The Universality Spectrum of Stable Unsuperstable Theories

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ABSTRACT

It is shown that if T is stable unsuperstable, and $\aleph_1 < \lambda = \text{cf}\lambda < 2^{\aleph_0}$, or $2^{\aleph_0} < \mu^+ < \lambda = \text{cf}\lambda < \mu^{\aleph_0}$ then T has no universal model in cardinality λ, and if e.g. $\aleph_\omega < 2^{\aleph_0}$ then T has no universal model in $\aleph_\omega$. These results are generalized to $\kappa = \text{cf}\kappa < \kappa(T)$ in the place of $\aleph_o$. Also: if there is a universal model in $\lambda > |T|$, T stable and $\kappa < \kappa(T)$ then there is a universal tree of height $\kappa + 1$ in cardinality λ.

§1. Introduction

We handle the universal spectrum of stable-unsuperstable first order theories. This continues [KjSh 409] and adds information the picture started up in [Sh 100]. The general subject addressed here is the universal model problem, which although natural and old, was not treated very extensively in the past. For background, motivation and history of the subject see the introduction to [KjSh 409], a paper in which unstable theories with the strict order property are handled (e.g. the class of linear orders).

When looking at a class K of structures together with a class of allowed embeddings — say all models of some first order theory (T) with elementary embeddings — we get a partial order: $A \leq B$ if there is a mapping of A into B in the class of allowed mappings.

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The universal model problem can be phrased, in this context, as a question about this partial order: is there in $\{M \in K : ||M|| \leq \lambda\}$ a "greatest" element — which we call "universal" — namely one such that all other elements $M \in K$, $||M|| \leq \lambda$ are smaller than or equal to it. This question can be elaborated: what is the cofinality, i.e. the minimal cardinality of a subcollection of elements such that every element is smaller than or equal to *one* of the elements in this subcollection? Can a universal object be found outside our collection? (for instance, is there a model of T of cardinality $\mu > \lambda$ such that every model of T of cardinality λ is elementarily embeddable into it). How does the existence, or nonexistence, of universal objects in one collection of structure influence the existence or non existence of universal objects in related collections?

In this paper we prove that if T is stable unsuperstable, and $\mu^+ < \lambda = \mathrm{cf}\lambda < \mu^{\aleph_0}$ then T has no model of cardinality λ into which all models of T of the same cardinality are elementarily embeddable, not even a family of models $\langle M_i : i < \lambda' \rangle$, $\lambda' < \mu^{\aleph_0}$, each of cardinality λ such that every model of T of cardinality λ is elementarily embedded into some model in the family. It follows from the theory of covering numbers that certain singular cardinals are also not in the universality spectrum of stable unsuperstable theories.

Also, it is shown that a certain theory (the "canonical" stable unsuperstable theory) is "minimal" with respect to the existence of universal models, namely that whenever some stable unsuperstable theory T has a universal model in cardinality λ, also this theory has one.

We mention here without proof that GCH implies that all first order theories have universal models in all uncountable cardinals (above the cardinality of the theory), and that the question whether $\aleph_1$ is in the universality spectrum of a countable, stable but not superstable theory is independent of $ZFC + 2^{\aleph_0} = \aleph_2$ (see [Sh 100] §2). At this point it is interesting to note that it is consistent that there is a universal graph in $\aleph_2 < 2^{\aleph_0}$ (see [Sh 175a]), but it is not consistent to have a universal model for some countable, stable unsuperstable T. So in this respect, stable unsuperstable theories are not "$\leq$" all unstable theories.

In subsequent papers, universality spectrums of some classes of infinite abelian groups, and complementary consistency results to the negative results known so far will be dealt with. (Note: if T is countable first order, stable unsuperstable theory, and $\mu = \Sigma_n \mu_n^{\aleph_0}$, then there is a universal model for T in μ; if, say, $\beth_\omega^+ < \mu < \beth_{\omega+1}$ there isn't; and we do not settle here the case $\mu = \beth_\omega^+$.)

We assume some familiarity with the definitions of stability and superstability, as well as with fundamentals of forking theory (to be found in e.g. [[Sh-c],III).

Lastly, those reader who speculate that $\ddot\smile$ * is some inexplicable whim of the laser printer are wrong. Smileys indicate ends of proofs, and replace the old and square boxes.

§2. Preliminaries and Setup

Having fixed attention on a given T, we work in some "monster model" C, which is a

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* We thank Martin Goldstern for the smiley TEXnology and for $\bigcup\!\!\!\!\!|$.

big saturated model, of which all the models we are interested in are elementary submodels of smaller cardinality.

2.1 Definition: for a complete first order theory T,

(0) A model $M \models T$ is *$\prec$-universal in cardinality* λ if $||M|| = \lambda$ and for every $N \models T$ such that $||N|| = \lambda$ there is an elementary embedding $h : N \to M$. It is $<$-universal, if we omit "elementary" from the definition.

(1) $\mathrm{Univ}(T, \prec) = \{\lambda : \lambda \text{ is a cardinal and } T \text{ has a universal model in } \lambda\}$ is the *universality spectrum* of T.

(2) $\mathrm{Univ}_p(T, \prec)$ is the family of pairs (λ, μ) such that there is a family of μ models of T each of cardianlity λ, such that any model of T of cardianlity λ can be elelmentarily embeded into one of them.

(3) $\mathrm{Univ}_t(T, \prec)$ is the family of triples (λ, κ, μ) such that there is is a family of μ models of T each of cardianlity $\leq \kappa$ and any model of T of cardianlity λ can be elemntarily embeded into one of them.

In this paper "universal" means $\prec$-universal and $\mathrm{Univ}(T)$ means $\mathrm{Univ}(T, \prec)$ unless otherwise stated.

2.2 Definition:

(0) A Theory T is *stable in* λ if for every model N and set $A \subseteq N$, $|A| \leq \lambda \Rightarrow |S(A)| \leq \lambda$. For equivalent definitions see [Sh-g],II.2.13.

(1) $\kappa(T)$, the cardinal of T is as defined in [Sh-c],III,§3. We recall from [Sh-c]III that for

a countable, complete first order T, T is stable unsuperstable iff $\kappa(T) = \aleph_1$.

(2) The notation $\mathbf{a} \underset{B}{\overset{N}{\bigcup\!\!\!\!\!\!\bigcup}} A$ means "the type of $\mathbf{a}$ over the set A in the model N does not fork over the set B". The notation $\mathbf{a} \underset{B}{\overset{N}{\not\bigcup}} A$ means "the type of $\mathbf{a}$ over the set A in the model N forks over the set B". When the model N in which the relation of forking exists is clear from context, it is omitted.

By small bold faced letters we shall denote finite sequences of elements from a model. Following a widely spread abuse of notation we shall not write $\mathbf{a} \in |N|^{<\omega}$, but write $\mathbf{a} \in N$, and even refer to $\mathbf{a}$ as "element". This is perfectly all right with what is about to be done here, because we may add the finite sequences as elements into the model and work in T^{eq} (or C^{eq}), or replace a type of an n-tuple by a 1-type when necessary.

The forking facts which we shall need are summarized in the following quotation from [Sh-c] p.84:

2.3 Theorem:

(0) (finite character of forking) If $\mathbf{a} \underset{B}{\not\bigcup} A$ then there is some *finite* set $A' \subseteq A$ such that $\mathbf{a} \underset{B}{\not\bigcup} A'$. Also $\mathbf{a} \underset{B}{\not\bigcup} A$ iff $\mathbf{a} \underset{B}{\not\bigcup} A \cup B$.

(1) (symmetry) $\mathbf{a} \underset{A}{\bigcup\!\!\!\!\!\!\bigcup} A \cup \mathbf{b}$ iff $\mathbf{b} \underset{A}{\bigcup\!\!\!\!\!\!\bigcup} A \cup \mathbf{a}$.

