# A PROOF OF SHELAH'S PARTITION THEOREM

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In 1965 Erdős, Hajnal and Rado proved ([EHR], Theorem 42) that for a cardinal $\mu$ of countable cofinality the following partition relation holds:

$$\binom{\mu^+}{\mu} \longrightarrow \binom{\mu}{\mu}^{1,1}_2 \tag{1}$$

This partition relations means: for every function $C : \mu^+ \times \mu \to 2$ there are sets $A \subseteq \mu^+$ and $B \subseteq \mu$ such that $|A| = |B| = \mu$ and $c$ is constant on $A \times B$.

It is still unknown whether the GCH implies the following partition relation:

$$\binom{\aleph_{\omega_1+1}}{\aleph_{\omega_1}} \longrightarrow \binom{\aleph_{\omega_1}}{\aleph_{\omega_1}}^{1,1}_2 \tag{2}$$

But recently Shelah proved that (2) holds under a cardinal arithmetic assumption: namely, if $\aleph_{\omega_1}$ is a strong limit and $2^{\aleph_{\omega_1}} > \aleph_{\omega_1+1}$. Shelah's theorem holds actually for every strong limit singular of uncountable cofinality that violates the GCH.

The following is a self contained presentation of Shelah's recent proof.

The proof here is arranged differently from the proof in the appendix of the forthcoming [Sh 513] so that no use of other results of Shelah is made, except for Fact 2 below, which comes from pcf theory. In other words, we avoid here using the ideal $I[\lambda]$ from [Sh 420] and the tools from [Sh 108].

The pcf theory needed to obtain 2 below for $\mu$ which is not a fixed point is available in either [BM] or [J]; for the remaining case of a fixed point $\mu$ we refer the reader to [S], or the survey paper [K].

A discussion of how pcf theory implies Fact 2 below is found in a few paragraphs at the beginning of the proof. We assume in those paragraphs that the reader is familiar with basic pcf definitions like pcf, tcf and generators $B_\lambda[A]$ for pcf. However, these are not needed in the proof at all. Thus, if the reader is willing to assume Fact 2 below, those paragraphs may be skipped.

**1. Theorem**: Suppose that $\mu$ is a strong limit singular cardinal of uncountable cofinality and $2^\mu > \mu^+$. Then

$$\binom{\mu^+}{\mu} \longrightarrow \binom{\mu^+}{\mu+1}^{1,1}_{<\mathrm{cf}\,\mu} \tag{2}$$

The relation $\binom{\mu^+}{\mu} \longrightarrow \binom{\mu^+}{\mu+1}^{1,1}_{<\mathrm{cf}\,\mu}$ means: for every coloring $c$ of $\mu^+ \times \mu$ by less than $\mathrm{cf}\mu$ many colors there are $A \subseteq \mu^+$ with $\mathrm{otp}\, A = \mu + 1$ and $B \subseteq \mu$ with $\mathrm{otp}\, B = \mu$ such that $c$ is constant on $A \times B$.

*Proof*: The theorem follows by using the pigeon hole principle from the following statement: for every function $c : (\mu^+ \times \mu) \to \theta$, for $\theta < \mathrm{cf}\mu$, there are $A \subseteq \mu^+$ and $B \subseteq \mu$ with $\mathrm{otp}\, A = \mu + 1$ and $\mathrm{otp}\, B = \mu$ such that the function $c|(A \times B)$ does not depend on the first coordinate.

Let $\kappa$ denote $\mathrm{cf}\mu$. Fixed an increasing sequence $\overline{\mu} = \langle \mu_i : i < \kappa \rangle$ cofinal in $\mu$ such that $\mu_0 > \kappa$. We discuss briefly the pcf facts we need. The material presented in either [BM] or [J] is enough for the case $\mu$ is not a fixed point of the aleph function.

