,1]^{\mathbb{R}} : |Rng(f)| < \aleph_0\}$.

- A is not open. Let $f \in A$ be arbitrary, and let $U \subseteq [0,1]^{\mathbb{R}}$ be a basic open neighbourhood of f, with a finite support $\{i_1, \dots, i_n\} = F \subseteq \mathbb{R}$. Let

$$g(x) = \begin{cases} f(i_1) & if\ x < i_1 \\ tf(i_j) + (1-t)f(i_{j+1}) & if\ x \in [i_j, i_{j+1})\,,\ t = \frac{i_{j+1}-x}{i_{j+1}-i_j},\ 1 \leq j < n \\ f(i_n) & if\ x \geq i_n \end{cases}$$

  Clearly $g \in U$. Also

$$|Rng(g)| = \left| \{t \in [0,1] : \min_{j=1,\dots,n} \{f(i_j)\} \leq t \leq \max_{j=1,\dots,n} \{f(i_j)\}\} \right| = \aleph$$

  and thus $g \notin A$, and A is not open.
- A is not closed. Let $f \in [0,1]^{\mathbb{R}} \setminus A$, and let $f \in U$ be a basic neighbourhood with a finite support $F = \{i_1, \dots, \imath_n\}$. Let

$$g(x) = \begin{cases} f(i_1) & if\ x < \imath_1 \\ f(i_j) & if\ i_j \leq x < i_{j+1},\ 1 \leq j < n \\ f(i_n) & if\ i_n \leq x \end{cases}$$

  Again, $g \in U$, and $|Rng(g)| = |F| < \aleph_0$, thus $g \in A$ and $[0,1]^{\mathbb{R}} \setminus A$ is not open.
- A is path-connected. For every $f, g \in A$ the exist a path $[0,1] \ni t \overset{\varphi}{\mapsto} tf + (1-t)g$ such that φ is continuous from f to g. We shall show that $\varphi([0,1]) \subseteq A$. let $t \in (0,1)$ be arbitrary: $|Rng(\varphi(t))| = |Rng(tf + (1-t)g)| \leq |Rng(tf)| + |Rng((1-t)g)| < \aleph_0$, and for $t \in \{0,1\}$ the claim is trivial.

(d) $A = \{f \in [0,1]^{\mathbb{R}} : \forall r \in \mathbb{R}\, f(r) = f(-r)\}$

- A is closed. Let $f \notin A$, there exists some $r_0 \in \mathbb{R}$ such that $f(r_0) \neq f(-r_0)$. Denote $\delta = \left|\frac{f(r_0) - f(-r_0)}{3}\right|$. By applying a similar proof to 4.b it can be shown that the set $U = \{\eta \in [0,1]^{\mathbb{R}} : \eta(r_0) \in B(f(r_0), \delta) \wedge \eta(-r_0) \in B(f(-r_0), \delta)\}$ is open and $U \subseteq [0,1]^{\mathbb{R}} \setminus A$, so $[0,1]^{\mathbb{R}} \setminus A$ is open, and A is closed.

- $[0,1]^{\mathbb{R}}$ is connected, as a product of connected spaces, thus $A \neq \emptyset, [0,1]^{\mathbb{R}}$ is not clopen, and thus not open.
- A is path-connected, by the path $t \overset{\varphi}{\mapsto} tf + (1-t)g$ is contiuous, from g to f, and $\forall t \in [0,1], \forall r \in \mathbb{R} : \varphi(t)(r) = tf(r) + (1-t)g(r) = tf(-r) + (1-t)g(r) = \varphi(t)(-r)$, so $\varphi([0,1]) \subseteq A$.