RULES AND REALS

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ABSTRACT. A "k-rule" is a sequence $\vec{A} = ((A_n, B_n) : n < \omega)$ of pairwise disjoint sets B_n, each of cardinality $\leq k$ and subsets $A_n \subseteq B_n$. A subset $X \subseteq \omega$ (a "real") follows a rule $\vec{A}$ if for infinitely many $n \in \omega$, $X \cap B_n = A_n$.

There are obvious cardinal invariants resulting from this definition: the least number of reals needed to follow all k-rules, $\mathfrak{s}_k$, and the least number of k-rules without a real following all of them, $\mathfrak{r}_k$.

Call $\vec{A}$ a *bounded* rule if $\vec{A}$ is a k-rule for some k. Let $\mathfrak{r}_\infty$ be the least cardinality of a set of bounded rules with no real following all rules in the set.

We prove the following: $\mathfrak{r}_\infty \geq \max(\text{cov}(\mathbb{K}), \text{cov}(\mathbb{L}))$ and $\mathfrak{r} = \mathfrak{r}_1 \geq \mathfrak{r}_2 = \mathfrak{r}_k$ for all $k \geq 2$. However, in the Laver model, $\mathfrak{r}_2 < \mathfrak{b} = \mathfrak{r}_1$.

An application of $\mathfrak{r}_\infty$ is in Section 3: we show that below $\mathfrak{r}_\infty$ one can find proper extensions of dense independent families which preserve a pre-assigned group of automorphism. The original motivation for discovering rules was an attempt to construct a maximal homogeneous family over ω. The consistency of such a family is still open.

INTRODUCTION

In the present paper we present new cardinal invariants which resulted from investigations of homogeneous families. These numbers have intrinsic interest (in fact we regard it as surprising that those numbers have not been discovered earlier).

In Section 1 we introduce "k-rules" and the notion of when a real r follows a k-rule $\vec{A}$. A *k-rules* is a sequence $\vec{A} = ((A_n, B_n) : n < \omega)$ of pairwise disjoint sets B_n, each of cardinality $\leq k$, and subsets $A_n \subseteq B_n$. A subset $X \subseteq \omega$ (a "real") *follows* a rule $\vec{A}$ if for infinitely many $n \in \omega$, $X \cap B_n = A_n$.

A rule $\vec{A}$ is *bounded* if it is a k-rule for some $k \in \omega$.

The obvious cardinal invariants related to rules are the following: the least number of reals needed to follow all k-rules, $\mathfrak{s}_k$, and the least number of k-rules with no real following all of them, $\mathfrak{r}_k$. Let $\mathfrak{r}_\infty$ be the least number of bounded rules with no real following all of them.

We compare the $\mathfrak{r}_k$s and $\mathfrak{r}_\infty$ among themselves and to well known cardinal invariants: covering of category, covering of Lebesgue measure, $\mathfrak{r}$, $\mathfrak{b}$, $\mathfrak{d}$ and the evasion numbers $\mathfrak{e}_k$ which were studied by Blass and Brendle. We prove:

(–) $\max(cov(\mathbb{K}), cov(\mathbb{L})) \leq \mathfrak{r}_\infty$;
(–) $\mathfrak{r} = \mathfrak{r}_1 \geq \mathfrak{r}_2 = \mathfrak{r}_k$ for all $k \geq 2$;
(–) $\mathfrak{s}_2 \leq \mathfrak{e}_2$;
(–) $\mathfrak{r}_\infty \leq \min(\mathfrak{r}_2, \mathfrak{d})$.

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In Section 2 we prove the consistency of $\mathfrak{r}_2 < \mathfrak{b}$.

In Section 3 we show that below $\mathfrak{r}_\infty$ one can properly extend an independent family of subsets of ω preserving a prescribed group of automorphisms. This is the relevance of $\mathfrak{r}_\infty$ to the behavior of homogeneous families under inclusion, which was the original motivation for the discovery of rules.