(2) (transitivity) if $A \subseteq B \subseteq C$ and $\mathbf{a} \underset{B}{\bigcup\!\!\!\!\!\!\bigcup} C$ and $\mathbf{a} \underset{A}{\bigcup\!\!\!\!\!\!\bigcup} B$ then $\mathbf{a} \underset{A}{\bigcup\!\!\!\!\!\!\bigcup} C$

(3) Let $B \subseteq A$, then: $\mathbf{b} \underset{B \cup A}{\bigcup\!\!\!\!\!\!\bigcup} A \cup \mathbf{a}$ and $\mathbf{a} \underset{B}{\bigcup\!\!\!\!\!\!\bigcup} A$ iff $\mathbf{a}\frown\mathbf{b} \underset{B}{\bigcup\!\!\!\!\!\!\bigcup} A$

(4) When M is a model, the type p does not fork over $|M|$ iff p is finitely satisfiable in

M.

(5) if $A \subseteq B \subseteq C$, $p \in S(B)$ does not fork over A, then there is some $q \in S(C)$,$p \subseteq q$ and q does not fork over A.

(6) ([Sh-c] p.113) If $p \in S(|M|)$ is definable over A where $A \subseteq M$ then p does not fork over A.

We need a few facts about sets of indiscernibles. We denote sets of indiscernibles by $\mathbf{I}$ and $\mathbf{J}$. We say that $\mathrm{tp}(\mathbf{I}) = \mathrm{tp}(\mathbf{J})$ if for every n, formula φ and elements $\mathbf{a}_1, \cdots, \mathbf{a}_n \in \mathbf{I}, \mathbf{b}_1, \cdots, \mathbf{b}_n \in \mathbf{J}$, $\mathrm{tp}(\mathbf{a}_1, \cdots, \mathbf{a}_n, \emptyset) = \mathrm{tp}(\mathbf{b}_1, \cdots, \mathbf{b}_n, \emptyset)$.

2.4 Theorem:

(1) ([Sh-c],III,4.13, p.77) If T is stablle, $\varphi(\overline{x}, \overline{y})$ a formula, then there is some natural number $n(\varphi)$ such that for every set of indiscernibles $\mathbf{I}$ and parameters $\mathbf{c}$, either $|\{\mathbf{a} \in \mathbf{I} : \models \varphi(\mathbf{a}, \mathbf{c})\}| < n(\varphi)$ or $|\{\mathbf{a} \in \mathbf{I} : \models \neg\varphi(\mathbf{a}, \mathbf{c})\}| < n(\varphi)$

(2) ([Sh-c],III, 1.5 p.89) Let I be an infinite set of indicernibles. $\mathrm{Av}_\Delta(\mathbf{I}, A)$, the average of $\mathbf{I}$ over the set of formulas Δ and over the set of parameters A, is the set of all fomulas $\varphi(\overline{x}, \mathbf{c})$ such that $\varphi \in \Delta$, $\mathbf{c} \in A$ and $\models \varphi(\mathbf{a}, \mathbf{c})$ for all but finitely many $\mathbf{a} \in \mathbf{I}$.

(3) ([Sh-c],III,, 3.5 p. 104) If $\mathbf{J}$ is an indiscernible set over A, B is any set, then there is $\mathbf{I} \subseteq \mathbf{J}$ such that $\mathbf{J} - \mathbf{I}$ is indiscernible over $A \cup B \cup \bigcup \mathbf{I}$, and

(a) $|\mathbf{I}| \leq \kappa(T) + |B|$.

(b) If $|B| < \mathrm{cf}(\kappa(T))$ then $|\mathbf{I}| < \kappa(T)$. (The interesting case is when $|\mathbf{J}|$ is large enough with relation to $|B|$.)

(4) ([Sh-c],III,4.17 p.117) If $\mathbf{I}, \mathbf{J}$ are infinite indiscernible sets, and $\mathrm{Av}(\mathbf{I}, \bigcup \mathbf{I}) = \mathrm{Av}(\mathbf{J}, \bigcup \mathbf{I})$ and $\mathrm{Av}(\mathbf{J}, \bigcup \mathbf{J}) = \mathrm{Av}(\mathbf{I}, \bigcup \mathbf{J})$ then $\mathrm{Av}(\mathbf{I}, C) = \mathrm{Av}(\mathbf{J}, C)$.

(5) ([Sh-c] p.112),III, 4.9 If Δ is finite and $p \in S^m(|M|)$, then for every type $q \in S^m(B)$ extending p which does not fork over M there is an infinite Δ-indiscernible set $\mathbf{I} \subseteq M$ such that $q = \mathrm{Av}_\Delta(\mathbf{I}, B)$.

(5) ([Sh-c],III,1.12 p.92) For every β and set A there is an indiscernible sequence $\mathbf{I}$ over A and based on A (= for every B $\mathrm{Av}(\mathbf{I}, B)$ does not fork over A) such that $\mathbf{b} \in \mathbf{I}$.

The interested reader is welcome to inquire [Sh-c] for more details and/or results.

We recall some combinatorics which we need:

2.5 Definition:

Suppose λ is a regular uncountable cardinal, and $S \subseteq \lambda$ is stationary.

(1) A sequence $\overline{C} = \langle c_\delta : \delta \in S \rangle$ is a *club guessing sequence on S* if c_δ is a club (=closed unbounded subset) of δ for every $\delta \in S$ and for every club E of λ the set $S_E = \{\delta \in S : c_\delta \subseteq E\}$ is stationary.

(2) For $\overline{C}$ as in (1), $\mathrm{id}^a(\overline{C}) \stackrel{\text{def}}{=} \{A \subseteq S :$ there is a club $E \subseteq \lambda$ such that $\delta \in A \cap S \Rightarrow c_\delta \not\subseteq E\}$ is a λ-complete proper ideal.

(3) a sequence $\langle P_\delta : \delta \in S \rangle$, $S \subseteq \lambda$, is a *weak* club gueesing sequence if $P_\delta = \langle c_i^\delta : i < i(\delta) \rangle$, $i(\delta) \leq \lambda$, for each $i < i(\delta)$, c_i^δ is a club of δ and for every club $E \subseteq \lambda$, the set $S_E = \{\delta \in S : (\exists i < i(\delta))(c_i^\delta \subseteq E)\}$ is stationary. The existence of a weak club

guessing sequence is clearly equivalent to the existence of a sequence $\langle c_\beta : \beta < \lambda \rangle$ such that $c_\delta \subseteq \beta$ and for every club $E \subseteq \lambda$ the set $\{\alpha < \lambda : (\exists \beta)(\sup c_\beta = \alpha) \mathbin{\&} c_\beta \subseteq E\}$ is stationary. We call such a sequence also a weak club guessing sequence.

(4) If $\overline{P} = \langle P_\delta : \delta \in S \rangle$ is a weak club guessing sequence, then $\mathrm{id}^a(\overline{P}) = \{A \subseteq S : (\exists E)(E \subseteq \lambda$ is club such that $(\forall \delta \in E \cap S)(\neg \exists i < i(\delta))(c_i^\delta \subseteq E)\}$ is a proper λ-complete ideal. $\smile_{2.5}$

2.6 Fact:

(1) If $\lambda = \mathrm{cf}\lambda > \aleph_1$ then there are a stationary $S \subseteq \lambda$ and a club guessing sequence $\overline{C} = \langle c_\delta : \delta \in S \rangle$ on S such that for every $\delta \in S$ the order type of c_δ is ω.

(2) If κ is regular and uncountable, $\kappa^+ < \lambda = \mathrm{cf}\lambda$, then there are sequences $\overline{C} = \langle c_\delta : \delta \in S \rangle$, $S \subseteq \lambda$ stationary, and $\langle P_\alpha : \alpha \in \lambda \rangle$ such that $\mathrm{otp}\, c_\delta = \kappa$, $\sup c_\delta = \delta$, $\overline{C}$ is a club guessing sequence, $|P_\alpha| < \lambda$ and for every $\delta \in S$ and $\alpha \in \mathrm{nacc}\, c_\delta$, $c_\delta \cap \alpha \in P_\alpha$.