If $\mu$ is not a fixed point of the aleph function, then there is an end segment $A$ of the regular cardinals below $\mu$ that satisfies $\min A > |A|^+$, and even $\min A > 2^{|A|}$. Because $2^\mu \geq \mu^{++}$ and $2^\mu = \max \mathrm{pcf}\, A$ by Theorem 5.1 in [BM], we conclude, as $\mathrm{pcf}\, A$ is an interval of regular cardinals (by Corollary 2.2 in [BM]), that $\mu^{++} \in \mathrm{pcf}\, A$. Therefore $\mu^{++} = \mathrm{tcf}\, \Pi B_{\mu^{++}}[A]/J_{<\mu^{++}}[A]$ (see 4.4 in [BM]). By subtracting $B_{\mu^+}[A]$ from $B_{\mu^{++}}[A]$, if necessary, we may assume that those two generators are disjoint. Consequently, the ideal $J_{<\mu^{++}}$ on $B_{\mu^{++}}$ equals $J_{<\mu^+}$ on it. However, since $\mu$ is a strong limit, the latter is nothing but the ideal of bounded sets on $A$. The existence of generators $B_\lambda[A]$ is proved in 4.6 and 7.9 of [BM].

Thus, in this case, for some increasing sequence of regular cardinals $\langle \lambda_i : i < \kappa \rangle$ with limit $\mu$ we have

$$\mu^{++} = \mathrm{tcf}\, \Pi_{i<\kappa} \lambda_i / J^{bd}_\kappa$$

where $J^{bd}_\kappa$ is the ideal of bounded subsets of $\kappa = \mathrm{cf}\mu$.

This is true for every strong limit singular cardinal $\mu$ of uncountable cofinality $\kappa$ if $2^\mu > \mu^+$ — even if $\mu$ is a fixed point — by two theorems of Shelah we now refer to. By Theorem 5.4 in Chapter 2 of [S] it follows that $\mu^{++}$ is in pcf $A$ for *some* set or regular cardinals $A \subseteq \mu$ and Theorem 1.6(2) in chapter 8 of [S] assures us that if $\mu^{++}$ is represented as a possible cofinality in this way, then it is also represented as the true cofinality of $\Pi_{i<\kappa}\lambda_i/J^{bd}_\kappa$ for some set $A \subseteq \mu$ or regular cardinals.

Let us summarize the pcf facts we shall need in the proof:

**2. Pcf Fact**: There is an increasing sequence of regular cardinals $\overline{\lambda} = \langle \lambda_i : i < \kappa \rangle$ with $\sup \overline{\lambda} = \mu$ such that $\Pi_i \lambda_i/J^{bd}_\kappa$ is $\mu^{++}$-directed.

Since we are dealing with $J^{bd}_\kappa$ we may thin out $\overline{\lambda}$ and assume that $\lambda_i > 2^{\mu_i^+}$.

Suppose that $c : (\mu^+ \times \mu) \to \theta$ is given for some $\theta < \kappa$. We need to produce $A$ and $B$ as above. These sets will be constructed in $\kappa$ many approximations, after some preparation.

Fix a function $F$ from $[\mu^+]^2$ to $\kappa$ that such that for all $i < \kappa$ the set $a_i^\alpha := \{\beta < \alpha : F(\alpha, \beta) \leq i\}$ has cardinality at most $\mu_i^+$. Thus $\alpha = \bigcup_{i<\kappa} a_i^\alpha$.

Let $\chi$ be a sufficiently large regular cardinal. We define by double induction on $\mu^+ \times \kappa$ a matrix $\{M_{\alpha,i} : \alpha < \mu^+, i < \kappa\}$ of elementary submodels of $(H(\chi), \in)$, satisfying:

(0) $M_{\alpha,i} \prec (H(\chi), \in)$, $||M_{\alpha,i}|| = 2^{\mu_i^+}$ and ${}^{\mu_i^+}M_{\alpha,i} \subseteq M_{\alpha,i}$ ($M_{\alpha,i}$ is closed under sequences of length $\mu_i^+$).