An open problem which the second author and S. Shelah have tried to tackle in the past is the existence of a maximal homogeneous family of subsets of ω. This problem is still open. Even the consistency of a maximal homogeneous family is not known. However, $\mathfrak{r}_\infty$ sheds some light of the potential size of a maximal homogeneous family - if one such exists. If one defines $\mathfrak{ex}$ as the least cardinality of a dense independent family which is not extendible to a homogeneous family, then if $\mathfrak{r}_\infty \leq \mathfrak{ex}$ then any dense independent family of cardinality below $\mathfrak{r}_\infty$ can be extended to a homogeneous family of size $\mathfrak{r}_\infty$. The number $\mathfrak{ex}$ will be studies elsewhere.

1. Rules

Definition 1.1.
- A *rule* is a sequence $\vec{A} = (A_n, B_n : n \in \omega)$, where the sets B_n are disjoint and finite, and for all n, $A_n \subseteq B_n \subseteq \omega$.
- We say that $X \in [\omega]^\omega$ *follows* the rule $\vec{A}$ if there are infinitely many n with $X \cap B_n = A_n$; otherwise X is said to *avoid* $\vec{A}$.
- For $k \in \omega$ we say that $\vec{A}$ is a k-rule if all sets B_n have size $\leq k$. We say that $\vec{A}$ is a *bounded* rule if $\vec{A}$ is a k-rule for some k.
- More generally, for any function $f : \omega \to \omega$ we say that $\vec{A}$ is an f-rule if for all n, $|B_n| \leq f(n)$. We say that f is a "slow" function if

  $$\sum_{n=0}^{\infty} 2^{-f(n)} = \infty,$$

  and we say that $\vec{A}$ is a slow rule if it is an f-rule for some slow f.

Definition 1.2.
1. For $k \in \omega$ let $\mathfrak{r}_k := \min\{|\mathfrak{R}| :$ there is no X which follows all k-rules from $\mathfrak{R}$ $\}$. (similarly $\mathfrak{r}_f$, when $f : \omega \to \omega$).
2. Dually, let $\mathfrak{s}_k := \min\{|\mathfrak{S}| :$ every rule is followed by some $X \in \mathfrak{A}$ $\}$.
3. We let $\mathfrak{r}_\infty = \min\{|\mathfrak{R}| :$ there is no X which follows all bounded rules from $\mathfrak{R}\}$.

We remark that 2^k trivially bounds the least cardinality of a set of rules with the property that every real follows some rule in the set.

Recall that the "splitting" number $\mathfrak{s}$ and the "reaping" number $\mathfrak{r}$ are defined as follows:

Definition 1.3. If $s, X \in [\omega]^\omega$, then we say that s "splits" X if s divides X to two infinite parts, i.e., $s \cap X$ and $(\omega - s) \cap X$ are both infinite.
- $\mathfrak{s} := \min\{|\mathfrak{S}| : \mathfrak{S} \subseteq [\omega]^\omega$, every $X \in [\omega]^\omega$ is split by some $s \in \mathfrak{S}$ $\}$
- $\mathfrak{r} := \min\{|\mathfrak{R}|$: $\mathfrak{R} \subseteq [\omega]^\omega$, there is no $X \in [\omega]^\omega$ which splits all $r \in \mathfrak{R}$ $\}$

Fact 1.4. $\cdots \leq \mathfrak{r}_3 \leq \mathfrak{r}_2 \leq \mathfrak{r}_1 = \mathfrak{r}$, and $\mathfrak{s} \leq \mathfrak{s}_2 \leq \mathfrak{s}_3 \leq \cdots$. However, $\mathfrak{s}_1 = 2$, witnessed by $\mathfrak{S} = \{\emptyset, \omega\}$.

