REPRESENTING EMBEDDABILITY AS SET INCLUSION

June 1995

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ABSTRACT. **A few steps are made towards representation theory of embeddability among uncountable graphs. A monotone class of graphs is defined by forbidding countable subgraphs, related to the graph's end-structure. Using a combinatorial theorem of Shelah it is proved:**

- **The complexity of the class in every regular uncountable $\lambda > \aleph_1$ is at least $\lambda^+ + \sup\{\mu^{\aleph_0} : \mu^+ < \lambda\}$**
- **For all regular uncountable $\lambda > \aleph_1$ there are 2^λ pairwise non embeddable graphs in the class having strong homogeneity properties.**
- **It is characterized when some invariants of a graph $G \in \mathcal{G}_\lambda$ have to be inherited by one of fewer than λ subgraphs whose union covers G.**

All three results are obtained as corollaries of a *representation theorem* **(Theorem 1.10 below), that asserts the existence of a surjective homomorphism from the relation of embeddability over isomorphism types of regular cardinality $\lambda > \aleph_1$ onto set inclusion over all subsets of reals or cardinality λ or less. Continuity properties of the homomorphism are used to extend the first result to all singular cardinals below the first cardinal fixed point of second order.**

The first result shows that, unlike what Shelah showed in the class of all graphs, the relations of embeddability in this class is not independent of negations of the GCH.

§0 Introduction

The study of embeddability among infinite structures has a long tradition of invoking combinatorics. One well known example is Laver's use of Nash-Williams' combinatorial results to show that embeddability among countable order types is well quasi ordered [L]. In the study of embeddability among uncountable structures, the most prominent combinatorial principle has been the Generalized Continuum Hypothesis (GCH), which asserts that every infinite set has the least possible number of subsets. Hausdorff proved as early as 1914 using the GCH that in every infinite power there is a universal linear ordering, that is, one in which every linear ordering is embedded as a subordering. Jonsson [Jo] used the GCH to prove that classes of structures satisfying a list of 6 axioms have universal structures in all uncountable powers. See also [R] for graph theory and [MV] for model theory.

Finer combinatorial principles have come from Jensen's work in Gödel's universe of constructible sets [Je]. Thus, for example, Macintyre [M] uses Jensen's diamond — a principle stronger than CH — to prove that no abelian locally finite group of size $\aleph_1$ is embeddable in all universal locally finite groups of size $\aleph_1$, and Komjath and Pach [KP1] use the same principles to prove that there is no universal graph in power $\aleph_1$ among all graphs omitting K_{ω,ω_1}.

A common property of the combinatorial principles mentioned above is that they are not provable from the usual axioms of Set Theory. Easton [E] showed that the GCH can fail for all regular cardinal. Magidor [Ma] showed the GCH could fail at $\aleph_\omega$ with GCH below it and Foreman and Woodin [FW] showed that the GCH could fail everywhere (both using large cardinals). In Spite of this, the common impression among mathematicians working in areas having intimate relations to infinite cardinals, like infinite graph theory, infinite abelian groups, and model theory, remained that the GCH was a useful assumption, while its negations were not.

In the context of embeddability this impression was fortified by Shelah's independence results. Shelah showed that universal structures in uncountable powers may or may not exist under negations of the GCH. Thus, while GCH implies the existence of universal graphs in all infinite λ, the assumption $\lambda < 2^{\aleph_0}$ for regular uncountable λ does not determine the existence or non existence of a universal graph in λ (see [S3], [Me] and [K]).

Shelah's independence results [S1,2,3] created the expectation that the existence of a universal structure in a class of structures in uncountable cardinalities would always be independent of negations of GCH, unless the existence was trivial (because the class of structures is "dull"). See, for example, [KS] for results about the class of K_{ω_1}-free graphs that support this expectation.