(3) Suppose $\mu^+ < \lambda = \mathrm{cf}\lambda$ and $\mathrm{cf}\mu \leq \mu$. Then there is a weak club guessing sequence $\overline{C} = \langle c_\beta : \beta < \lambda \rangle$ such that for every $\beta < \lambda$ the order type of c_β is μ and for every $\alpha < \lambda$ the set $\{c_\beta \cap \alpha\}$ has cardinality smaller than λ.

Proof: See [Sh-g], [Sh 365] or the appendix to [KjSh 409] for a proof of (1) and see [Sh 420]§1 for the proofs of (2) and (3). $\smile_{2.6}$

On covering numbers see [Sg-g]. We refer the reader to [KjSh 409],§4, for a detailed exposition of covering numbers of singular cardinals, in particular to Theorem 4.5 there. Here we quote

2.7 Definition: $\mathrm{cov}(\lambda, \mu, \theta, \sigma)$ is the minimal size of a family $A \subseteq [\lambda]^{<\mu}$ which satisfies that for all $X \in [\lambda]^{<\theta}$ there are less than σ members of A whose union covers X.

2.8 Theorem: If μ is not a fix point of the second order, i.e. $|\{\lambda < \mu : \lambda = \aleph_\lambda\}| = \sigma < \mu$, and $\sigma + \mathrm{cf}\mu < \kappa < \mu$, then $\mathrm{cov}(\mu, \kappa^+, \kappa^+, \kappa) = \mu$. $\smile_{2.8}$

For example, for every $\aleph_n$ it is true that $\mathrm{cov}(\aleph_\omega, \aleph_{n+1}, \aleph_{n+1}, \aleph_n) = \aleph_\omega$.

§3. The Machinery

In this section T denotes a first order, countable, stable but unsuperstable theory.

3.1 Definition: Suppose that $N \models T$, and that $\overline{N} = \langle N_i : i < \lambda \rangle$ is given, $N_i \prec N_{i+1} \prec N$, $||N_i|| < \lambda$ and $N_j = \cup_{i<j} N_j$ for limit j. $\overline{N}$ is called a *representation* of N.

Suppose $c \subseteq \lambda$ is of limit order type and is enumerated (continuously and increasingly) by $\langle \alpha_i : i < \mathrm{otp}\, c \rangle$.

(0) for an element $\mathbf{a} \in N$, $\mathrm{Inv}_{\overline{N}}(\mathbf{a}, c) \overset{\mathrm{def}}{=} \{\alpha_i : \mathbf{a} \underset{N_{\alpha_i}}{\overset{N}{\uplus}} N_{\alpha_{i+1}}\}$;

(1) $\mathrm{Inv}^*_{\overline{N}}(\mathbf{a}, c) \overset{\mathrm{def}}{=} \{i : \mathbf{a} \underset{N_{\alpha_i}}{\overset{N}{\uplus}} N_{\alpha_{i+1}}\}$

(2) $P(\overline{N}, c) \overset{\mathrm{def}}{=} \{\mathrm{Inv}_{\overline{N}}(\mathbf{a}, c) : \mathbf{a} \in N\}$

(3) $P^*(\overline{N}, c) \overset{\mathrm{def}}{=} \{\mathrm{Inv}^*_{\overline{N}}(\mathbf{a}, c) : \mathbf{a} \in N\}$

3.2 Definition: Suppose that $\overline{C} = \langle c_\delta : \delta \in S \rangle$ is a club guessing sequence on some staitonary $S \subseteq \lambda$ and that that $\overline{N}$ is as in 3.1.

(4) $\mathrm{INV}^a(\overline{N}, \overline{C})$ is the sequence $\langle P(\overline{N}, c_\delta) : \delta \in S \rangle$ modulo the ideal $\mathrm{id}^a(\overline{C})$.

(4) Assuming that for all $\delta \in S$, $\mathrm{otp}\, c_\delta$ is some fixed $\delta(*)$, $\mathrm{INV}^b(\overline{N}, \overline{C}) \overset{\mathrm{def}}{=} \{Y \subseteq \delta(*) :$

$\{\delta \in S : Y \in P^*(\overline{N}, c_\delta)\} \notin \mathrm{id}^a(\overline{C})\}$

(5) Under the assumptions of (4), $\mathrm{INV}^c_{(\overline{N},\overline{C})} \stackrel{\mathrm{def}}{=} \{Y \subseteq \delta(*) : \{\delta \in S : Y \notin P^*(\overline{N}, c_\delta)\} \in \mathrm{id}^a(\overline{C})\}$

3.3 Remark: We shall not use 3.2 much, but our results can be interpreted as saying that those invariants do not depend on the representation $\overline{N}$ but just on the model N, and that we can prove non universality by just looking at one of these invariants.

3.4 Lemma: Suppose $\lambda = \mathrm{cf}\lambda > \aleph_1$, N, M are models of T, $||N|| = ||M|| = \lambda$ with given representations $\overline{N}, \overline{M}$. If $h : N \to M$ is an elementary embedding, then there is some club $E \subseteq \lambda$ such that for every $\mathbf{a} \in N$ and $c \subseteq E$, $\mathrm{Inv}_{\overline{N}}(\mathbf{a}, c) = \mathrm{Inv}_{\overline{M}}(h(\mathbf{a}), c)$.

Proof: Let $E_h = \{i < \lambda : \mathrm{ran}(h^{-}N_i) \subseteq M_i\}$. Clearly E_h is a club of λ. Dnote by N^*_i the set $\mathrm{ran}(h^{-}N_i)$. So for $\delta \in E_h$, N^*_i is the universe of an elementary submodel of M_i. Denote by N^* the image of N under h.

3.5 Claim: The set $E_1 = \{\delta \in E_h : (\forall \mathbf{a} \in M_\delta)(\mathbf{a} \underset{N_\delta(*)}{\bigcup\!\!\!\!|}\ N^*)\}$ is a club.

Proof: As T is countable, stable but unsuperstable, $\kappa(T) = \aleph_1$. Therefore for every $\mathbf{a} \in M$, there is a countable set $A_{\mathbf{a}} \subseteq N^*$ such that $\mathbf{a} \underset{A_{\mathbf{a}}}{\bigcup\!\!\!\!|}\ N^*$. Let $i(\mathbf{a})$ be the least i such that $A_{\mathbf{a}} \subseteq N^*_i$. For $\alpha \in E_h$ let $j(\alpha)$ be the least $j \in E_h$ such that for all $\mathbf{a} \in [\alpha]^{<\omega}$, $i(\mathbf{a}) \leq j$. $E' = \{\delta \in E_h : \alpha < \delta \Rightarrow j(\alpha) < \lambda\}$ is club. If $\delta \in E'$ and $\mathbf{a} \in [\delta]^{<\omega}$, then $A_{\mathbf{a}} \subseteq N_\delta(*)$. So as $\mathbf{a} \underset{A_{\mathbf{a}}}{\bigcup\!\!\!\!|}\ N^*$, also $\mathbf{a} \underset{N_\delta(*)}{\bigcup\!\!\!\!|}\ N^*$. So $E' \subseteq E_1$. E_1 is closed, for if $\delta \in \mathrm{acc}\, E_1$ and $\mathbf{a} \in M_\delta$, then there is some $\alpha < \delta$ such that $\sup \mathbf{a} < \alpha$ and $\alpha < i < \delta$ such that $i \in E_1$. $\mathbf{a} \underset{N^*_i}{\bigcup\!\!\!\!|}\ N^*$, therefore $\mathbf{a} \underset{N_{\delta(*)}}{\bigcup\!\!\!\!|} N^*$. $\ddot\smile_{3.5}$

Let $\langle \alpha_i : i < \lambda \rangle$ be its increasing enumeration of E_1. We show that for every $\mathbf{a} \in N$ and $i < \lambda$, $\mathbf{a} \underset{N_{\alpha_i}}{\overset{N}{\bigcup\!\!\!\!\!-}} N_{\alpha_{i+1}} \Leftrightarrow h(\mathbf{a}) \underset{M_{\alpha_i}}{\overset{M}{\bigcup\!\!\!\!\!-}} M_{\alpha_{i+1}}$.