Theorem 1.5. (a) *Let $(N, \in)$ be a model of ZFC* (a large enough fragment of ZFC). If a real X follows all rules from N, then X is Cohen over N. (Conversely, a Cohen real over N follows all rules from N.)*

(b) *If X is random over N, then X follows all slow rules from N (so in particular, all bounded rules).*

(c) $\max(cov(\mathbb{K}), cov(\mathbb{L})) \leq \mathfrak{r}_\infty$. *(*$\mathrm{cov}(\mathbb{K})$ *is the smallest number of first category sets needed to cover the real line.* $\mathrm{cov}(\mathbb{L})$ *is defined similarly using measure zero sets.)*

Proof. (a): Assume that $X \subseteq \omega$ follow all rules from N. We claim that χ_X, the characteristic function of X, is a Cohen real over N, that is, the set $\{\chi_X \restriction n : n \in \omega\}$ is generic for the forcing notion ${}^{<\omega}2$.

To verify this claim, consider any nowhere dense tree $T \subseteq {}^{<\omega}2$ in N. We have to check that χ_X is not a branch of T.

Using the fact that T is nowhere dense (and T in N) we can by induction (in N!) find sequences $(n_i : i < \omega)$ and $(\eta_i : i < \omega)$ such that for all $i < \omega$ we have:

1. $n_i < n_{i+1}$, $\eta_i \in {}^{[n_i, n_{i+1})}2$
2. For all $\nu \in {}^{n_i}2$, $\nu \cup \eta_i \notin T$.

Now let $B_i := [n_i, n_{i+1})$, $A_i = \{k : \eta_i(k) = 1\}$. Our assumption was that X follows the rule $(A_i, B_i : i \in \omega)$. So for some i we have $X \cap B_i = A_i$, hence $\chi_X \supseteq \nu_i$. Hence χ_X is not a branch of T.

This concludes the proof of (a).

The converse to (a) is obvious.

(b) is also easy: Let $X_n := \{X : X \cap B_n \neq A_n\}$. For $n \neq m$, the sets X_n and X_m are independent (in the probabilistic sense), and $\mu(X_n) = 1 - 2^{-f(n)}$, where μ is the Lebesgue measure on $\mathcal{P}(\omega) \simeq {}^{\omega}2$. Hence $\mu(\bigcap_{n>m} X_n) = \prod_{n>m}(1 - 2^{-f(n)}) = 0$.

(c) follows from (a) and (b).

□

Theorem 1.6 (Shelah). *For $k \geq 2$, $\mathfrak{r}_k = \mathfrak{r}_2$ (and similarly, $\mathfrak{s}_2 = \mathfrak{s}_k$).*

Proof. We will show that $\mathfrak{r}_k = \mathfrak{r}_{k+1}$: Let N_0 be sufficiently closed (say, a model of ZFC*, but closed under some recursive functions is sufficient) of size $< \mathfrak{r}_k$; we have to show that there is a real that follows all $k+1$-rules from N_0.

We define a sequence $(N_i, C_i : i \leq k)$ such that $N_i \cup \{C_i\} \subseteq N_{i+1}$, each N_i is sufficiently closed and of the same cardinality as N_0, and C_i follows all k-rules from N_i.

Now let C be the "average" of the C_i: $m \in C$ iff m is in "most" of the C_i's, or formally:

$$C := \{m \in \omega : |\{i \leq k : m \in C_i\}| > (k+1)/2\}$$

Now we check that C indeed follows all $k+1$-rules from N_0.

Let $(A_n, B_n : n \in \omega)$ be a $k+1$-rule in N_0. For $0 \leq i \leq k$ we let $(A^i_n, B^i_n : n \in \omega)$ be the k-rule obtained by removing the each ith element of B_n. That is, letting $\{b^0_n, \dots, b^k_n\}$ be the increasing enumeration of B_n we let $B^i_n := B_n \setminus \{b^i_n\}$, $A^i_n := A_n \cap B^i_n$.

Let $E_0 := \omega$. For $0 \leq i \leq k$ let

$$E_{i+1} := \{n \in E_i : B^i_n \cap C_i = A_n \cap B^i_n\},$$

i.e., E_{i+1} is the set of indices on which C_i follows the rule $(A^i_n, B^i_n : n \in E_i)$. Note that $E_i \in N_i$ and $C_i \in N_{i+1}$. By the choice of C_i we know that each E_{i+1} is infinite.

