μ-Complete Souslin Trees on μ^+

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ABSTRACT

We prove that $\mu = \mu^{<\mu}$, $2^\mu = \mu^+$ and "there is a non reflecting stationary subset of μ^+ composed of ordinals of cofinality $< \mu$" imply that there is a μ-complete Souslin tree on μ^+.

Introduction The old problem of the existence of Souslin trees has attracted the attention of many (see [Je] for history). While the $\aleph_1$ case is settled, the consistency of $GCH + SH(\aleph_2)$ is still an open question. Gregory showed in [G] that GCH + "there is a non reflecting stationary set of ω-cofinal elements of ω_2" implies the existence of an $\aleph_2$-Souslin tree. Gregory's result showed that the consistency strength of $GCH + SH(\aleph_2)$ is at least that of the existence of a Mahlo cardinal. Without GCH, the consistency of $CH + SH(\aleph_2)$ is known from [LvSh 104]. In [ShSt 279] the equiconsistency of the existence of a weakly compact cardinal with "every $\aleph_2$-Aronszajn tree is special" is shown. In [ShSt 154] it is shown that under CH, the consistency strength of "there are no $\aleph_1$-complete $\aleph_2$-Souslin trees" is at least that of an inaccessible cardinal.

We show how a Souslin tree which is μ-complete (μ regular) can be constructed on a cardinal μ^+ from a certain combinatorial principle (Theorem 2 below), and then show how this principle may be gotten from GCH and a non reflecting stationary set of ordinals with cofinality $< \mu$ in μ^+ (Theorem 3 below). As a corollary (Corollary 5 below), GCH + "there is a non reflecting stationary set of ω-cofinal elements of ω_2" implies the existence of an $\aleph_1$-complete Souslin tree on $\aleph_2$.

As mentioned in [G], 1.10(3), CH and the existence of a diamond sequence on $\{\delta < \aleph_2 : \mathrm{cf}\delta = \aleph_1\}$ imply the existence of a Souslin tree on $\aleph_2$ which is $\aleph_1$-complete. The construction of such a tree is by induction on levels, where at a stage of countable cofinality all branches are realized (for $\aleph_1$-completeness), while at stages of cofinality $\aleph_1$ the diamond is consulted to realize only a part of the cofinal branches in a way which kills all future big antichains. The combinatorial principle used in Theorem 2 to construct a μ-complete Souslin tree on μ^+ under GCH can be viewed as a weaker substitute for a diamond sequence on $\{\delta < \mu^+ : \mathrm{cf}\delta = \mu\}$: instead of using a guess at a single guessat stage of cofinality μ, we use unboundedly many guesses, each at a level of cofinality $< \mu$.

Unlike a diamond sequence on the stationary set of critical cofinality, this principle makes sense also in the case of an inaccessible cardinal (where there is no "critical cofinality"). This principle is closely related to club guessing (see [Sh-g] and [Sh-e]), which was discovered while the second

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author was trying to prove some results in Model Theory. This principle continues the principle that appears in [AbShSo 221], in which Souslin trees on successors of singulars are treated.

We learned from the referee that Gregory presented in the seventies in a seminar at Buffalo a construction of a countably complete Souslin tree on $\aleph_2$ from GCH and a square, but that this was not written.

1. Notation: : (1) If C is a set of ordinals, then $\operatorname{acc} C$ is the set of accumulation points of C and $\operatorname{nacc} C =^{df} C \setminus \operatorname{acc} C$. By T_α we denote the α-th level of the tree T and by $T(\alpha)$ we denote $\bigcup_{b<\alpha} T_\beta$.

2. Theorem: Suppose that

(a) $\lambda = \mu^+ = 2^\mu$, $\mu = \mu^{<\mu}$;

(b) $S^* \subseteq \{\alpha \in \lambda : \operatorname{cf}\alpha = \mu\}$ and $\overline{C} = \langle c_\delta : \delta \in S^* \rangle$, $\delta = \sup c_\delta$, c_δ is a closed set of limit ordinals.