As $E \subseteq E_1$, for every α_i and $\mathbf{b} \in M_\delta$,

$$\mathbf{b} \underset{N^*_{\alpha_i}}{\overset{M}{\bigcup\!\!\!\!|}} N^*$$

This can be written as

$$\star_1 \qquad M_{\alpha_i} \underset{N^*_{\alpha_i}}{\overset{M}{\bigcup\!\!\!\!|}} N^*$$

By monotonicity, for a given $\mathbf{a} \in N$,

$$\star_2 \qquad M_{\alpha_i} \underset{N^*_{\alpha_i}}{\overset{M}{\bigcup\!\!\!\!|}} h(\mathbf{a})$$

Symmetry of non-forking gives

$$\star_3 \qquad h(\mathbf{a}) \underset{N^*_{\alpha_i}}{\overset{M}{\bigcup\!\!\!\!|}} M_{\alpha_i}$$

Suppose now, first, that

$$\mathbf{a} \underset{N_{\alpha_i}}{\overset{N}{\bigcup\!\!\!\!|}} N_{\alpha_{i+1}}.$$

As h is an elementary embedding,

$$\star_4 \qquad h(\mathbf{a}) \underset{N^*_{\alpha_i}}{\overset{M}{\bigcup\!\!\!\!|}} N^*_{\alpha_{i+1}}$$

By $\star_3$, $h(\mathbf{a}) \underset{N^*_{\alpha_{i+1}}}{\bigcup\!\!\!\!\bigcup} M_{\alpha_{i+1}}$ (we omit M, in which we work from now on). By $\star_4$, and the transitivity of non forking, $h(\mathbf{a}) \underset{N^*_{\alpha_i}}{\bigcup\!\!\!\!\bigcup} N_{\alpha_{i+1}}$. By monotonicity, $h(\mathbf{a}) \underset{M_{\alpha_i}}{\bigcup\!\!\!\!\bigcup} M_{\alpha_{i+1}}$.

For the other direction, suppose that $h(\mathbf{a}) \underset{M_{\alpha_i}}{\bigcup\!\!\!\!\bigcup} M_{\alpha_{i+1}}$. By monotonicity, $h(\mathbf{a}) \underset{M_{\alpha_i}}{\bigcup\!\!\!\!\bigcup} N^*_{\alpha_{i+1}}$. By $\star_3$ and the transitivity of non forking, $h(\mathbf{a}) \underset{N^*_{\alpha_i}}{\bigcup\!\!\!\!\bigcup} N_{\alpha^*_{i+1}}$, which is what we want. $\smile_{3.4}$

3.6 Corollary: Suppose $\overline{N}$ and $\overline{M}$ are as above and that $h : N \to M$ is an elementary embedding. Let E be the club given by the previous lemma. If $c \subseteq E$ then

(1) for every $\mathbf{a} \in N$, $\mathrm{Inv}_{\overline{N}}(\mathbf{a}, c) = \mathrm{Inv}_{\overline{M}}(h(\mathbf{a}), c)$;

(2) $P(\overline{N}, c) \subseteq P(\overline{M}, c)$.

We will need a slight generalization of 3.4:

3.7 Lemma: Suppose $N \models T$ is with universe λ, $\overline{N}$ is a representation of N. Suppose $L \prec M$ are models of T, L is of cardinality λ, its universe is B and $\overline{L}$ is a representation. If $h : N \to M$ is an elementary embedding, then there is some club $E \subseteq \lambda$ such that for every $c \subseteq E$ and $\mathbf{a} \in h^{-1}(B)$, $\mathrm{Inv}_{\overline{N}}(\mathbf{a}, c) = \mathrm{Inv}_{\overline{L}}(h(\mathbf{a}), c)$.

Proof: Denote by N^*_i, N^* the images of N_i, N under h respectively. Let $A_i = |N^*_i| \cap B$. Let $A = \cup A_i$. We prove

3.8 Claim: There is a club $E_1 \subseteq \lambda$ such that $i \in E$ implies $N^*_i \underset{A_i}{\bigcup\!\!\!\!\bigcup} A$ and $L_i \underset{A_i}{\bigcup\!\!\!\!\bigcup} A$

Proof: Same as in 3.5. $\smile_{3.8}$

For the rest of the proof, show, precisely as in 3.5, that

$$h(\mathbf{a}) \underset{B_{\alpha_i}}{\bigcup\!\!\!\!|}\, B_{\alpha_{i+1}} \Leftrightarrow h(\mathbf{a}) \underset{A_{\alpha_i}}{\bigcup\!\!\!\!|}\, A_{\alpha_{i+1}} \Leftrightarrow h(\mathbf{a}) \underset{N^*_{\alpha_i}}{\bigcup\!\!\!\!|}\, N^*_{\alpha_{i+1}}$$

When $\langle \alpha_i : i < \operatorname{otp} c\rangle$ is the enumeration of c. $\smile$ 3.7

3.9 Lemma: (the construction Lemma) Let λ be uncountable and regular. Suppose that $\overline{C}$ is a club guessing sequence on some stationary $S \subseteq \lambda$ and for every $\delta \in S$, $\operatorname{otp} c_\delta = \mu$ for some fixed μ with $\operatorname{cf}\mu = \aleph_0$. Suppose $Y \subseteq \mu$ is a given set of order type ω. Then there is a model $M \models T$ of cardinality λ and a representation $\overline{M}$ such that for every $\delta \in S$, $Y \in P^*(\overline{N}, c_\delta)$.

Proof: We work in the monster model, C. By $\kappa(T) > \aleph_0$, there is some $\mathbf{b}$ and M with the property that for every finite set A, $\mathbf{b} \overset{C}{\underset{A}{\not\bigcup}} M$. Pick by induction on n a finite sequence $\mathbf{a}_n$ such that

(i) $\mathbf{a}_n$ is a proper initial segment of $\mathbf{a}_{i+1}$;

(ii) $\mathbf{b} \underset{\mathbf{a}_n}{\not\bigcup} \mathbf{a}_{n+1}$.

Let $\mathbf{a}_0 = \langle\rangle$. The induction step: as $\mathbf{b} \underset{\mathbf{a}_n}{\not\bigcup} M$, by the choice of $\mathbf{b}$, and the finite character of forking, there is some finice $\mathbf{c} \in M$ such that $\mathbf{b} \underset{\mathbf{a}_n}{\not\bigcup} \mathbf{c}$. Let $\mathbf{a}_{n+1} = \mathbf{a}_n\widehat{\ }\mathbf{c}$. By monotonicity, $\mathbf{b} \underset{\mathbf{a}_n}{\not\bigcup} \mathbf{a}_{n+1}$.

Now, we know that $\mathbf{b} \underset{\cup_n \mathbf{a}_n}{\bigcup\!\!\!\!|}\, \cup_n \mathbf{a}_n$. By the existence of non-forking extensions, we may assume that $\mathbf{b} \underset{\cup_n \mathbf{a}_n}{\bigcup\!\!\!\!|}\, M$.

We construct now by induction on $i < \lambda$ a continuous increasing chain of models with the following properties:

(1) $\alpha_0 = \omega$ and $\alpha_{i+1} = \alpha_i + |\alpha_i|$; if i is limit, $\alpha_i = \sup_{j<i} \alpha_j$.

(2) The universe of N_i is α_i. $N_i \models T$ and $N_i \prec N_{i+1}$. If i is limit, $\overline{N}_i$ is the representation $N_i = \bigcup_{j<i} N_j$.

(3) For every $\eta \in {}^{<\omega}i$, strictly increasing, $\mathbf{a}_\eta \in N_i$. $\mathrm{tp}(\mathbf{a}_{\eta\restriction 0}\frown\mathbf{a}_{\eta\restriction 1}\frown\cdots\frown\mathbf{a}_{\eta\restriction \lg\eta}) = \mathrm{tp}(\mathbf{a}_0\frown\mathbf{a}_1\frown\cdots\frown\mathbf{a}_{\lg\eta})$;

(4) If $\eta = \nu\frown\langle i\rangle$, then $\mathbf{a}_\eta \underset{\mathbf{a}_\nu}{\overset{N_{i+1}}{\bigcup\!\!\!\!|}} N_i$.