(1) $\alpha, c, \overline{\mu}, \overline{\lambda}$ and $F$ belong to $M_{\alpha,i}$ and $\{M_{\beta,j} : (\beta, j) <_{lx} (\alpha, i)\}$ belongs to $M_{\alpha,i}$.

There is no problem to choose $M_{\alpha,i}$ so that is satisfies the conditions above using the Skolem-Löwenheim theorem and $(2^{\mu_i})^{\mu_i} = 2^{\mu_i}$.

We make a few simple observations about this array of models:

**3. Fact**:

(0) If $(\beta, j) <_{lx} (\alpha, i)$ and $\beta \in M_{\alpha,i}$ then $M_{\beta,j} \in M_{\alpha,i}$.

(1) If $M_{\beta,j} \in M_{\alpha,i}$ and $j \leq i$ then $M_{\beta,j} \subseteq M_{\alpha,i}$ and hence $M_{\beta,j} \prec M_{\alpha,i}$.

(2) $M_{\alpha,j} \prec M_{\alpha,i}$ for all $\alpha < \mu^+$ and $j < i < \kappa$.

(3) $\alpha \subseteq \bigcup_i M_{\alpha,i}$ for all $\alpha < \mu^+$

(4) For all $\beta < \alpha < \mu^+$ for an end segment of $i < \kappa$ it holds that $M_{\beta,i} \subseteq M_{\alpha,i}$ and hence $M_{\beta,i} \prec M_{\alpha,i}$.

*Proof*: Clause (0) is follows from the demand that $\{M_{\beta,j} : (\beta, j) <_{lx} (\alpha, i)\} \in M_{\alpha,i}$ and the fact that $\kappa \subseteq \mu_i \subseteq M_{\alpha,i}$, so $i \in M_{\alpha,i}$, and therefore $M_{\beta,j}$ is definable from parameters in $M_{\alpha,i}$. Being an elementary submodel, $M_{\alpha,i}$ contains every set definable from parameters in $M_{\alpha,i}$.

To see clause (1) suppose that $M_{\beta,j} \in M_{\alpha,i}$ and that $j \leq i$. By elementarity of $M_{\alpha,i}$ there is a bijection $\varphi : 2^{\mu_i^+} \to M_{\beta,j}$ in $M_{\alpha,i}$. As $2^{\mu_i^+} \subseteq M_{\alpha,i}$, also $\mathrm{ran}\varphi \subseteq M_{\alpha,i}$ and hence $M_{\beta,j} \subseteq M_{\alpha,i}$. Since also $M_{\beta,j} \prec (H(\chi), \in)$ and $M_{\alpha,i} \prec (H(\chi), \in)$, necessarily $M_{\beta,j} \prec M_{\alpha,i}$ and (1) holds.

Clause (2) follows from the previous two and the fact that $\alpha \in M_{\alpha,i}$.

To prove (3) use the fact that $a_i^\alpha \in M_{\alpha,i}$ and also $a_i^\alpha \subseteq M_{\alpha,i}$ for all $i < \kappa$. Therefore for all $i \geq F(\alpha, b)$ it holds that $\beta \in M_{\alpha,i}$. Thus (3) holds.

The last clause follows from the previous ones.

A conclusion of those facts is the following:

**4. Fact**: The sequence $\overline{M}_\alpha = \langle M_{\alpha,i} : i < \kappa\rangle$ is increasing in $\prec$, $\alpha \subseteq \bigcup_i M_{\alpha,i}$ and if $\beta < \alpha < \mu^+$ then $\overline{M}_\beta \in_{J_\kappa^{bd}} \overline{M}_\alpha$, $\overline{M}_b \subseteq_{J_\kappa^{bd}} \overline{M}_\alpha$, and even $\overline{M}_b \prec_{J_\kappa^{bd}} \overline{M}_\alpha$, namely for all sufficiently large $i < \kappa$ we have that $M_{\beta,i} \in M_{\alpha,i}$, $M_{\beta,i} \subseteq M_{\alpha,i}$ and $M_{\beta,i} \prec M_{\alpha,i}$.