(c) For every $\delta \in S^*$ and $\alpha \in \operatorname{nacc} c_\delta$, $P_{\delta,\alpha} \subseteq \mathcal{P}(\alpha)$, $|P_{\delta,\alpha}| \le \operatorname{cf}\alpha$, and if $\alpha \in S^*$, then $|P_{\delta,\alpha}| < \mu$;

(d) For every set $A \subseteq \lambda$ and club $E \subseteq \lambda$, there is a stationary $S_{A,E} \subseteq S^*$ such that for every $\delta \in S_{A,E}$, $\delta = \sup\{\alpha \in \operatorname{nacc} c_\delta : A \cap \alpha \in P_{\delta,\alpha} \wedge \alpha \in E\}$;

(e) If $\delta, \delta^* \in S^*$, $\delta \in \operatorname{acc} c_{\delta *}$, then there is some $\alpha < \delta$ such that $\langle P_{\delta,\beta} : \beta \in \operatorname{nacc} c_\delta \wedge \beta > \alpha \rangle = \langle P_{\delta *,\beta} : \beta \in (\operatorname{nacc} c_\delta \cap \delta) \wedge \beta > \alpha \rangle$.

(f) for every $\gamma < \lambda$, $|\{\langle P_{\delta,\alpha} : \alpha \in C_\delta \cap \gamma \rangle : \delta \in S\}| \le \mu$.

Then there is a μ-complete Souslin tree on λ.

Discussion: Condition (d) is the prediction demand. It says that for every club E and a set A there is a stationary set of δ-s, such that for unboundedly many non-accumulation points α of c_δ two things happen: $\alpha \in E$ it and $A \cap \alpha$ is guessed by $P_{\delta,\alpha}$.

Proof: We assume, without loss of generality, that for every $\delta \in S^*$ and $\alpha \in c_\delta$, $\alpha = \mu\alpha$. By induction on $\alpha < \lambda$ we construct a tree $T(\alpha)$ of height α such that:

(i) The universe of $T(\alpha)$ is $\mu(\alpha+1)$, the β-th level in $T(\alpha)$, T_β, consists of the elements $[\mu\beta, \mu(\beta+1))$, and every $x \in T_\beta$ for $\beta < \alpha$ has an extension in T_γ for every $\gamma < \alpha$. Every $x \in T(\alpha)$ such that $\operatorname{Lev}(x) + 1 < \alpha$ has at least two immediate successors in T_α.

(ii) $T(\alpha)$ is μ-complete;

(iii) For $\alpha < \beta$, $T(\alpha) = T(\beta)\ |T(\alpha)|$

Also, we define a partial function (which, intuitively speaking, chooses branches which help us in preserving the maximality of small antichains that occur along the way):

(iv) For every $x \in T(\alpha)$ and a sequence $t = \langle P_{\delta,\beta} : \beta \in \operatorname{nacc} c_\delta \cap \alpha \rangle$ such that $\sup(c_\delta \cap \alpha) < \alpha$ and $\operatorname{Lev}(x) < \max(c_\delta \cap \alpha)$ and $\max(c_\delta \cap \alpha) \in \operatorname{nacc} c_\delta$, $y(x,t)$ is defined, and is an element in the level $\max(c_\delta \cap \alpha)$ which extends x and has the property that for every $A \in P_{\delta, \max(c_\delta \cap \alpha)}$ which is a maximal antichain of $T(\max c_\delta \cap \alpha)$, there is an element of A below $y(x,t)$.

(v) If the sequence s extends the sequence t and $y(x,t), y(x,s)$ exist, then $T(\alpha) \models y(x,t) < y(x,s)$.

(vi) For every increasing sequence $\langle t_i : i < i^* \rangle$ there is an upper bound (in the tree order) to $\langle y(x, t_i) : i < i^* \rangle$.

The last demand is:

(vii) If $\alpha = \delta + 1$, $\delta \in S^*$ then every $y \in T_\delta$ satisfies that there is some $\delta^* \geq \delta \in \operatorname{acc} c_{\delta^*}$ and $x \in T(\delta)$, such that y is the least upper bound (in the tree order) of $\langle y(x, t_\alpha) : \alpha \in \operatorname{nacc} c_{\delta^*} \cap \delta \wedge \alpha_x < \alpha < \delta \rangle$ where α_x is the least in $\operatorname{nacc} c_{\delta^*}$ such that $\alpha_x > \operatorname{Lev}(x)$, and $t_\alpha = \langle P_{\delta,\beta} : \beta \in \operatorname{nacc} c_{d^*} \wedge \beta \leq \alpha \rangle$.