We conclude the proof by showing that for $n \in E_{k+1}$ we have $A_n = B_n \cap C$. Let $n \in E_{k+1}$ (so also $n \in E_i$ for all $i \leq k$), and $m \in B_n$. Say $m = b_n^j$. Then for $i \neq j$ we have $m \in B_n^i$, so $m \in A_n^i \Leftrightarrow m \in C_i$.

Hence the cardinality of the set $\{i \leq k : m \in C_i\}$ is either in $\{0,1\}$ (iff $m \notin A_n$) or in $\{k, k+1\}$. In any case we get $m \in C$ iff $m \in A_n$. So $A_n = B_n \cap C$.

□

Theorem 1.7. $\mathfrak{r}_\infty \geq \min(\mathfrak{r}_2, \mathfrak{d})$. *In particular, if* $\mathfrak{r}_2 \leq \mathfrak{d}$ *then* $\mathfrak{r}_\infty = \mathfrak{r}_2$.

Proof. Let N be a model of cardinality $< \min(\mathfrak{r}_2, \mathfrak{d})$. We will find a real X following all bounded rules from N.

Define sequences $(N_i : i < \omega)$, $(X_i : i < \omega)$ satisfying the following conditions:

1. $N_0 = N$.
2. N_i is a model of ZFC*, $N_{i-1} \cup \{X_i\} \subseteq N_i$.
3. $|N_i| = |N_0|$.
4. X_i follows all i-rules (and hence also all j-rules for $j \leq i$) from N_{i-1}.

Let N_ω be a model of size N_0 containing $(N_i : i < \omega)$ and $(X_i : i < \omega)$. Since $|N_\omega| < \mathfrak{d}$ we can find a strictly increasing function f that is not dominated by any function from N_∞.

Define $X \subseteq \omega$ by requiring $X \cap (f(i-1), f(i)] = X_i \cap (f(i-1), f(i)]$. We claim that X follows all bounded rules from N.

To complete the proof, consider an arbitrary k-rule $(A_n, B_n : n \in \omega)$ from N. We may assume $\min \bigcup_n B_n > f(k)$. We define sequences $(E_i : k \leq i < \omega)$ satisfying the following conditions for all $i \geq k$.

1. $\forall n \in E_i \; B_n \cap X_i = A_n$.
2. $E_i \in N_i$.
3. $E_{i+1} \subseteq E_i$.

We can carry out this construction, because $(A_n, B_n : n \in E_i)$ is a rule in N_i, so we just choose E_{i+1} to witness that X_{i+1} follows this rule.

Now let $n_i := \min E_i$. Clearly the function $i \mapsto n_i$ is in N_∞. So we can find infinitely many j such that $f(j) > \max B_{n_j}$.

We claim that for each such j, $X \cap B_{n_j} = A_{n_j}$. For all $i \in [k, j]$ we have $n_i \in E_j$, so $X_i \cap B_j = A_j$. Note that $B_j \subseteq [f(k), f(j))$, so we also have $X \cap B_j = A_j$.

□

Problem 1.8. Is $\mathfrak{r}_\infty < \mathfrak{r}_2$ consistent?

We remark that in the random real model we have $\mathfrak{r}_2 = \mathrm{cov}(\mathbb{L}) = \mathfrak{c} = \mathfrak{r}_\infty$, $\mathfrak{d} = \aleph_1$. So one cannot hope to prove $\mathfrak{r}_2 \leq \mathfrak{d}$.

We now consider the invariant that is dual to $\mathfrak{r}_k$, and we compare it with the well-known "evasion" number.

Definition 1.9. (π, D) is a k-predictor, if D is infinite, $\pi = (\pi_n : n \in D)$, π_n a function from ${}^n k$ to k.