(5) If $i = \delta \in S$ satisfies that for every $j < \delta$, $\alpha_j < \delta$, then there is some element $\mathbf{b} \in N_{\delta+1}$ such that $\{\alpha_i \in c_\delta : \mathbf{b} \underset{N_{\alpha_i}}{\overset{N_{\delta+1}}{\bigcup\!\!\!\!\!-}} N_{\alpha_{i+1}}\} = Y$.

At the induction stage, when given $\nu \in {}^{<\omega}i$ and increasing, denoting by η the sequence $\nu\frown i$, we should say who $\mathbf{a}_\eta$ is. There is an elementary mapping h such that for every $k < \lg\eta$, $h(\mathbf{a}_k) = \mathbf{a}_{\nu\restriction k}$. Therefore $h[\mathrm{tp}(\mathbf{a}_{\lg\eta}, \cup_{l<\lg\eta})]$ is a complete type over $\bigcup_{l<\lg\eta} \mathbf{a}_{\eta\restriction l}$. By the existence of non forking completion of a partial non forking type, there is some type p over N_i which does not fork over $\bigcup_{l<\lg\eta} \mathbf{a}_{\eta\restriction l}$. Let $\mathbf{a}_\eta$ realize p in N_{i+1}.

In case $i = \delta \in S_0^\lambda$ is as in (5), we should also take care of (5).

Let $Y(\delta) = \langle \alpha^d_{i(n)} : i \in Y\rangle$. Let $\eta = \langle \alpha^\delta_{i(n)} : n \in \omega\rangle$, and let h be an elementary mapping such that $h(\mathbf{a}_l) = \mathbf{a}_{\eta\restriction l}$. Then in $N_{\delta+1}$ we add an element $\mathbf{b}_\delta$ which realizes $h(\mathrm{tp}(\mathbf{b}, \cup_{l<\omega}\mathbf{a}_l))$ and $\mathbf{b}_\delta \underset{\cup_{l<\omega}\mathbf{a}_{\eta\restriction l}}{\overset{N_{\delta+1}}{\bigcup\!\!\!\!|}} N_\delta$ (due the existence of non forking extentions of types). We have to show

3.10 Lemma: $\{\alpha_i : \mathbf{b} \underset{N_{\alpha_i}}{\overset{N_{\delta+1}}{\bigcup\!\!\!\!\!-}} N_{\alpha_{i+1}}\} = Y(\delta)$.

We first need

3.11 Lemma: If $\eta^\delta(k) = \alpha_m$, then $\cup_{l<\omega}\mathbf{a}_{\eta\restriction l} \underset{a_{\eta\restriction k}}{\bigcup\!\!\!\!|} N_{\alpha_m}$.

Proof: By induction on r, $\quad k \leq r < \omega$, we see that $\cup_{l \leq r} \mathbf{a}_{\eta^\frown l} \underset{\mathbf{a}_{\eta}{}^\frown k}{\biguplus} N_{\alpha_m}$.

$r = k$: $\cup_{l \leq k} \mathbf{a}_l \underset{\mathbf{a}_{\eta^\frown k}}{\biguplus} N_{\alpha_m}$ is trivial, as $\cup_{l \leq k} \mathbf{a}_{\eta^\frown l} = \mathbf{a}_{\eta^\frown k}$.

$r+1$: By the induction hypothesis,

$$(a) \qquad \cup_{l \leq r} \mathbf{a}_{\eta^\frown l} \underset{\mathbf{a}_{\eta^d{}^\frown k}}{\biguplus} N_{\alpha_m}$$

By the construction,

$$(b) \qquad \mathbf{a}_{\eta^\frown (r+1)} \underset{\mathbf{a}_{\eta^\frown r}}{\biguplus} N_{\eta(r)}$$

Monotonicity gives

$$(c) \qquad \mathbf{a}_{\eta^\frown (r+1)} \underset{\mathbf{a}_{\eta}{}^\frown (r)}{\biguplus} N_{\alpha_m}$$

(a) and (c) give

$$\mathbf{a}_{\eta^\frown (r+1)} \underset{\mathbf{a}_{\eta^\frown k}}{\biguplus} N_{\alpha_m}$$

By the finite character of non forking Lemma 3.11 is proved. $\smile_{3.11}$

Suppose now, first, that $\alpha_i \notin Y(\delta)$. Let $\eta(k-1) < \alpha_i < \eta(k)$. We know that $\mathbf{b} \underset{\cup_{l<\omega} \mathbf{a}_{\eta}{}^\frown l}{\biguplus} N_\delta$. So by monotonicity $\mathbf{b} \underset{\cup_{l<\omega} \mathbf{a}_{\eta}{}^\frown l}{\biguplus} N_{\alpha_{i+1}}$. By 3.11, $\cup_{l<\omega} \mathbf{a}_{\eta}{}^\frown l \underset{\mathbf{a}_{\eta^\frown (k)}}{\biguplus} N_{\eta(k)}$. By monotonicity and the fact that $\mathbf{a}_{\eta^\frown k} \in N_{\alpha_i}$ we get $\cup_{l<\omega} \mathbf{a}_{\eta^\frown l} \underset{N_{\alpha_i}}{\biguplus} N_{\alpha_{i+1}}$. By transitivity of non forking, $\mathbf{b}_\delta \underset{N_{\alpha_i}}{\biguplus} N_{\alpha_{i+1}}$, namely $\alpha_i \notin \mathrm{Inv}_{\overline{N}_{\delta+1}}(\mathbf{b}_\delta, c_\delta)$.

For the other direction: suppose that $\alpha_i \in Y(\delta)$ and that $\alpha_i = \eta(k)$. We know that $\mathbf{b} \underset{\mathbf{a}_\eta{}^\frown k}{\not\!\perp} \mathbf{a}_\eta{}^\frown(k+1)$. Therefore by monotonicity, $\mathbf{b} \underset{\mathbf{a}_\eta{}^\frown k}{\not\!\perp} N_{\alpha_{m+1}}$. By 3.11, as in the previous case, $\mathbf{b} \underset{\mathbf{a}_\eta{}^\frown k}{\perp} N_{\alpha_i}$. Suppose to the contrary that $\mathbf{b} \underset{N_{\alpha_i}}{\perp} N_{\alpha_{i+1}}$. Then by transitivity we get $\mathbf{b} \underset{\mathbf{a}_\eta{}^\frown k}{\perp} N_{\alpha_{i+1}}$ — a contradiction. $\smile_{3.10,3.9}$

We will need also

3.12 Lemma: (the second construction Lemma)

Suppose λ is uncountable regular, and $\overline{C} = \langle c_\beta : \beta < \lambda \rangle$ is a weak club guessing sequence, such that for every $\beta < \lambda$ the order type of c_β is some fixed μ with $\mathrm{cf}\mu = \aleph_0$. Suppose that $Y(*) \subseteq \mu$ is given and of order type ω_0. There is some model M of T with universe λ and representation $\overline{M}$ such that $Y(*) \in P^*(\overline{M}, c_\beta)$ for every $\beta < \lambda$.

Proof: The proof is essentially the same as this of 3.9. The only difference is in the construction: we add the witness not in stage $\delta + 1$ but in stage $\beta + 1$, where $\sup c_\beta = \delta$.

$\smile_{3.12}$

3.13 Lemma:(the third construction lemma) Suppose T is a stable first order theory, $\mathrm{cf}\lambda = \lambda \geq |T|$, $\mathrm{cf}\kappa = \kappa < \kappa(T)$ and $\overline{C}, \overline{P}$ are as in 2.6 (2). Suppose $Y(*) \subseteq \kappa$ is given. Then there is a model $M \models T$ of cardinality λ and representation $\overline{M}$ such that fore every $\delta \in S$, $Y(*) \in P^*(\overline{M}, c_\delta)$.