For every $\alpha < \mu^+$ and $i < \kappa$ define $f_\alpha(i) = \sup M_{\alpha,i} \cap \lambda_i$. As we assumed that $\lambda_i > 2^{\mu_i^+} = ||M_{\alpha,i}||$, it follows by the regularity of $\lambda_i$ that $f_\alpha(i) \in \lambda_i$, for all $i < \kappa$ and therefore $f_\alpha \in \Pi\lambda_i$ for all $\alpha < \mu^+$.

Furthermore, if $\beta < \alpha < \mu^+$ then from some $i_{\alpha,\beta} < \kappa$ onwards $M_{\beta,i} \in M_{\alpha,i}$ and therefore (as $\overline{\lambda} \subseteq M_{\alpha,i}$) $f_\beta(i) \in M_{\alpha,i}$ and hence $f_\beta(i) < f_\alpha(i)$ on an end segment of $\kappa$, or $f_\beta <_{J_\kappa^{bd}} f_\alpha$. Thus $\overline{f} = \langle f_\alpha : \alpha < \mu^+\rangle$ is increasing in $<_{J_\kappa^{bd}}$.

Use Fact 2 above to find a bound $f^* \in \Pi\lambda_i$ to $\overline{f}$ in $\leq_{J_\kappa^{bd}}$.

Using $f^*$ and the coloring $c$, define $g_\alpha(i) = c(\alpha, f^*(i))$ for all $\alpha < \mu^+$ and $i < \kappa$. The function $g_\alpha$ specifies the $c$-type of $\alpha$ over the sequence $\langle f^*(i) : i < \kappa\rangle$.

As there are only $\theta^\kappa < \mu^+ = \mathrm{cf}\mu^+$ many possible such types, we find a function $g^* : \kappa \to \theta$ so that $A := \{\alpha < \mu^+ : g_\alpha = g^*\}$ is unbounded in $\mu^+$.

Let us find now by induction on $\zeta < \mu^+$ an increasing continuous chain of elementary submodels $\overline{N} = \langle N_\alpha : \zeta < \mu^+\rangle$ satisfying:

(0) $\mu \subseteq N_\zeta \prec (H(\chi), \in)$ and $||N_\zeta|| = \mu$

(1) $A$, $g^*$ and $\{M_{\alpha,i} : \alpha < \mu^+, i < \kappa\}$ belong to $N_0$

Let $E = \{\zeta < \mu^+ : \zeta = N_\zeta \cap \mu^+\}$. This is a club of $\mu^+$.

By induction on $i < \kappa$ we choose a strictly increasing sequence of ordinals $\delta_i < \mu^+$ satisfying:

(a) $\delta_i \in \operatorname{acc} E$ (that is, $\delta_i$ is an accumulation point of $E$) and

(b) $\operatorname{cf}\delta_i = \mu_i^+$.

Observe that $\delta_i > \sup\{\delta_\nu : \nu < i\}$ for all $i < \kappa$, because $\operatorname{cf}\delta_i = \mu_i^+$. This enables us to choose $\alpha(i) \in \delta_i \setminus \sup\{\delta_\nu : \nu < i\}$ for every $i < \kappa$.

We also observe that if $\alpha \in N_\zeta$ then $M_{\alpha,i} \prec N_\zeta$ for $i < \kappa$. Therefore, if $\zeta \in E$, then $M_{\alpha,i} \prec N_\zeta$ for all $\alpha < \zeta$ and $i < \kappa$.

Pick $\alpha(*) \in A \setminus \sup\{\delta_i : i < \kappa\}$.

We define now by induction on $i < \kappa$ sets $A_i, B_i$ and an index $j(i) < \kappa$ such that the following conditions hold:

(a) $j(i) > i$ and $i_1 < i_2 \Rightarrow \lambda_{j(i_1)} < \mu_{j(i_2)}$

(b) For any two ordinals $\sigma < \tau$ in the set $\{\delta_\nu : \nu \leq i\} \cup \{\alpha_\nu : \nu \leq i\} \cup \{\alpha(*)\}$ it holds that $\overline{M}_\sigma \prec \overline{M}_\tau$ and $f_\sigma < f_\tau$ on the end segment $(j(i), \kappa)$ of $\kappa$.