The understanding of negations of the GCH at singular cardinals has changed dramatically in the last five years. The most fascinating development in this area is Shelah's bound on the exponents of singular cardinals. Shelah proved the following magnificent theorem, formulated here, though, in a way Shelah himself resents:

0.1 Theorem: *If $2^{\aleph_n} < \aleph_\omega$ for all n then $2^{\aleph_\omega} < \aleph_{\omega_4}$.*

Knowing that by Cohen's results no bound can be put on the exponent of a regular cardinal, this theorem is exceptionally thrilling. A short proof of it can be found in [J].

The formulation Shelah prefers is the following:

0.1a Theorem: $\mathrm{cf}\,([\aleph_\omega]^{\aleph_0}, \subseteq) < \aleph_{\omega_4}$

This formulations says that the cofinality of the partial ordering of set inclusion over countable subsets of $\aleph_\omega$ is ALWAYS smaller than $\aleph_{\omega_4}$, no matter how large $\aleph_\omega^{\aleph_0}$ may be. In other words, this theorem exposes a robust *structure* of the partial ordering of set inclusion, which is affected by negations of the GCH in a limited way only. The reader will verify that 0.1a implies 0.1. A proof of this theorem is in Shelah's recent book on Cardinal arithmetic [S]. In this book Shelah reduces the problem of computing the exponent of a singular cardinal to an algebra of reduced products of regular cardinal, and uses a host of new and sophisticated combinatorics to analyze the structure of such reduced products.

A common property of the combinatorial principles Shelah uses in [S] and in later works on cardinal arithmetic, is that they *are* proved in ZFC, the usual axiomatic framework of set theory. This is necessary, since 0.1a (unlike the conclusion of 0.1) is an absolute theorem, namely assumes nothing about cardinal arithmetic.

In this paper we use some of Shelah's combinatorics to expose robust connections between the structure of embeddability over a monotone class of infinite graphs and the relation of set inclusion. This is done by means of a *representation theorem*, that asserts the existence of a surjective homomorphism from the former relation onto the latter. One corollary is that the structure of embeddability over the class we shall study — which is defined by imposing restrictions on the the graph's end-structure — is not independent of negations of GCH, but also information that is not related to cardinal arithmetic is obtained.

Shelah's ZFC combinatorics on uncountable cardinals was found useful in the study of embeddability in several papers. In [KjS1] it was shown that if $\lambda > \aleph_1$ is regular and $\lambda < 2^{\aleph_0}$ then there is no universal linear ordering in λ. In other words, an appropriate negation of CH determines negatively the problem of existence of a universal linear ordering in power λ. Similar results were proved for models of first order theories [KjS2]; infinite abelian groups [KjS3] and [S4] and metric spaces [S5]. But so far no application was found for infinite graphs, in spite of the existing rich and active theory of universal graphs.

The theory of universal graphs, that began with Rado's construction [R] of a countable strongly universal graph, has advanced considerably since, especially in studying universality over monotone classes (see [DHV] for motivation for this). A monotone class of graphs is always of the form $\mathrm{Forb}\,(\Gamma)$, the class of all graphs omitting a some class Γ of "forbidden" configurations as subgraphs. A good source for the development of this theory is the survey paper [KP1], in which the authors suggest a generalization of universality, which they name "complexity": the complexity of a class of graphs is the least number of members in the class needed to embed as induced subgraphs all members in the class. The complexity is 1 exactly when a strongly universal graphs exists in the class.

The paper is organized as follows. In Section 1 a class of graphs is specified by forbidding countable configurations related to the graph's end-structure, and it is noted that by a generalization of a theorem by Diestel, Halin and Vogler the complexity of the resulting class $\mathcal{G}$ at power λ is at least λ^+. A surjective homomorphism is now constructed from the relation of (weak) embeddability over $\mathcal{G}_\lambda$ for regular $\lambda > \aleph_2$ onto the relation of set-inclusion over all subsets of reals of cardinality $\leq \lambda$. Combining both results, $\max\{\lambda^+, 2^{\aleph_0}\}$ is set as a lower bound for the complexity of $\mathcal{G}_\lambda$ for regular $\lambda > \aleph_1$.

In Section 2 a certain continuity property of the homomorphism from Section 1 is proved, and is used to extend the lower bound from Section 1 to all singular cardinals below the first fixed point of second order. In this Section the representation Theorem is stated in its full generality, generalizing Theorem 1.8 to higher cardinals.