We say that $f \in {}^\omega k$ evades (π, D) if there are infinitely many $\ell \in D$ such that $f(\ell) \neq \pi_\ell(f \restriction \ell)$.

$$\mathfrak{e}_k := \min\{|N| : \forall \pi \, \exists f \in N : f \text{ evades } \pi\}$$

Brendle in [1] investigated these and other cardinal invariants and showed [remarked] that all $\mathfrak{e}_k$ are equal to each other.

The following construction connects rule with predictors.

Definition 1.10. Let $R = (A_n, B_n : n \in \omega)$ be a 2-rule. Define a 2-predictor (π_R, D_R) as follows:

1. $D_R = \{\max B_n : n \in \omega\}$
2. If $\ell = \max B_n$, and $|A_n| = 1$, then $\pi_\ell(f) = f(\min B_n)$ for all $f \in {}^\ell 2$. Otherwise, $\pi_\ell(f) = 1 - f(\min B_n)$.

Lemma 1.11. *Let $X \subseteq \omega$. If χ_X evades π_R, then either X or $\omega \setminus X$ follows R.*

Proof. Let $\ell_n := \max B_n$, $i_n = \min(B_n)$ for all n.

X evades π_R, so there are infinitely many n such that $X(\ell_n) \neq \pi_{\ell_n}(X \restriction \ell_n)$.

Case 1: There are infinitely many such n where in addition $|A_n| = 1$.

So for each such n, $X(\ell_n) \neq \pi_{\ell_n}(X \restriction \ell_n) = X(i_n)$. So $X(\ell_n) \neq X(i_n)$, so $X \cap B_n$ must be either A_n or $B_n \setminus A_n$. One of the two alternatives holds infinitely often. Hence, either there are infinitely many n such that $X \cap B_n = A_n$, or there are infinitely many n such that $(\omega \setminus X) \cap B_n = A_n$.

Case 2: There are infinitely many such n with $X(\ell_n) \neq \pi_{\ell_n}(X \restriction \ell_n)$, where in addition $|A_n| = 2$, i.e., $A_n = B_n$. So for each such n, $X(\ell_n) \neq \pi_{\ell_n}(X \restriction \ell_n) = 1 - X(i_n)$. So $X(\ell_n) = X(i_n)$, so $X \cap B_n$ must be either B_n or $\emptyset$. One of the two alternatives holds infinitely often. So again we either get infinitely many n such that $X \cap B_n = A_n$, or infinitely many n such that $(\omega \setminus X) \cap B_n = A_n$.

Case 3: For infinitely many n as above we have $A_n = \emptyset$. Similar to the above. □

Corollary 1.12. $\mathfrak{s}_2 \leq \mathfrak{e}_2$

Proof. Let N be a model (of set theory) witnessing $\mathfrak{e}_2$, i.e., for every 2-predictor π there is a function $f \in N$ evading π.

Let R be any 2-rule. There is $X \in N$ evading π_R, so either X or $\omega \setminus X$ (both in N) follows R. □

Remark 1.13. $\mathfrak{s} \leq \mathfrak{e}_2$ is known. Brendle showed that $\mathfrak{s} < \mathfrak{s}_2$ is consistent (unpublished).

2. Consistency of $\mathfrak{r}_2 < \mathfrak{r}$

We show here in contrast to the result in previous section that $\mathfrak{r}$ is not provably equal to $\mathfrak{r}_2$. Moreover, whereas $\mathfrak{b} \leq \mathfrak{r}$ is provable in ZFC (see [6] for a collection of results on cardinal invariants), we show that $\mathfrak{r}_2 < \mathfrak{b}$ is consistent with ZFC.

The following definition is standard:

Definition 2.1.
1. S is a *slalom* iff $\mathrm{dom}(S) = \omega$ and for all $n \in \omega$, $S(n)$ is a finite set of size n.
2. If f is a function with $\mathrm{dom}(f) = \omega$, S a slalom, then we say that S *captures* f iff $\forall^\infty n\ f(n) \in S(n)$.
3. Let $M \subseteq N$ be sets (typically: models of ZFC). We say that N has the *Laver property* over M iff:

For every function $H \in {}^\omega\omega \cap M$, for every function $f \in {}^\omega\omega \cap N$ satisfying $f \leq H$ there is a slalom $S \in M$ that captures f.