We first show that this construction, once carried out, yields a μ-complete Souslin tree on λ. The completeness of $T = \cup T(\alpha)$ is clear from the regularity of λ. Suppose that $A \subseteq \lambda$ is a maximal antichain of T of size λ. Let E be the club of points $\delta < \lambda$ such that $T\ \delta = T(\delta)$ and $A\ \delta$ is a maximal antichain of $T(\delta)$. Pick a point $\delta \in S^*$ such that $\delta = \sup\{\alpha \in \operatorname{nacc} c_\delta : \alpha \in E \wedge A\ \alpha \in P_{\delta,\alpha}\}$. As $|T(\delta)| < \lambda$ there is an element $a \in A$, $\operatorname{Lev}(a) > \delta$. Let y be the unique such that $\operatorname{Lev}(y) = \delta$ and $y < a$. Then by demand (vii), there is some $\delta^* \geq \delta$ and $x \in T(\delta)$ such that y is the least upper bound (in the tree order) of $\langle y(x, t_\alpha) : \alpha \in \operatorname{nacc} c_{\delta^*} \cap \delta \wedge \alpha > \operatorname{Lev}(x) \rangle$. There is some $\alpha^* < \delta$ such that $\langle P_{\delta,\beta} : \alpha < \beta < \delta \wedge \beta \in \operatorname{nacc} c_\delta \rangle = \langle P_{\delta^*,\beta} : \alpha^* < \beta \in \operatorname{nacc} c_{\delta^*} \cap \delta \rangle$. Pick some $\alpha \in \operatorname{nacc} c_\delta$ such that $\alpha > \max\{\operatorname{Lev}(x), \alpha^*\}$, $\alpha \in E$ and $A\ \alpha \in P_{\delta,\alpha_i}$. So $\alpha \in \operatorname{nacc} c_{\delta^*}$ and $A \cap \alpha \in P_{\delta^*,\alpha}$. Then the unique $x' < y$ with $\operatorname{Lev}(x') = \alpha$ (which equals $y(x, \langle P_{\delta^*,\gamma} : \gamma \in (\operatorname{nacc} c_{\delta^*} \cap (\alpha+1)) \rangle))$ is above an element $a' \in A\ \alpha$. But $x' < a$ — a contradiction to the fact that A is an antichain.

Next let us show that we can carry out the construction by induction. When $\alpha = \beta + 1$ and β is a successor or zero, add two immediate successors to every point in the β-th level. When β is limit, $\operatorname{cf}\beta < \mu$, add an element above every infinite branch. This addition amounts to the total of $\mu^{<\mu} = \mu$ points. If, in addition, $\beta \in \operatorname{nacc} c_\delta$ for some $\delta \in S^*$, then for every $x \in T(\beta)$ define $y(x, \langle P_{\delta,\gamma} : \gamma \in \operatorname{nacc} c_\delta \wedge \gamma \leq \beta \rangle)$ as follows: let $\gamma_0 = \max(c_\delta \cap \beta)$. When $\operatorname{Lev}(x) < \gamma_0$ set x_0 as the supremum (in $T(\alpha)$) of $\langle y(x, \langle P_{\delta,\alpha} : \alpha \leq \alpha^* \rangle) \rangle : \alpha^* \leq \gamma_0 \wedge \alpha^* \in \operatorname{nacc} c_\delta \rangle$; else, $x_0 = x$. As $|P_{\delta,\beta}| \leq \operatorname{cf}\beta$, we can in $\operatorname{cf}\beta$ steps choose a cofinal branch above x_0 which has a point above an element from A for every $A \in P_{\delta,\beta}$ which is a maximal antichain of T_β. Let the required y be the supremum of this branch.

If β is a limit and $\operatorname{cf}\beta = \mu$, distinguish two cases: case (a): $\beta = \delta \in S^*$. So we should satisfy demand (vii), namely add bounds precisely to those branches which for some $\delta^* \geq \delta$ in S^*, $\delta \in \operatorname{acc} c_{\delta^*}$, are of the form $\langle y(x, t_\gamma) : \gamma \in \operatorname{nacc} c_{\delta^*} \cap \beta \rangle$ where $t_\gamma = \langle P_{\delta^*,\zeta} : \zeta \leq \gamma \wedge \zeta \in \operatorname{nacc} c_{\delta^*} \rangle$. By (f) this costs only the addition of μ new elements. If, in addition, there is some $\delta' \in S^*$ such that $\delta \in \operatorname{nacc} c_{\delta'}$, we should define $y(x, \langle P_{\delta',\gamma} : \gamma \in \operatorname{nacc} c_{\delta'} \wedge \gamma \leq \delta \rangle)$ for all $x \in T(\delta)$. This presents no problem: as $|P_{\delta',\delta}| < \mu$, we attach to each x some x_0 such that $x_0 = x$ or $T_\delta \models x_0 > x$ and such that x_0 is above members from every maximal antichain in $P_{\delta',\delta}$; now $y(x, \langle P_{\delta',\gamma} : \gamma \in \operatorname{nacc} c_{\delta'} \wedge \gamma \leq \delta \rangle)$ will be the point in level δ above x_0 we obtained anyway to satisfy demand (vii).