In Section 3 it is proved that in every regular $\lambda > \aleph_1$ there are 2^λ pairwise non mutually embeddable elements in $\mathcal{G}_\lambda$, each of which being "small" in the sense that it is mapped by the homomorphism to a finite set. No cardinal arithmetic assumptions are made in this Section and in Section 4.

In Section 4 a decomposition theorem is proved for a proper subclass of $\mathcal{G}_\lambda$, $\lambda > \aleph_1$ regular, which is also defined by forbidding countable configurations. The Theorem gives a necessary and sufficient condition to when the invariant of a graph G in the class is inherited by at least one subgraphs from a collection of $< \lambda$ subgraphs whose union covers G.

NOTATION A graph G is a pair $\langle V, E\rangle$ where V is the set of vertices and $E \subseteq [V]^2$ is the set of edges. By $G[v]$ we denote the *neighbourhood* of $v \in V$ in G, namely $\{u \in V : \{v, u\} \in E\}$. A graph G is *bipartite* if there is a partition $G = G_1 \cup G_2$ of G to two (non-empty) disjoint independent vertex sets, each of which is called a *side*.

An ordinal is a set which is well ordered by $\in$. A *cardinal* is an initial ordinal number. The cardinality $|A|$ of a set A is the unique cardinal equinumerous with A. The *cofinality* $\operatorname{cf}\lambda$ of a cardinal λ is the least cardinal κ such that λ can be represented as a union of κ sets, each of cardinality less than λ. A cardinal λ is singular if $\operatorname{cf}\lambda < \lambda$ and is *regular* if $\operatorname{cf}\lambda = \lambda$. If κ, κ' are cardinals we denote by K_κ the complete graphs on κ vertices and by $K_{\kappa,k'}$ the complete bipartite graphs with κ vertices in one side and κ' in the other.

If G_1 is isomorphic to a subgraph of G_2 we write $G_1 \leq_w G_2$ and we write $G_1 \leq G_2$ if G_1 is isomorphic to an *induced* subgraph of G_2. We also say the G_1 is *embeddable* (embeddable as an induced subgraph) if $G_1 \leq_w G_2$ ($G_1 \leq G_2$).

Classes of graphs will be denoted by $\mathcal{G}$ and Γ and are always assumed to be closed under isomorphism. A class of graphs $\mathcal{G}$ is *monotone* if $G_1 \leq_w G_2 \in \mathcal{G} \Rightarrow G_1 \in \mathcal{G}$. If Γ is a set of graphs then $\operatorname{Forb}(\Gamma)$ is the class of all graphs without a subgraph in Γ. Let $\mathcal{G}_\lambda$ be the set of all isomorphism types of $\mathcal{G}$ whose cardinality is λ. The relations $\leq$ and $\leq_w$ are reflexive and transitive, and therefore $\langle \mathcal{G}_\lambda, \le \rangle$ and $\langle \mathcal{G}_\lambda, \le_w \rangle$ are quasi-ordered sets for all classes $\mathcal{G}$ and cardinals λ.

Let $\mathrm{cp}\,\mathcal{G}_\lambda$, the *complexity* of $\mathcal{G}_\lambda$, be the least cardinality of a subset $D \subseteq \mathcal{G}_\lambda$ with the property that for every $G \in \mathcal{G}_\lambda$ there exists $G' \in D$ such that $G \le G'$; the *weak complexity* is defined by replacing $\le$ by $\le_w$ (see [KP]). Clearly, $\mathrm{wcp}\,\mathcal{G} \le \mathrm{cp}\,\mathcal{G}$ for any class $\mathcal{G}$.

The complexity $\mathrm{cp}\,\mathcal{G}_\lambda$ is 1 iff there is a graph $G^* \in \mathcal{G}_\lambda$ with the property that every member of $\mathcal{G}_\lambda$ is isomorphic to an induced subgraph of G^*. Such a graph G^* is called *universal in* λ (or, sometimes, "strongly universal") for the class $\mathcal{G}$. $\mathrm{wcp}\,\mathcal{G}_\lambda = 1$ is equivalent to the existence of a *weakly universal* element in $\mathcal{G}_\lambda$.