4. A forcing notion P has the Laver property iff $\Vdash_P$ "V^P has the Laver property over V."

Before we formulate the main lemma, we need the following easy claim:

Claim 2.2. Let $k > 2^n$. If $X \subseteq {}^k2$, $|X| = n$ then there are $i < j$ in k such that for all $f \in X$, $f(i) = f(j)$.

Proof. For $i < j$, $f \in X$, define an equivalence relation $\sim_f$ by: $i \sim_f j \iff f(i) = f(j)$. Let $i \sim j$ iff $i \sim_f j$ for all f in X. Since each $\sim_f$ has at most 2 equivalence classes, $\sim$ has at most 2^n classes, so there are $i \neq j$, $i \sim j$. $\square$

Lemma 2.3. *Assume that $(N, \in)$ is a model of (some large fragment of) ZFC, and that V has the Laver property over N.*

Then every real avoids some 2-rule from N.

Proof. Let $a_0 = 0$, $a_{n+1} = a_n + 2^n + 1$. The sequence $(a_n : n \in \omega)$ is in N.

For any $X \in \mathcal{P}(\omega)$, we will find a rule in N which X does not follow.

Let $\chi_X \in {}^\omega 2$ be the characteristic function of X. Define $X^* := (\chi_X \restriction [a_n, a_{n+1}) : n \in \omega)$. Note that there are only 2^{2^n+1} many possibilities for $\chi_X \restriction [a_n, a_{n+1})$.

Since V has the Laver property over N there is a sequence $\vec{S} = (S_n : n \in \omega) \in N$, $S_n \subseteq {}^{[a_n, a_{n+1})}2$, $|S_n| \leq n$, and for all $n > 0$, $\chi_X \restriction [a_n, a_{n+1}) \in S(n)$. By the above claim we can find $i_n < j_n$ in $[a_n, a_{n+1})$ such that for all $z \in S(n)$, $z(i_n) = z(j_n)$. Since the sequence $\vec{S}$ is in N, we can find such a sequence $(i_n, j_n : n < \omega)$ in N.

Define a 2-rule $(A_n, B_n : n \in \omega) \in N$ by $A_n = \{i_n\}$, $B_n = \{i_n, j_n\}$. Since $i_n \in X$ iff $j_n \in X$, X does not follow this rule.

$\square$

Lemma 2.4. (a) *Let $\bar{P} = (P_i, Q_i : i < \aleph_2)$ be a countable support iteration of proper forcing notions such that for each i we have $\Vdash_i$ "Q_i has the Laver property." The P_{ω_2}, the countable support limit of $\bar{P}$, also has the Laver property.*

(b) *Laver forcing is proper and has the Laver property.*

(c) *Laver forcing adds a real that dominates all reals from the ground model.*

Proof. These facts are well known and (at least for the case where each Q_i is Laver forcing) appear implicitly or explicitly in Laver's paper [4]. $\square$

Conclusion 2.5. Let P_{ω_2} be the limit of a countable support iteration of Laver forcing over a model V_0 of GCH. Then $\Vdash_{P_{\omega_2}} \mathfrak{b} = \mathfrak{r} = \omega_2$ and $\mathfrak{r}_2 = \omega_1$.

Proof. Let $V_{\omega_2} = V^{P_{\omega_2}}$. $V_{\omega_2} \models \mathfrak{b} = \omega_2$ is well known. (Let f_i be the real added by the ith Laver forcing, then $(f_i : i < \omega_2)$ is a strictly increasing and cofinal sequence in ${}^\omega\omega$.