(c) $A_i \subseteq A \cap \delta_i$, $\operatorname{otp} A_i = \mu_i^+$ and $A_i \in M_{\delta_i, j(i)}$.

(d) $B_i \subseteq \lambda_{j(i)} \setminus \sup\{\lambda_{j(\nu)} : \nu < i\}$, $\operatorname{otp} B_i = \lambda_{j(i)}$ and $B_i \in M_{\delta_i, j(B_i)}$ for some $j(B_i) < \kappa$. Also, $B_\nu \in M_{\delta_i, j(i)}$ for all $\nu < i$.

(e) If $\alpha \in \bigcup_{\nu \leq i} A_i \cup \{\alpha(*)\}$ and $\beta \in B_\nu$ for some $\nu \leq i$ then $c(\alpha, \beta) = g^*(j(\nu))$.

If the induction is carried out successfully, then by (e) it follows that if $\alpha \in A = \bigcup_{i<\kappa} A_i \cup \{\alpha(*)\}$ and $\beta \in B = \bigcup_{i<\kappa} B_i$ then $c(\alpha, \beta) = g^*(j(i))$ for the (unique) first $i$ satisfying $\lambda_{j(i)} > \beta$. From (c) and (d) it follows that $\operatorname{otp} A = \mu + 1$ and $\operatorname{otp} B = \mu$. Thus $A, B$ are as required by the theorem.

Suppose, then, that $A_\nu, B_\nu$ and $j(\nu)$ are defined for all $\nu < i$ and satisfy the conditions above.

Since $\alpha(i) > \nu$ for every $\nu < i$, there is some $j(\nu) < \kappa$ such that $B_\nu, A_\nu, j(\nu) \in M_{\alpha(i), j}$ for $j \geq j(\nu)$. Let $j_0 < \kappa$ be large enough so that $B_\nu, A_\nu, j(\nu) \in M_{\alpha(i), j_0}$ for all $\nu < i$ and so that $\mu_{j_0} > \lambda_{j(\nu)}$ for all $\nu < i$. This can be done as there are less than $\kappa$ many $\nu$-s.

We have, then, $B_\nu \in M_{\alpha(i),j_0}$ for all $\nu < i$ or $\{B_\nu : \nu < i\} \subseteq M_{\alpha(i),j_0}$. As $M_{\alpha(i),j_0}$ is closed under sequences of length at most $\mu_{j_0}^+ > \kappa$ we also have that $\langle B_\nu : \nu < i\rangle \in M_{\alpha(i),j_0}$. Similarly, $\langle A_\nu : \nu < i\rangle \in M_{\alpha(i),j_0}$ and $\langle j(\nu) : \nu < i\rangle \in M_{\alpha(i),j_0}$.

Since $\delta_i$ is an accumulation point of $E$ and has cofinality $\mu_i^+$, we can find an increasing sequence $\langle \zeta_\epsilon : \epsilon < \mu_i^+\rangle$ of elements of $E$ with $\zeta_0 > \alpha(i)$.

For every $\zeta_\epsilon$ in the sequence we chose, $\alpha(i) \in \zeta_\epsilon \subseteq N_{\zeta_\epsilon}$, and therefore $M_{\alpha(i),j_0} \prec N_{\zeta_\epsilon}$ and hence $\langle B_\nu : \nu < i\rangle, \langle j(\nu) : \nu < i\rangle \in N_{\zeta_\epsilon}$.