Case (b): $\mathrm{cf}\beta = \mu$ and $\beta \notin S^*$. Then when there is some δ such that $\beta \in \mathrm{acc}\, c_\delta$ we realize enough limits to obtain completeness under increasing sequences of the form $\langle y(x, t_i) : i < i^* \rangle$. By (f), we add thus $\leq \mu$ elements. If there is no such δ, just make sure, by adding μ points to the tree in level β, that above every $x \in T(\beta)$ there is a point in level β. This takes care also of (i). If there is some δ' such that $\beta \in \mathrm{nacc}\, c_{\delta'}$, then for every $x \in T(\beta)$ define $y(x, \langle P_{\delta',\gamma} : \gamma \in \mathrm{nacc}\, c_{\delta'} \wedge \gamma \leq \delta \rangle)$ exactly as in the case of smaller cofinality.$\dashv$2

We will show now how to obtain from a non-reflecting stationary set a special case of the prediction principle we used in the previous theorem. One should substitute $P_{\delta,\alpha}$ in the previous theorem by B_α from the next theorem to get the assumptions of the previous theorem.

3. Theorem: Suppose $\lambda = \mathrm{cf}\lambda > \aleph_1$, $S \subseteq \lambda$ is stationary, non-reflecting, and carries a diamond sequence $\langle A_\alpha : \alpha \in S \rangle$, S^* is a given non reflecting stationary subset of λ, $S^* \cap S = \emptyset$ and $\delta \in S^* \Rightarrow \mathrm{cf}\delta > \aleph_0$. Then there are sequences $\overline{C} = \langle c_\delta : \delta \in S^* \rangle$ and $\overline{B} = \langle B_\alpha : \alpha \in S \rangle$ such that:

(i) $B_\alpha \subseteq \alpha$;

(ii) $\sup c_\delta = \delta$ and c_δ is a closed set of limit ordinals;

(iii) if $\delta, \delta^* \in S^*$ and $\delta \in \mathrm{acc}\, c_{\delta^*}$, then there is some $\alpha < \delta$ such that $c_{\delta^*} \cap (\alpha, \delta) = c_\delta \cap (\alpha, \delta)$;

(iv) for every club $E \subseteq \lambda$ and set $X \subseteq \lambda$ there are stationarily many $\delta \in S^*$ such that $\delta = \sup\{\alpha \in \mathrm{nacc}\, c_\delta : \alpha \in S \cap E \wedge X \cap \alpha = A_\alpha\}$.

Proof: We fix some 1-1 pairing function $\langle -, - \rangle$ from $\lambda \times \omega_0$ onto λ and let $A^\alpha_n = \{\beta < \alpha : \langle \beta, n \rangle \in A\}$. We may assume that for every countable sequence $\overline{X} = \langle X_n : n < \omega \rangle$ of subsets of λ there are stationarily many $\alpha \in S$ such that for every n, $X_n \cap \alpha = A^\alpha_n$. Denote by $S(\overline{X})$, for a (finite or infinite) sequence of subsets of λ the stationary set $\{\alpha \in S : \bigwedge_n X_n \cap \alpha = A^\alpha_n\}$.

To every limit $\alpha < \lambda$ we attach a club of α, e_α, satisfying $e_\alpha \cap S = e_\alpha \cap S^* = \emptyset$, $\mathrm{otp}\, e_\alpha = \mathrm{cf}\alpha$ and e_α contains only limit ordinals whenever $\alpha \in S^*$. Let $\overline{C}_0 = \langle e_\delta : \delta \in S^* \rangle$. Suppose that $\overline{C}_n = \langle c^n_\delta : \delta \in S \rangle$ is a bad candidate for the job, namely that there are a club E_n and a set X_n such that for every $\delta \in E_n \cap S^*$ the set $\{\alpha \in \mathrm{nacc}\, c^n_\delta : \alpha \in S(X_n) \cap E_n\}$ is bounded below δ. (Surely, we may assume that E_n is as thin as we like — in particular that all its members are limits). Define $\overline{C}_{n+1}$ by induction on δ: For every $\gamma \in c^n_\delta$, we define $c^{n+1}_\delta \cap (\gamma, \min c^n_\delta \setminus (\gamma+1))$ (where (γ, β) denotes, as usual, an open interval of ordinals), and we let $c^{n+1}_\delta = c^n_\delta \cup \bigcup\{c^{n+1}_\delta \cap (\gamma, \min c^n_\delta \setminus (\gamma+1)) : \gamma \in c^n_\delta\}$. This is well defined, as every $\gamma \in c^n_\delta$ has a successor in c^n_δ. So denote by β the ordinal $\min c^n_\delta \setminus (\gamma+1)$, and let