Proof: We work in a monster model C and construct a sequence $\langle \mathbf{a}_\alpha : \alpha < \kappa \rangle$ and an element $\mathbf{b}$ such that $\mathbf{a}_\alpha$ is an infinite sequence, increasing with α, namely $\mathbf{a}_\alpha$ is is a proper initial segment of $\mathbf{a}_\beta$ whenever $\alpha < \beta$ and $\mathbf{b} \underset{\mathbf{a}_\alpha}{\bigcup} \mathbf{a}_{\alpha+1}$ for all $\alpha < \kappa$. This is possible because $\kappa < \kappa(T)$. Without loss of generality, $\mathbf{b} \underset{\cup_\alpha \mathbf{a}_\alpha}{\bigcup} \cup_\alpha \mathbf{a}_\alpha$. Let $Y(*)_\delta \subseteq c_\delta$ for $\delta \in S$ be the isomorphic image of $Y(*)$ under the enumeration of c_δ. We may assume, without loss of generality, that for every $\alpha \in \operatorname{nacc} c_\delta$ for $\delta \in S$, $Y(*)_\delta \cap \alpha \in P_\alpha$. Construct by induction on $\alpha < \lambda$ an elementary chain of models M_α with the following properties:

(1) : for every $\eta \in P_\alpha$, $\eta \in [\alpha]^{<\kappa}$ there is a sequence $\overline{\mathbf{a}}_\eta$ such that $\mathbf{a}_\eta(\beta) \in N_\alpha$ and $\operatorname{tp}(\cdots{}^\frown \mathbf{a}_\eta(\beta)^\frown \cdots) = \operatorname{tp}(\cdots{}^\frown \mathbf{a}(\beta)^\frown \cdots)$.

(2) If $\alpha = \delta \in S$ then there is an element $\mathbf{a}_\delta \in N_{\delta+1}$ such that $\operatorname{Inv}^*_{\overline{M}_{\delta+1}}(\mathbf{a}_\delta, c_d) = Y(*)$.

We let the reader verify that the analogs of 3.10 and 3.11 are true. $\smile_{3.13}$

§4. The Main Results

4.1 Theorem: Suppose T is a complete, countable, stable but unsuperstable first order theory, and that $\aleph_1 < \lambda = \operatorname{cf}\lambda < 2^{\aleph_0}$. Then $\lambda \notin \operatorname{Univ}(T)$. Furthermore, for every family $\{M_i\}_{i\in I}$, $M_i \models T$, $||M_i|| = \lambda$ and $|I| < 2^{\aleph_0}$ there is a model $N \models T$, $||N|| = \lambda$ and N is not elementarily embeddable into M_i for all $i \in I$.

Proof:

Clearly, it is enough to prove the "furthermore" part of the theorem. Suppose that $\{M_i\}_{i\in I}$ is a family of less then $2^{\aleph_0}$ moels of T, each of cardinality λ. Let $\overline{N}_i$ represent M_i. Use 2.6 part 1 to pick some club guessing sequence $\overline{C}$ on $S \subseteq \lambda$ with all c_δ of order type ω. Pick some set $Y(*) \subseteq \omega$ such that $Y(*) \notin \bigcup_{i\in I,\delta\in S} P^*(\overline{N}_i, c_\delta)$. This is possible, because the size of this union is smaller than $2^{\aleph_0}$. Use the construction lemma to get a model M of size λ and a representation $\overline{M}$ such that for every $\delta \in S$, $Y(*) \in P^*(\overline{M}, c_\delta)$. Suppose to the contrary that for some $i \in I$, $h : M \to M_i$ is an elementary embedding. By 3.4, there is a club $E \subseteq \lambda$ such that for every $\delta \in S$ such that $c_\delta \subseteq E$, $P(\overline{M}, c_\delta) \subseteq P(\overline{M}_i, c_\delta)$. Pick some $\delta_0 \in S_E$. So $Y(*) \in P^*(\overline{M}, c_{\delta_0}) \subseteq P^*(\overline{M}_i, c_{\delta_0})$ – a contradiction to $Y(*) \notin \bigcup_{i \in I, \delta \in S} P(\overline{M}_i, c_\delta)$. $\smile_{4.1}$

4.2 Theorem: Suppose $2^{\aleph_0} < \lambda = \mathrm{cf}\lambda < \lambda^{\aleph_0}$ and there are no μ_n such that $\lambda = (\Sigma \mu_n^{\aleph_0})^+$. Then if T is a stable unsuperstable theory, $|T| \leq \lambda$, then $\lambda \notin \mathrm{Univ}(T)$. Furthermore, for every family $\{M_i\}_{i \in I}$, $M_i \models T$, $||M_i|| = \lambda$ and $|I| < \lambda^{\aleph_0}$ there is a model $M \models T$, $||N|| = \lambda$ such that M is not elementarily embeddable into M_i for all $i \in I$.

Proof: Again, the "furthermore" part is enough.

Let μ be the least cardinal such that $\mu^{\aleph_0} > \lambda$. Since λ is uncountable and regular, $\lambda^{\aleph_0} = \cup_{\alpha < \lambda} \alpha^{\aleph_0}$. If for every cardinal $\kappa < \lambda$ $\kappa^{\aleph_0} = \kappa$, we should have had $\lambda^{\aleph_0} = \lambda$. Therefore μ is stricly smaller than λ. If $\mathrm{cf}\mu > \aleph_0$, then $\mu^{\aleph_0} = \cup_{\alpha < \mu} \alpha^{\aleph_0}$. By the minimality of μ, for every $\alpha < \mu$, $\alpha^{\aleph_0} \leq \lambda$. This contradicts $\mu^{\aleph_0} > \lambda$. We conclude that $\mathrm{cf}\mu = \aleph_0$. Lastly, if $\lambda = \mu^+$, then, μ being of cofinality ω, there would be μ_n increasing to μ such that $\mu_n^{\aleph_0} < \mu$. This contradicts the assumptions on λ.

Use 2.6 part 3 to pick some weak club guessing sequence $\overline{C} = \langle c_\beta : \beta < \lambda \rangle$ with all c_δ of order type μ. Suppose to the contrary that $\{M_i\}_{i \in I}$ is as stated above. By the assumption $\lambda < \mu^{\aleph_0}$, we can find some $Y(*) \subseteq \mu$ of order type ω such that $Y(*) \notin \bigcup_{i \in I, \beta < \lambda} P^*(\overline{M}_i, c_\beta)$. By the construction lemma there is some model M and representation $\overline{M}$ such that for every $\beta < \lambda$, $Y(*) \in P^*(\overline{M}, c_\beta)$. Suppose to the contrary that for some $i \in I$ there were an elementary embedding $h : M \to M_i$. By 3.4 there is a club $E \subseteq \lambda$ such that if $c_\beta \subseteq E$ then $P^*(\overline{M}, c_\beta) \subseteq P^*(\overline{M}_i, c_\beta)$. As $\overline{C}$ is a weak club guessing sequnce there is such a c_β, and the contradiction to the choice of $Y(*)$ follows as before. $\smile_{4.2}$

4.3 Theorem: Assume T is first order complete countable stable unsuperstable theory. Suppose μ is singular, and there is some $\sigma < \mu$ and $\kappa < \mu$ such that $\sigma^+ < \kappa = \mathrm{cf}\kappa$ and $\sigma^{\aleph_0} > \mathrm{cov}(\mu, \kappa^+, \kappa^+, \kappa)$, then there is no model of T in cardinality μ into which all models of T of cardinality κ are elementarily embeddable. In particular $\mu \notin \mathrm{Univ}(T)$.