For every $\epsilon < \mu_i^+$ the ordinal $\alpha(*)$ satisfies in $(H(\chi), \in)$ the following formula $\varphi(x, \zeta_\epsilon)$ (when substituted for $x$):

$$\varphi(x,\zeta_\epsilon) := x \in A \;\&\; x > \zeta_\epsilon \;\&\; (\forall \nu < i)(\beta \in B_\nu \Rightarrow c(x,\beta) = g^*(j(\nu))) \tag{1}$$

Since all the parameters in this sentence — namely $A$, $\langle B_\nu : \nu < i\rangle$, $\langle j(\nu) : \nu < i\rangle$, $c$, $g^*$ and $\zeta_\epsilon$ — belong to $N_{\zeta_{\epsilon+1}}$ and the latter is an elementary submodel of $(H(\chi), \in)$, there is an ordinal $\gamma_\epsilon \in N_{\zeta_{\epsilon+1}}$ such that $\varphi(\gamma_\epsilon, \zeta_\epsilon)$ holds. Clearly, $\zeta_\epsilon < \gamma_\epsilon < \zeta_{\epsilon+1} < \delta_i$.

Let $A_i' := \{\gamma_{\epsilon+1} : \epsilon < \mu_i^+\}$. We have shown that $A_i' \subseteq A \cap (\alpha(i), \delta_i)$ and every $\alpha \in A_i'$ satisfies that $c(\alpha, \beta) = g^*(j(i))$ for the first $i$ such that $\lambda_{j(i)} > \beta$. Each member of $A_i'$ belongs to $M_{\delta_i,j}$ for some $j < \kappa$, since $\delta_i \subseteq \bigcup_{j<\kappa} M_{\delta_i,j}$. Because $\mu_i^+ > \kappa$ is regular, there must be some index $j_1 < \kappa$ such that $A(i) = A'(i) \cap M_{\delta_i,j_1}$ has cardinality $\mu_i^+$. Let $A(i)$ be the set $A_i$ we need to define. This takes care of the first two parts in (c).

Let $j(i) \geq \max\{j_1, j_0\}$ be large enough so that $A_i \in M_{\delta_i,j_1}$ and $M_{\delta_i,j(i)} \prec M_{\alpha(*),j(i)}$, and also such that $f_{\delta_i}(j(i)) < f^*(j(i))$. Now the remaining part of (c), (a) and (b) are also satisfied.

Work now in $M_{\alpha(*),j(i)}$. We know that $\langle A_\nu : \nu < i\rangle, A_i, \alpha(*) \in M_{\alpha(*),j(i)}$ and that also the function $\nu \mapsto j(\nu)$ for $\nu < i$ belongs to $M_{\alpha(*),j(i)}$, because all functions from $\kappa$ to $\kappa$ belong to it.

Therefore the following set is definable in $M_{\alpha(*),j(i)}$:

$$B := \{\beta < \lambda_{j(i)} : c(\alpha,\beta) = g^*(j(i)) \text{ for all } \alpha \in \bigcup_{\nu \leq i} A_\nu \cup \{\alpha(*)\}\} \tag{2}$$

Observe that $f^*(j(i))$ belongs to the set $B$ defined in (2) because $\bigcup_{\nu\leq i} A_\nu \cup \{\alpha(*)\} \subseteq A$, but that since $f^*(j(i)) > f_{\delta_i}(j(i)) = \sup M_{\delta_i,j(i)} \cap \lambda_{j(i)}$ it does not belong to $M_{\delta_i,j(i)}$. This shows that $B$ has no bound in $M_{\delta_i,j(i)} \cap \lambda_{j(i)}$. We conclude, then, that $B$ is unbounded below $\lambda_{j(i)}$: being definable in $M_{\delta_i,j(i)}$, if there were a bound to $B$ below $\lambda_{j(i)}$ there would be one in $M_{\delta_i,j(i)}$; but there is not.

Using the same argument as before, we find some $j(B) < \kappa$ such that $B_i = B \cap M_{\delta_i,j(B)} \setminus \sup\{\lambda_{j(\nu)} : \nu < i\}$ belongs to $M_{\delta_i,j(B)}$ and has cardinality $\lambda_{j(i)}$. Now (d) and (e) are also satisfied.

This completes the induction, and the proof as well.

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