$$(*)\quad c^{n+1}_\delta \cap (\gamma, \beta) = \begin{cases} c^{n+1}_\beta \cap (\gamma, \beta) & \text{if } \beta \in S^* \\ \emptyset & \text{if } \beta \in S(X_0, \cdots, X_n) \cap E_n \\ \{\alpha : \gamma < \alpha < \beta \wedge (\exists \zeta \in e_\beta)(\alpha = \sup(\zeta \cap E_n))\} & \text{otherwise} \end{cases}$$

Note that for the definition to be consistent, $\beta \in c^n_\delta$ must always be limit (and this is indeed the case).

3.1 Lemma: Suppose that $\overline{C}_n$ is defined for $n \leq m$. Then for For every $n < m$ and $\delta \in S^*$:

(0) If $\alpha \in c^n_\delta$ then β is a limit ordinal;

(1) c^n_δ is closed.

(2) $c^n_\delta \subseteq c^{n+1}_\delta$.

(3) If $\alpha \in S^* \cap \operatorname{acc} c^n_\delta$, then c^n_α and $c^{n+1}_\delta \cap \alpha$ have a common end segment.

(4) If $\alpha \in c^{n+1}_\delta \cap S(X_0, \cdots, X_n) \cap E_n$, then $\alpha \in \operatorname{nacc} c^{n+1}_\delta$.

Proof: : (2) is true by the definition of c^{n+1}_δ for every n and $\delta \in S^*$. (0), (1), (3) and (4) are proved by induction on n and δ.

For $n = 0$ we know that $e_\delta = c^0_\delta$ is all limits and is closed, so (0) and (1) hold. (3) is vacuously true, because $e_\delta \cap S^* = \emptyset$, and (4) is vacuously true because $e_\delta \cap S = \emptyset$.

For $n + 1$:

(0): Suppose $\alpha \in c^{n+1}_\delta$. If $\alpha \in c^n_\delta$ then it is a limit ordinal by (0) and the induction hypothesis on n. If $\alpha \notin c^n_\delta$, let $\gamma = \sup c^n_\delta \cap \alpha$. Because of (1) and the induction hypothesis $\gamma < \alpha$. Let $\beta = \min c^n_\delta \setminus (\alpha + 1)$. If $\beta \in S^*$ then $c^{n+1}_\delta \cap (\gamma, \beta) = c^{n+1}_\beta \cap (\gamma, \beta)$. So $\alpha \in c^{n+1}_\beta$, and by the induction hypotheses on β, α is a limit ordinal. If $\beta \notin S^*$, the $c^{n+1}_\delta \cap (\gamma, \beta) = \{\alpha : \gamma < \alpha < \beta, \quad (\exists \zeta \in e_\beta)(\alpha = \sup \zeta \cap E_n)\}$. Therefore for some $\zeta \in e_\beta$ our α is $\sup(\zeta \cap E_n)$. Since E_n is a club, $\alpha \in E_n$. But E_n is a club of limits, so α is limit.

(1) Suppose that $\alpha \in \operatorname{acc} c^{n+1}_\delta$, and we wish to show $\alpha \in c^{n+1}_\delta$. If $\alpha \in \operatorname{acc} c^n_\delta$, then because of (1) and the induction hypothesis on n $\alpha \in c^n_\delta$ and (by (2)) $\alpha \in c^{n+1}_\delta$. Else, $\gamma = \sup \alpha \cap c^n_\delta$ and $\beta = \min c^n_\delta \setminus (a + 1)$, $\gamma < \alpha < \beta$. If $\beta \in S^*$ then $\alpha \in \operatorname{acc} c^{n+1}_\beta$. By the induction hypothesis on β and (1), $\alpha \in c^{n+1}_\delta$. Otherwise, α is a limit of $\langle a_i : i < i^* \rangle$ such that $\alpha_i = \sup \zeta_i \cap E_n \in c^{n+1}_\delta$. So clearly $\alpha \in E_n$. Let ζ^* be the minimal in e_β above α. So $\alpha = \sup \zeta^* \cap E_n$. Therefore $\alpha \in c^{n+1}_\delta$.