Suppose that A is a given infinite set. By $[A]^\lambda$ we denote the collection $\{B : B \subseteq A \;\&\; |B| = \lambda\}$ of all subsets of A whose cardinality is λ. If $B_1 \in [A]^\lambda$ is contained as a subset in $B_2 \in [A]^\lambda$ we write $B_1 \subseteq B_2$. Since $\subseteq$ is reflexive, transitive and antisymmetric, $\langle [A], \subseteq \rangle$ is a partially ordered set. Let $\mathrm{cov} A_\lambda$, the covering number of $[A]^\lambda$, be the least cardinality of a subset $D \subseteq [A]^\lambda$ with the property that for every $B \in [A]^\lambda$ there exists $B' \in D$ such that $B \subseteq B'$.

We remark that the least cardinality of a dominating subset is defined for every quasi-ordered set, and bears the name "cofinality"; but we stick here to the customary graph-theoretic and set theoretic existing terminologies and refer to the former as "complexity" and to the latter as "covering number".

0.2 Definition: Let λ be an uncountable regular cardinal. A *club* of λ is a closed (in the order topology) and unbounded subset of λ. Club sets generate a *filter* over λ, indeed a λ-*complete* filter: the intersection of fewer than λ subsets of λ, each of which contains a club, contains a club. A subset of λ is called *stationary* if its intersection with every club of λ is non empty. The ideal of all subsets of λ which are disjoint to some club of λ is the *non-stationary* ideal. Thus club sets are analogous to measure 1 sets, non-stationary sets are measure zero and stationary sets are positive measure (meet every measure 1 set). Let S^λ_κ be $\{\alpha : \alpha < \lambda \wedge \mathrm{cf}\,\alpha = \kappa\}$ and $S^\lambda_0 = \{\alpha : \alpha < \lambda \wedge \mathrm{cf}\,\alpha = \omega\}$.

We shall need the following combinatorial tool:

0.3 Theorem:(Shelah) *If $\lambda >$ is regular, μ a cardinal and $\mu^+ < \lambda$ then there is a stationary* *set* $S \subseteq \lambda$ *and a sequence* $\bar{C} = \langle c_\delta : \delta \in S \rangle$ *with* $\mathrm{otp}\, c_\delta = \mu$ *and* $\sup c_\delta = \delta$ *such that for every closed unbounded* $E \subseteq \lambda$ *the set* $N(E) := \{\delta \in S : c_\delta \subseteq E\}$ *is stationary.*

For a proof see I[Sh-e, new VI§2]= [Sh-e, old III§7].

A sequence $\bar{C}$ as in the theorem is called a "club guessing sequence". If the c_δ are thought of as "guesses", then the theorem says that for every club (measure 1) set stationarily many (positive measure) of the guesses are successful.

Suppose $\bar{C}$ is a club guessing sequence as above. We define two *guessing ideals* over λ, $\mathrm{id}^a(\bar{C})$ and $\mathrm{id}^b(\bar{C})$, as follows:

0.4 Definition:

(0) $X \in \mathrm{id}^a(\overline{C})$ iff for some club $E \subseteq \lambda$ it holds that $c_\delta \not\subseteq E$ for all $\delta \in S \cap E$.

(1) $X \in \mathrm{id}^b(\overline{C})$ iff for some club $E \subseteq \lambda$ it holds that $c_\delta \not\subseteq^* E$ for all $\delta \in S \cap E$, where $c_\delta \subseteq^*$ means that an end segment of c_δ is contained in E.

Thus a set $X \subseteq \lambda$ is in $\mathrm{id}^a(\bar{C})$ iff there are no stationarily many $\delta \in X$ such that c_δ is contained in E for some club E, and $X \subseteq \lambda$ is in $\mathrm{id}^b(\bar{C})$ iff there are no stationarily many $\delta \in X$ such that c_δ is almost (=except for a proper initial segment) contained in E for some club E.