By 2.4, V_{ω_2} has the Laver property over V_0. Hence, by 2.3, every real avoids some rule from V_0. So $\mathfrak{r}_2 \leq |{}^\omega\omega \cap V_0| = \aleph_1$. $\square$

3. Application to independent families

A family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ of subsets of ω is *independent* iff it generates a free boolean algebra in $\mathcal{P}(\omega)/\text{fin}$. Equivalently, for any two disjoint finite subsets of $\mathcal{F}$, the intersection of all members in the first set with all complements of members in the second set is infinite.

The following is an example of an independent family of size continuum over a countable set: $\{A_r : r \in \mathbb{R}\}$ where $A_r = \{p \in \mathbb{Z}[X] : p(r) > 0\}$.

A family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is *dense* iff for any two finite disjoint subsets of ω there are infinitely many members of $\mathcal{F}$ that contain the first set and are disjoint to the second.

An interesting (proper) subclass of the class of dense independent families over ω is the class of homogeneous families, which was introduced in [2]. Its study was continued in [3].

While every dense independent family is contained in a maximal dense independent family, this is not obvious (and perhaps false) for homogeneous families. The existence, even the consistency, of a maximal homogeneous family over ω is still open. In particular, an increasing union of homogeneous families needs not be homogeneous.

In the study of extendibility of homogeneous families, the following notion is fundamental: Let $G \subseteq \operatorname{Aut}\mathcal{F}$. We define $(\mathcal{F}, G) \leq (\mathcal{F}', G')$ iff $\mathcal{F} \subseteq \mathcal{F}'$, $G \subseteq G' \subseteq \operatorname{Aut}\mathcal{F}'$. The usefulness of $\leq$ is that unions of suitable $\leq$ chains *are* homogeneous (see [3] for a detailed account of direct limits in the category of homogeneous families).

We show now that below $\mathfrak{r}_\infty$ one can get proper $\leq$-extensions of independent families. This was our original motivation for discovering $\mathfrak{r}_\infty$.

Theorem 3.1. *Suppose $G \subseteq \operatorname{Aut}\mathcal{F}$, $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is dense independent and $|\mathcal{F}| + |G| < \mathfrak{r}_\infty$, then there exists $\mathcal{F}' \supsetneqq \mathcal{F}$ such that $(\mathcal{F}, G) \leq (\mathcal{F}', G)$*

Proof. Suppose that $G \subseteq \operatorname{Aut}\mathcal{F}$, $\mathcal{F}$ is dense independent and $|G| + |\mathcal{F}| < r_\infty$. We shall find a real $X \subseteq \omega$ such that $X \notin \mathcal{F}$ and $\mathcal{F} \cup G[X]$ is independent, where $G[X]$ is the orbit of X under G. This will suffice, since clearly $G \subseteq \operatorname{Aut}\mathcal{F} \cup G[X]$ for any real X.

It is a priori unclear why such X should exist. If for example there is some $\sigma \in G$ with finite support, then for no $X \subseteq \omega$ even the orbit $G[X]$ itself is independent. However, the following lemma takes care of this. Let $\operatorname{supp}(\sigma) = \{X \in \omega : \sigma(X) \neq X\}$ for a permutation $\sigma \in \operatorname{Sym}\omega$.

Lemma 3.2. *Suppose that $\mathcal{F}$ is dense independent and $\sigma \in \operatorname{Aut}\mathcal{F}$ is not the identity. Then there are distinct sets $B_n \in \mathcal{F}$ such that for all n, $B_{2n} - B_{2n+1} \subseteq \operatorname{supp}\sigma$.*

Remark 3.3. In particular, the supports of non-identity automorphisms form a filter base. This is the "strong Mekler condition" for $\operatorname{Aut}\mathcal{F}$ (see [5]).

Proof. Find $X \in \omega$ for which $\sigma(X) \neq X$. Find B_{2n}, B_{2n+1} by induction on n. Suppose B_k is chosen for $k < 2n$. By density, there are infinitely many $B \in \mathcal{F}$ for which $X \in B$, $f(X) \notin B$. Choose some such B so that B and $f[B]$ are not among $\{B_k : k < 2n\}$. Let $B_{2n} = B$ and $B_{2n+1} = f[B]$. Since $X \in B_{2n} - B_{2n+1}$, those sets are indeed distinct.