Before proving (3) we note:

3.2 Fact: Suppose $\gamma \in c^n_\delta$ and $\beta = \min c^n_\delta \setminus (\gamma + 1)$. If $\beta \notin S^*$ and $\alpha \in c^{n+1}_\delta \cap (\gamma, \beta)$ is a limit of c^{n+1}_δ, then $\alpha \in e_\beta$.

Indeed, if $\alpha = \sup\{\alpha(i) : i < i^*\}$, where $\alpha(i) = \sup \zeta(i) \cap E_n$ are elements in c^{n+1}_δ, $\alpha \in E_n$. Therefore every $\zeta(i) < \alpha$ (or else $\sup \zeta(i) \cap E_n \geq \alpha > \alpha(i)$). But $\zeta(i) \geq \alpha(i)$, so α is a limit of e_β. As $\alpha < \beta$ and e_β is closed, $\alpha \in e_\beta$.$\dashv$3.2

(3): Let $\alpha \in \operatorname{acc} c^{n+1}_\delta \cap S^*$, and we wish to show that c^{n+1}_δ and c^{n+1}_α have a common end segment. If $\alpha \in \operatorname{acc} c^n_\delta$, then by the induction hypothesis on n and (3), we know that c^n_δ and c^n_α have a common end segment; say they agree on the interval $(\alpha(0), \alpha)$. This means in particular that for every $\gamma \in c^n_\delta \cap (\alpha(0), \alpha)$, $\alpha \in c^n_\alpha$ and $\min c^n_\delta \setminus (\gamma + 1) = \min c^n_\alpha \setminus (\gamma + 1) =: \beta$. Therefore also $c^{n+1}_\delta \cap (\gamma, \beta) = c^{n+1}_\alpha \cap (\gamma, \beta)$, and consequently $c^{n+1}_\delta \cap (\alpha(0), \alpha) = c^{n+1}_\alpha \cap (\alpha(0), \alpha)$. So assume that $\alpha \notin \operatorname{acc} c^n_\delta$. The first possibility is that $\alpha \notin c^n_\delta$ altogether. In this case let $\gamma < \alpha < \beta$ assume their traditional roles as the last ordinal of c^n_δ below α and the first above. If $\beta \in S^*$, then by the induction hypothesis on β we know that c^{n+1}_β and c^{n+1}_α have a common end segment; but $c^{n+1}_\delta \cap (\gamma,\beta) = c^{n+1}_\beta \cap (\gamma,\beta)$, so it follows that c^{n+1}_δ and c^{n+1}_α have a common end segment.

If $\beta \notin S^*$, then by the Fact above, $\alpha \in e_\beta$ — contradiction to $e_\beta \cap S^*$ is empty. So this subcase is non existent.

The last case is: $\alpha \notin \operatorname{acc} c^n_\delta$ but $\alpha \in c^n_\delta$, or in short $\alpha \in \operatorname{nacc} c^n_\delta$. Let γ be the last element of $c^n_\delta \cap \alpha$. Then by $(*)$, $c^{n+1}_\delta \cap (\gamma,\alpha) = c^{n+1}_\alpha \cap (\gamma,\alpha)$.

(4): Suppose that $\alpha \in S(X_0,\cdot,X_n) \cap c^{n+1}_\delta$. We should see that $\alpha \in \operatorname{nacc} c^{n+1}_\delta$. Let $m \leq n+1$ be the minimal such that $\alpha \in c^m_\delta$. It is enough to prove that $\alpha \in \operatorname{nacc} c^m_\delta$, because by $(*)$ it is clear that if $\alpha \in S(X_0,\cdots,X_m) \cap \operatorname{nacc} c^m_\delta$ then α will remain a non-accumulation point in c^{m+1}_δ (because nothing will be added in the interval below it). So without loss of generality we may assume that $\alpha \in c^{n+1}_\delta \setminus c^n_\delta$. So denote by (γ,β), as usual, the unique minimal interval with end points in c^n_δ to which α belongs. First case: $\beta \in S^*$. So $\alpha \in c^{n+1}_\beta$; and by the induction hypothesis on β, $\alpha \in \operatorname{nacc} c^{n+1}_\beta$. So this case is done. Otherwise, $\beta \notin S^*$. So by the Fact above, if α were a limit of c^{n+1}_δ, it would be in e_β. But $\alpha \in S$, and therefore cannot be in e_β by the very choice of e_β. Therefore $\alpha \in \operatorname{nacc} c^{n+1}_\delta$. (This is where the non reflection of S is used in an essential way).$\dashv$3.1

3.3 Claim: : There is some $n < \omega$ for which $\overline{C}_n$ and $\langle A^\alpha_n : \alpha \in S\rangle$ are as required.