The ideal $\mathrm{id}^a(\bar{C})$ is a λ-complete ideal over λ and $\mathrm{id}^b(\bar{C})$ is normal. Also, $\mathrm{id}^b(\bar{C}) \subseteq \mathrm{id}^a(\bar{C})$.

0.5 Definition: Let ω be the set of natural numbers. Let Fin be the set of all finite subsets of ω. Two subsets $X, Y \subseteq \omega$ are *equivalent* mod Fin iff the symmetric difference $X \setminus Y \cup Y \setminus X \in \mathrm{Fin}$. By $\mathcal{P}(\omega)$ we denote the *power set* of ω and by $\bar{\mathcal{P}}(\omega)$ we denote $\mathcal{P}(\omega)/\mathrm{Fin}$ the set of all equivalence classes of subsets of ω modulo Fin.

Finally, we need a few definitions about reduced powers. A reduced power is a generalization of *ultra-power.*

0.6 Definition: Suppose that A is a structure, λ a cardinal and F a filter over λ. Let A^λ be the set of all functions from λ to the structure A and let A^λ/F be the reduced power of A modulo F.

§1 Representing embeddability as set inclusion

In [K] it is proved:

1.1 Theorem: *If $\mathcal{G}$ is a class of graphs that contains all $K_{\omega,\omega}$-free incidence graphs of $A \subseteq \mathcal{P}(\omega)$ and the cofinality of the continuum is $\aleph_1$, then* $\mathrm{cp}\,\mathcal{G}_\lambda > 2^{\aleph_0}$ *for all uncountable* $\lambda < 2^{\aleph_0}$.

In particular, if $\mathrm{cf}\, 2^{\aleph_0} = \aleph_1$ there is no universal graph (in the class of all graphs) in any uncountable $\lambda < 2^{\aleph_0}$.

On the other hand, Shelah proved in [S3]:

1.2 Theorem: *If λ is regular uncountable, it is consistent that $\lambda < 2^{\aleph_0}$ and that a universal graph in power λ exists.*

Mekler [Me] generalized Shelah's result to more general classes of structures.

Both result together can be understood as follows: a singular $2^{\aleph_0}$ affects the structure of embeddability in a broad spectrum of classes of infinite graphs below the continuum, but a large regular $2^{\aleph_0}$ may have no effect on the class of all graphs and the classes handled by Mekler and Shelah.

It is reasonable to ask if for *some* "reasonably defined" class of graphs for which the structure of embeddabilty below $2^{\aleph_0}$ is influenced by the size of $2^{\aleph_0}$. In this section we show that forbidding certain countable configurations gives rise to a class with such a desired connection. The configurations we forbid are related to the *end structure* of graphs.

1.3 Definition: A *ray* in a graph G is a 1-way infinite path. A *tail* of a ray $R \subseteq G$ is an infinite connected subgraph of R. Two rays in G are *tail-equivalent* iff they share a common tail. Tail-equivalence is an equivalence relation on rays.

We mention in passing that tail-equivalence is a refinement of *end-equivalence.* For more on both relations see [D].

1.4 Definition: Let $\mathcal{G}$ be the class of all graphs G satisfying that for every $v \in G$ the induced subgraph of G spanned by $G[v]$ has at most one ray up to tail-equivalence.

1.5 Claim: *There is a non-empty set Γ of countable graphs, each containing an infinite path, such that* $\mathcal{G} = \mathrm{Forb}\,(\Gamma)$.

Proof: : Let Γ' be the set of all countable graphs that contain (at least) two tail-inequivalent rays and let Γ be all graphs obtained by choosing an element from Γ' and joining a new vertex to all its vertices. If a graph G contains a subgraph in Γ then $G \notin \mathcal{G}$. Conversely, Suppose that $G \notin \mathcal{G}$. Let $v \in G$ be a vertex such that there are two rays $R_1, R_2 \subseteq G[v]$ which are not tail-equivalent. Let $G' \subseteq G$ be the induced subgraph spanned by $\{v\} \cup R_1 \cup R_2$. Now $G' \in \Gamma$ and so $G \notin \mathrm{Forb}\,(\Gamma)$.