Proof: We may assume that $\mathrm{cf}\sigma = \aleph_0$. Suppose to the contrary that $M \models T$ is of cardinality μ and that every $N \models T$ of cardinality κ is elementarily embeddable into it. Without loss of generality the universe of M is μ. Let $\theta \stackrel{\text{def}}{=} \mathrm{cov}(\mu, \kappa^+, \kappa^+, \kappa)$, and let $\langle B_i : i < \theta \rangle$ demonstrate the definition of θ. Without loss of generality, each B_i is the universe of some $M_i \prec M$ of cardinality κ. By 2.6 part 3 pick some weak club guessing sequence $\overline{C}$ with all c_β of order type σ. Pick a presentation $\overline{M}_i$ for every M_i. Pick some $Y(*) \subseteq \mu$ of order type ω such that $Y(*) \notin \bigcup_{i<\theta, \beta<\kappa} P^*(\overline{M}_i, c_\delta)$, and use 3.9 to construct a model $N \models T$ of cardinality κ presetation $\overline{N}$ such that for every $\beta < \kappa$, $Y(*) \in P^*(\overline{M}, c_\beta)$. For every $\beta < \kappa$ there is some element $\mathbf{a}_\beta$ such that $\mathrm{Inv}^*_{\overline{N}}(\mathbf{a}_\beta, c_\beta) = Y(*)$. Suppose that $h : N \to M$ is an elementary embedding. There is some set of indices $X \subseteq \theta$ such that $|X| < \kappa$ and $\mathrm{ran}h \subseteq \cup_{i \in X} B_i$. Since $\mathrm{id}^a(\overline{C})$ is κ-complete, there is a set $S' \subseteq S$, $S' \notin \mathrm{id}^a(\overline{C})$, and a fixed $i_0 \in X$ such that $(\forall \delta \in S')(f(\mathbf{a}_\delta) \in B_{i_0})$. Denote B_{i_0} by B for notational simplicity, and let $L \prec M$ be the model with universe B. Use 3.7 to get the usual contradiction. $\smile_{4.3}$

§5. Generalizations

We wish now to generalise the discusstion of stable unsuperstable theories — namely those T with $\kappa(T) = \aleph_1$ — to stable theories with $\kappa(T)$ arbitrary.

5.1 Theorem: Suppose that T is stable and that $\lambda \geq |T|$ is an uncountable regular cardinal. Suppose that $\kappa < \kappa(T)$, and $\kappa^+ < \lambda < 2^\kappa$. Then $\lambda \notin \mathrm{Univ}(T)$. Furthermore, for every family $\{M_i\}_{i\in I}$ with $|I| < 2^\kappa$ of models of T. each of cardinality λ, there is a model $M \models T$ of cardinality λ which is not elementarily embeddable into M_i for all $i \in I$.

Proof: By 2.6 part 2 there is a club guessing sequence $\overline{C} = \langle c_\delta : \delta \in S\rangle$ on some stationary set $S \subseteq \lambda$ and a sequence $\overline{P} = \langle P_\alpha : \alpha < \lambda\rangle$ such that the order type of each c_δ is κ, for every $\alpha \in S$ and $\alpha \in \mathrm{nacc}\, c_\delta$, $c_\alpha \cap \alpha \in P_\alpha$, and each P_α has cardinality $< \lambda$. Pick a $Y(*) \subseteq \kappa$ such that $Y(*) \notin \bigcup_{\delta\in S, i\in I} \mathrm{Inv}^*(\overline{M}_i, c_\delta)$ and use the third construction lemma to find a model $M \models T$ of cardinality λ and a representation $\overline{M}$ such that for every $\delta \in S$, $Y(*) \in P^*(\overline{M}, c_\delta)$. Suppose to the contrary that there are $i \in I$ and an elementary embedding $h : M \to M_i$. By 3.4 and the fact that $\overline{C}$ guesses clubs we obtain the usual contradiction. $\smile_{5.1}$

5.2 Theorem: Assume $\kappa = \mathrm{cf}(\kappa) < \kappa(T)$, $\kappa \leq \mu$, $\mu^+ < \lambda = \mathrm{cf}(\lambda) < \chi < \mu^\kappa$. Suppose also that T is first order complete and $\kappa < \kappa(T)$. Then there is no model M of T of cardianlity χ universal for models of T of cardianlity λ.

Proof: : similar.

5.3 Remark: This means that $(\lambda, 1, \chi) \notin \mathrm{Univ}_t(T, \prec)$.

§6. A theory with a maximal universality spectrum

In [KjSh 409], 5.5 it was shown that whenever $\lambda \in \mathrm{Univ}(T)$, T a theory having the strict order property, then there is a universal linear order in λ. We prove now an analogous theorem for stable unsuperstable theories.

6.1 Definition: for a cardinal κ,

(1) $T_\kappa = Th(\langle {}^\kappa\omega, E_\zeta\rangle_{\zeta<\kappa})$ where $\eta E_\zeta \nu \Leftrightarrow \eta{}^\frown\zeta = \nu{}^\frown\zeta$ (so T is a first order complete theory of cardianlity κ with $\kappa(T) = \kappa^+$, and in fact is the canonical example of such a theory)

(2) K_κ is the class of all trees of height $\kappa + 1$.

(3) K^+_κ is the class of all trees of height $\kappa + 1$ such that above every member thee is one of height κ

6.2 Fact: $\mathrm{Univ}(K_\kappa, <) \cap (\kappa, \infty) = \mathrm{Univ}(K^+_\kappa, <) \cap (\kappa, \infty) = \mathrm{Univ}(T_\kappa, \prec) \cap (\kappa, \infty) = \mathrm{Univ}(T_\kappa, \leq)$.

Proof: Easy exercise. $\smile_{6.2}$

6.3 Theorem: Suppose that T is stable, $\kappa = \mathrm{cf}\kappa < \kappa(T)$, $\kappa \leq \lambda$ and $|T| < \lambda \in \mathrm{Univ}(T)$. Then $\lambda \in \mathrm{Univ}(T_\kappa)$.

6.4 Remark: Similarly for $\mathrm{Univ}_p, \mathrm{Univ}_t$

Proof: Without loss of generality, $|T| = \kappa$, for this may only increase $\mathrm{Univ}(T)$. So $|T| < \lambda$. Suppose that $N \models T$ is universal in power λ. We define a model M which we shall prove to be universal in λ for K^+_κ. By $\kappa < \kappa(T)$ we can find an element $\mathbf{a}$ and an elementary chain $\langle M_i : i \le \lambda \rangle$ such that $\mathbf{a} \underset{M_i}{\bigcup\!\!\!\!\smile} M_{i+1}$. Let M^+_κ be such that $M_\kappa \prec M^+_\kappa$ and such that there is $\mathbf{I} \subseteq M^+_\kappa$, $|\mathbf{I}| = \lambda$ and $\mathbf{I}$ an indiscrenible set based on M_κ, i.e. $\mathrm{Av}(\mathbf{I}, C)$ extends the type of $\mathbf{a}$ over M_κ but does not fork over M_κ.

The universe of M will be $B = \{p \in S^1(N) : p = \mathrm{Av}(\mathbf{J}, N)$ for some $\mathbf{J},\ \mathbf{J} \subseteq N,\ |\mathbf{J}| = \lambda,\ \mathrm{tp}(\mathbf{J}) = \mathrm{tp}(\mathbf{I})\}$.