Proof: : Suppose not. Let $\overline{X}_\omega = \langle X_n : n < \omega\rangle$. Let $E = \cap_n E_n$ and $E' = \operatorname{acc}(S(\overline{X}_\omega) \cap E)$. So E' is a club. Pick some $\delta \in S^* \cap E'$. For every n there is a bound below δ of the set $\{\alpha \in \operatorname{nacc} c^n_\delta : \alpha \in S(X_n) \cap E_n\}$. As $\operatorname{cf}\delta > \aleph_0$, let $\alpha^* < \delta$ bound $\alpha(n)$ for all n. Let $\delta > \beta > \alpha^*$ be in $S(\overline{X}_\omega) \cap E$. So for every n, $X_n \cap \beta = A^\beta_n$ and $\beta \in E_n$. If $\beta \in c^n_\delta$ for some n, then by (4) $\beta \in \operatorname{nacc} c^n_\delta$ — a contradiction to $\beta > \alpha(n)$. So $\beta \notin c^n_\delta$ for all n. Therefore for every n we may define $(\gamma(n),\beta(n))$ as the minimal interval with ends in c^n_δ which contains β.

3.4 Claim: $\beta(n+1) < \beta(n)$.

Proof: By its definition, $\beta(n) \in \operatorname{nacc} c^n_\delta$. In the case $\beta(n) = \delta^* \in S^*$, there are clearly elements in $c^{n+1}_{\delta^*}$ above β and below $\beta(n)$, so the claim is obvious. The case $\beta(n) \in S(X_0,\cdots,X_n)$ is impossible because of (4). In the remaining case, $c^{n+1}_\delta \cap (\gamma(n),\beta(n)) = \{\alpha : \gamma(n) < \alpha < \beta(n), (\exists\zeta \in e_{\beta(n)})(\alpha = \sup \zeta \cap E_n)\}$. Let $\zeta^* > \beta$ be in $e_{\beta(n)}$. As $\beta \in E \subseteq E_n$, $\sup \zeta^* \cap E_n \geq \beta$. But the right hand side of this inequality belongs to c^{n+1}_δ, while β does not; therefore $\sup \zeta^* \cap E_n > \beta$. So we see that there are elements of c^{n+1}_δ in $(\beta,\beta(n))$, therefore the least of them, namely $\beta(n+1)$ is smaller than $\beta(n)$.$\dashv$3.4

This is clearly a contradiction. We conclude that after finitely many steps, $\overline{C}_{n+1}$ cannot be defined due to the lack of a counterexample. This means that $\overline{C}_n$ and $\langle B_\alpha : \alpha \in S\rangle$ where $B_\alpha = A^n_\alpha$ satisfy (i), (ii) and (iv). By (3) above, they satisfy (iii) as well. $\dashv$3.3

This shows that after finitely many corrections all the requirements are satisfied, and our theorem is proved.$\dashv$3

4. Theorem: If the e_δ we pick in the proof of Theorem 3 satisfy the additional condition that for every $\gamma < \lambda$ the set $\{e_\delta \cap \gamma : \alpha \in \lambda \text{ is limit}\}$ has cardinality smaller then λ, then the resulting good $\overline{C} = \langle c_\delta : \delta \in S^* \rangle$ satisfies that for every $\gamma < \lambda$, $|\{c_\delta \cap \gamma : \delta \in S^*\}| < \lambda$

Proof: Let $\gamma < \lambda$ be given. We must show that $|\{c^n_\delta \cap \gamma : \delta \in S^*\}| \le \mu$. Let $N \prec H(\chi, \in)$ for some large enough χ, $|N| < \lambda$, $\gamma \subseteq N$, $\gamma \in N$, $\{e_\alpha \cap \gamma : \alpha < \lambda \text{ is limit}\} \subseteq N$, $\langle e_\alpha : \nu < \lambda \text{ is limit}\rangle \in N$, and $E_n, X_n \in N$ for every n.

We shall see that for every n and δ, $c^n_\delta \cap \gamma \in N$. Since $|N| < \lambda$, this is enough.

First we notice that if $\delta < \gamma$ then $c_\delta \in N$ and by elementarity also $c^n_\delta \in N$ for every n. Now we use induction on n to show that for every $\delta > \gamma$, $c^n_\delta \cap \gamma \in N$. For $n = 0$: if $\delta > \gamma$ then $c^0_\delta \cap \gamma = e_\delta \cap \gamma \in N$ by the assumptions on N. For $n+1$ we use induction on δ. Suppose, then, that for all $\delta' < \delta$, $c^{n+1}_{\delta'} \cap \gamma \in N$.