If $X \in B_{2n} - B_{2n+1}$ then $f(X) \in B_{2n+1}$ and therefore $X \neq f(X)$. □

Let M be a transitive model of a sufficiently large fragment of ZFC with $\mathcal{F}, G \in M$ and $G \subseteq M, \mathcal{F} \subseteq M$. Let X be a real that satisfies all bounded rules from M. Clearly, $X \notin M$, therefore $X \notin \mathcal{F}$.

We need to show that every boolean combination over $\mathcal{F} \cup G[X]$ is infinite. Suppose that

$$C := A \cap \sigma_0[X] \cap \cdots \cap \sigma_{n-1}[X] \cap (\omega - \sigma_n[X]) \cap \cdots \cap (\omega - \sigma_{m-1}[X])$$

is a boolean combination over $\mathcal{F} \cup G[X]$, where $\sigma_i \in G$ for $i < m$, and A is some boolean combination over $\mathcal{F}$. Clearly, $A \in M$.

Set $N = \binom{m}{2}$ and let $(\tau_i : i < N)$ be a list of all $\sigma_k \circ \sigma_\ell^{-1}$ for $k < \ell < m$. By induction on $k \le N$ find distinct B_{2k}, B_{2k+1} which do not participate in A and so that $B_{2k} - B_{2k+1} \subseteq \operatorname{supp} \tau_k$. This is possible by Lemma 3.2.

The intersection $D = A \cap \bigcap_{k<N} B_{2k} - \bigcup_{k<N} B_{2k+1}$ is infinite and belongs to M. Define by induction an m-rule $(A_j, B_j : j < \omega)$ as follows: suppose $(A_k, B_k : k \le j)$ are defined. Find a point $X_n \in D$ such that $B = \{\sigma_k^{-1}(X) : k < m\}$ is disjoint from $\bigcup_{\ell \le n} B_\ell$ and $X_n \notin \bigcup_{\ell \le n} B_\ell$. Let B_{n+1} be B and let $A_{n+1} = \{\sigma_\ell^{-1}(X) : \ell < n\}$.

The rule we defined obviously belongs to M. Since X satisfies all bounded rules from M, there are infinitely many n for which $X \cap B_n = A_n$. For each such n, $X_n \in C$. □

Theorem 3.4. *Suppose that every dense independent family of cardinality below $\mathfrak{r}_\infty$ extends to a homogeneous family. Then every dense independent family of cardinality smaller than $\mathfrak{r}_\infty$ can be extended to a homogeneous family of size $\mathfrak{r}_\infty$.*

Proof. Suppose $\mathcal{F}$ is given dense independent family and $|\mathcal{F}| < \mathfrak{r}_\infty$. Construct a $\le$-chain $((\mathcal{F}_i, G_i) : i < \mathfrak{r}_\infty)$ as follows: $\mathcal{F}_0 = \mathcal{F}$ and $G_0 = \{\mathrm{id}\}$. If i is limit then $\mathcal{F}_i = \bigcup_{j<i} \mathcal{F}_j$ and $G_i = \bigcup_{j<i} G_j$. Let $(\mathcal{F}_{i+1}, G_{i+1})$ be a proper $\le$-extension of $(\mathcal{F}_i, G_i)$ for a successor ordinal $i+1 < \mathfrak{r}_\infty$, which exists by Theorem 3.1. We may assume without loss of generality that $\mathcal{F}_{i+1}$ is homogeneous and that G_{i+1} acts homogeneously on $\mathcal{F}_{i+1}$ (See [2] or [3] for the definitions).

The union $\bigcup_{i<\mathfrak{r}_\infty} \mathcal{F}_i$ is a homogeneous extension of $\mathcal{F}$ as required. □

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