6.5 Lemma: $|B| \le \lambda$

Proof: Suppose to the contrary that there are λ^+ types $\langle p_i : i < \lambda^+ \rangle$ and λ^+ indiscernible sets $\mathbf{J}_i \subseteq N$, $|\mathbf{J}_i| = \lambda$ such that $p_i = \mathrm{Av}(\mathbf{J}_i, N)$. Pick a representation $\overline{N} = \langle N_\alpha : \alpha < \lambda \rangle$ of N as an elementary chain. For every $i < \lambda$ there is some $\alpha_i < \lambda$ such that $|\mathbf{J}_i \cap N_{\alpha_i}| \ge \aleph_0$. Also, by 2.4 (3) it follows that there is some $c_i \in \mathbf{J}_i$ which realizes $\mathrm{Av}(\mathbf{J}_i, N_{\alpha_i})$. By the pigeon hole principle there are some $i < j < \lambda$ such that $\alpha_i = \alpha_j$ and $c_i = c_j$. This contradics the fact that $p_i \ne p_j$ by 2.4 (4). $\smile_{6.5}$

By 2.4,(5), for every $p \in S(M_\kappa)$ and a finite set of formulas Δ there is an infinite set of indiscernibles $\mathbf{I} \subseteq M_\kappa$ such that $p = \mathrm{Av}_\Delta(\mathbf{I}, M_\kappa)$. By the stability of T and 2.4,(1), there is some n_Δ such that for every $\mathbf{J} \subseteq \mathbf{I}$ which satisfies $|\mathbf{J}| > 2n_\Delta$,

$$(*) \qquad (\forall \mathbf{b} \in M_\kappa)(\forall \varphi \in \Delta)(\varphi(\overline{x}, \mathbf{b}) \in p \Leftrightarrow |\{\mathbf{c} \in \mathbf{J} : \neg\varphi(\mathbf{c}, \mathbf{b})\}| \le n_\Delta$$

For every $\varphi \in L$ there is a minimal α_φ such that there is a set $\mathbf{J}'_\varphi \subseteq M_{\alpha_\varphi}$ of size

$> 2n_{\{\varphi\}}$ which satisfies $(*)$. Clearly, as $\mathbf{J}'_\varphi$ is finite, α_φ is a non limit ordinal. By 2.4 there is an infinite $\mathbf{J}_\varphi \subseteq M_{\alpha_\varphi}$ with $\mathrm{Av}(\mathbf{J}, M_{\alpha_\varphi}) = p^{\frown} M_{\alpha_\varphi}$. By 2.4, $p^{\frown} M_{\alpha_\varphi} = \mathrm{Av}(\mathbf{J}, M_{\alpha_\varphi})$.

If $\sup\{\alpha_\varphi : \varphi \in L\} = \alpha^* < \kappa$, then p were definable over M_{α^*}, and therefore, by 2.3, would not fork over M_{α^*}, contrary to its choice. We can, therefore, find a sequence of formulas $\langle \varphi_\zeta : \zeta < \kappa\rangle$ with $\langle \alpha_{\varphi_\zeta} : \zeta < \kappa\rangle$ strictly increasing. We shall assume, by re-enumeration, if necessary, that $\alpha_{\varphi_\zeta} = \zeta + 1$.

We define now the relations on our universe B. For every pair $p_1, p_2 \in B$ and φ_ζ the following is an equivalence relation: $p_1 E^\zeta p_2 \Leftrightarrow p_1{}^{\frown}\{\varphi_\xi : \xi \leq \zeta\} = p_2{}^{\frown}\{\varphi_\xi : \xi \leq \zeta\}$. Clearly, these are κ nested equivalence relations. We view the structure we defined as a tree of height $\kappa + 1$ with no short branches, i.e. is a member ofK^+_χ.

To show that M is universal, we will show that for every tree S of size λ with all branches of length $\kappa + 1$ we can find a model of T, N_S of the same cardinality, such that the elementary embedding of N_S into the universal model N will give an embedding of S into M.

We work by induction on $i \leq \kappa$ and for every $\eta \in {}^i\lambda$ construct an elementary embedding $f_\eta : M_i \to C$ with image M_η. We demand:

(1) $\nu \triangleleft \eta \Rightarrow f_\nu \subseteq f_\eta$.

(2) for every $\eta \in {}^i\lambda$ and $\alpha < \lambda$, $M_{\eta^\frown\langle\alpha\rangle} \underset{M_\eta}{\biguplus} \bigcup\{M_{\nu^\frown\langle\beta\rangle} : \nu^\frown\langle\beta\rangle \in {}^i\lambda,\ \nu^\frown\langle\beta\rangle \neq \eta^\frown\langle\alpha\rangle\} \cup M_\eta$

At limit i we take unions. For $i+1$: $M_{\eta^\frown\langle\alpha\rangle}$ exists by 2.3.

For every $\eta \in {}^\kappa\lambda$ extend f_η to $f^+_\eta : M^+_\nu \to C$.

6.6 Claim: Suppose that $\nu \neq \eta \in {}^{\kappa}\lambda$ and that ζ is the least such that $\eta(\zeta) \neq \nu(\zeta)$. Suppose that $N \prec C$ and that $M^+_\eta, M^+_\nu \subseteq N$. Then $\mathrm{Av}(\mathbf{I}_\eta, N) E^\xi \mathrm{Av}(\mathbf{I}_\nu, N) \Leftrightarrow \xi < \zeta$.

Proof: Let $\xi < \kappa$. Let $f \in \mathrm{Aut}(C)$ map M^+_η onto M^+_ν. For simplicity we assume that $f_\eta{}^- M_\zeta = f_\nu{}^- M_\zeta = id$. We know that $M_\eta \underset{M_\zeta}{\overset{C}{\bigcup\!\!\!\!\bigcup}} M_\nu$.

First case: $\zeta > \xi$. $(*)$ gives a definition of $\mathrm{Av}_{\varphi_\xi}(\mathbf{I}, N)$ with set of parameters $\mathbf{J}'_{\varphi_\xi}$. In C this definition gives, with respective sets of parameters $\mathbf{I}_\eta, \mathbf{I}_\nu$, the types p_1, p_2, which extend, respectively, $\mathrm{Av}_{\varphi_\xi}(\mathbf{I}_\eta, M^+_\eta)$ and $\mathrm{Av}_{\varphi_\xi}(\mathbf{I}_\nu, M^+_\nu)$. Let $\mathbf{I}_\eta$ be $\mathbf{I}_1$ and let $\mathbf{I}_\nu$ be $\mathbf{I}_2$.

6.7 Fact: for $l = 1, 2$, $\mathrm{Av}_{\varphi_\xi}(\mathbf{I}_l, C) = \mathrm{Av}_{\varphi_\xi}(\mathbf{J}_{\varphi_\xi}, C)$.

Proof: Suppose to the contary that $\mathbf{c} \in C$ demonstraes otherwise. Then by 2.4, there is someh is $\mathbf{I}'_l \subseteq \mathbf{I}_l$ of size $< \kappa(T)$ such that the set $(\mathbf{I}_l \setminus \mathbf{I}'_l) \cup \{\mathbf{c}\}$ is independent over M_l. Thereforee $\mathbf{c} \underset{M_l}{\bigcup\!\!\!\!\bigcup} M_l + \cup(\mathbf{I}_l \setminus \mathbf{I}'_l)$. By 2.3(4), the type of $\mathbf{c}$ is finitely satisfiable over M_l. There is finite information saying that $\varphi(-, \mathbf{c})$ behaves ddifferently in $\mathrm{Av}_{\varphi_\xi}(\mathbf{J}_{\varphi_\xi}, C)$ than in $\mathrm{Av}|\varphi_\xi(\mathbf{I}_l, C)$. So there is a counterexaample inside M_l — a contradiction. $\smile_{crux}$

By this fact we conclude that $\mathrm{Av}(\mathbf{I}_\eta, C) E^\xi \mathrm{Av}(\mathbf{I}_\nu, C)$.

Second case: $\xi \geq \zeta$. We extend $\mathrm{Av}(\mathbf{I}_\eta, M_\eta)$ to a non-forking extention $p \in S(C)$. So $p \underset{M_\zeta}{\bigcup\!\!\!\!\bigcup} M_\nu$. In particular $P^- \varphi_\xi \underset{M_\zeta}{\bigcup\!\!\!\!\bigcup} M_\nu$. Therefore there is some $\mathbf{J}'_\zeta$ as in $(*)$ — contradiction to $\alpha_\xi = \xi + 1$. $\smile_{ccc}$

Suppose now that S is a given tree in K^+_κ of size λ. Without losssof generality, $S < {}^{\kappa}\lambda$. Pick a model $N_S \prec C$ such that for every $\eta \in S$, $M_\eta \subseteq N$ and such that $||N|| \leq \lambda$. An elementary embedding of N_S into N easily gives an embedding of S into M. $\smile_{6.3}$

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