We need the easy

4.1 Fact: If (α_0, α_1) is a minimal interval of $c^n_\delta \cap (\gamma+1)$ then $c^{n+1}_\delta \cap (\alpha_0, \alpha_1) \in N$.

Proof: By $(*)$ above, the definition of $c^{n+1}_\delta \cap (\alpha_0, \alpha_1)$ depends only on e_{α_1}, E_n and (if case there is such) $c^{n+1}_{\alpha_1}$. All these objects are in N, so also $c^{n+1}_\delta \cap (\alpha_0, \alpha_1) \in N$.$\dashv$4.1

Denote $\gamma(\delta) = \sup c^n_\delta \cap \gamma$. So $\gamma(\delta) \le \gamma$. If $\gamma(\delta) = \gamma$, then $c^{n+1}_\delta \cap \gamma = c^n_\delta \cap \gamma \cup \bigcup_I c^{n+1}_\delta \cap I$ where I runs over all minimal intervals of $c^n_\delta \cap (\gamma+1)$. So by the Fact above we are done. Else, $\gamma(\delta) < \gamma$. In this case define $\beta(\delta) = \min c^n_\delta \setminus \gamma$. If $\beta(\delta) = \gamma$ then again we are done by the Fact. The remaining case is $\gamma(\delta) < \gamma < \beta(\delta)$. By the same Fact, $c^{n+1}_\delta \cap \gamma(\delta) \in N$. If $\beta(\delta) \in S^*$, then $c^{n+1}_\delta \cap (\gamma(\delta), \gamma) = c^{n+1}_{\beta(\gamma)} \cap (\gamma(\delta), \gamma)$. By the induction hypothesis, and since $\beta(\delta) < \delta$, the latter set is in N, and we are done. If $\beta(\delta) \notin S^*$, then either nothing is added into $(\gamma(\delta), \beta(\delta))$ (when $\beta(\delta) \in S(X_0, \cdots, X_n)$), or $c^{n+1}_\delta \cap (\gamma(\delta), \beta(\delta)) = \{\alpha : \gamma(\delta) < \alpha < \beta(d)\ (\exists \zeta \in e_{\beta(\delta)})(\alpha = \sup E_n \cap \zeta)\}$. So in this definition N might not know who $\beta(\delta)$ is, but $e_{\beta(\delta)} \cap \gamma \in N$. Therefore, denoting by α^* the last member in $E_n \cap \gamma$, we can determine in N the set $c^{n+1}_\delta \cap \alpha^*$. As to whether α^* itself is in this set or not, we need knowledge which is not available in N, but who cares, as long as both possibilities are in N. $\dashv$4

5. Corollary: If there is a non-reflecting stationary set $S \subseteq \{\alpha < \mu^+ : \mathrm{cf}\alpha < \mu\}$, and $2^\mu = \mu^+$, $\mu^{<\mu} = \mu$, then there is a μ-complete Souslin tree on μ^+.

6. Remark: This improves the result by Gregory in [G].

Proof: It is known (see [G] 2.1) that if $S \subseteq \{\delta \in \mu^+ : \mathrm{cf}\delta < \mu\}$ is stationary, then $\mu = \mu^{<\mu}$ implies $\diamondsuit(S)$. As S is non reflecting, we can, for every limit $\alpha < \mu^+$, choose a closed set e_α, $\alpha = \sup e_\alpha$ and $\mathrm{otp}\, e_\alpha = \mathrm{cf}\alpha$ such that $e_\alpha \cap S = \emptyset$. $\mu = \mu^{<\mu}$ implies that for every $\gamma < \mu^+$ the set $\{e_\alpha \cap \gamma : \alpha < \lambda,\ \alpha \text{ is limit}\}$ is of cardinality at most μ. Use Theorem 3 and Theorem 4 to obtain the assumptions of Theorem 2, S being the given non reflecting stationary set and S^* being $\{\delta < \lambda : \mathrm{cf}\delta = \mu\}$. By Theorem 2 there is an μ-complete Souslin tree on μ^+.$\dashv$5

7. Problem: (1) Can the existence of such a tree be proved in $ZFC + GCH$?

(2) Can a Souslin tree on $\aleph_2$ be constructed from GCH and *two* stationary sets, each composed

of ordinals of countable cofinality, which do not reflect simultaneously? By [Mg] this would raise the consistency strength of $GCH+SH(\aleph_2)$ to the consistency of the existence of a weakly compact cardinal.

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