Graphs with forbidden countable configuration that contain an infinite path were considered by Diestel, Halin and Vogler in [DHV] for $\lambda = \aleph_0$. They prove (Theorem 4.1):

1.6 Theorem: (Diestel-Halin-Vogler) *Let Γ be a non-empty set of countable graphs each containing an infinite path. Then $\mathcal{G}_{\aleph_0} = \mathcal{G}_{\aleph_0}(\Gamma)$ has no universal element.*

This Theorem applies to our class $\mathcal{G}$ by 1.5 above, and because every forbidden configuration in Γ contains an infinite path. The following is a straightforward generalization of Theorem 1.6, and is included only for completeness of presentation's sake:

1.7 Theorem: *Let Γ be a non-empty set of countable graphs each containing an infinite path. Then* $\mathrm{wcp}\,\mathrm{Forb}_{\lambda}(\Gamma) \geq \lambda^+$ *for all infinite cardinals λ.*

Proof: By induction on $\alpha < \lambda^+$ define graphs G_α as follows: G_α is obtained by joining a vertex w_α to a disjoint union of G_β for all $\beta < \alpha$ (such that w_α is not in this union). For all $\lambda \leq \alpha < \lambda^+$ the graph G_α contains no infinite path and therefore belongs to $\mathrm{Forb}_{\lambda}(\Gamma)$. Suppose that $\mathcal{F}$ is a collection of λ graphs and $f_\alpha : G_\alpha \to G(\alpha)$ is an embedding of G_α into some graph $G(\alpha) \in \mathcal{F}$. By the pigeon hole principle there is a fixed graph $G \in \mathcal{F}$ such that $G = G(\alpha)$ for λ^+ many $\alpha < \lambda^+$. A second use of the pigeon hole principle gives a vertex $w(0) \in G$ and an unbounded set $X \subseteq \lambda^+$ such that $w(0) = f_\alpha(w_\alpha)$ for all $\alpha \in X$. This implies, by the construction of the G_α's, that $G[w(0)]$ contains as subgraphs copies of G_α for unboundedly many $\alpha < \lambda^+$ and therefore of all $\alpha < \lambda^+$. Repeating this argument a set $\{w(n) : n < \omega\} \subseteq G$ is found that spans in G a copy of K_ω. Therefore G contains all countable configurations and therefore does not belong to $\mathrm{Forb}_{\lambda}(\Gamma)$. $\triangle$

Thus for every infinite λ we have $\mathrm{wcp}\,\mathcal{G}_\lambda \geq \lambda^+$. Also Theorem 1.1 from the previous section applies to $\mathcal{G}$, because $\mathcal{G}$ contains all bipartite graphs; thus (setting $\theta = \aleph_0$), $\mathrm{cp}\,\mathcal{G}_\lambda \geq {2^{\aleph_0}}^+$ if $\mathrm{cf}\, 2^{\aleph_0} \leq \lambda$ (we use here $\mathrm{wcp} \leq \mathrm{cp}$).

The virtue of $\mathcal{G}$ is, nevertheless, that $\operatorname{wcp} \mathcal{G}_\lambda \geq \max\{\lambda^+, 2^{\aleph_0}\}$ regardless to the cardinality of the continuum for all regular $\lambda > \aleph_1$ (and many singular λ, as seen in the next section). This is a corollary of the following:

1.8 Theorem: *If $\lambda > \aleph_1$ is regular then there is a surjective homomorphism $\Phi : \langle \mathcal{G}_\lambda, \leq_w \rangle \to ([\mathbf{R}]^{\leq\lambda}, \subseteq)$ from the relation of embeddability over $\mathcal{G}_\lambda$ onto subsets of reals or cardinality at most λ partially ordered by inclusion.*

Thus the relation of embeddability among members of $\mathcal{G}_\lambda$ is at least as complicated as inclusion among subsets of reals of cardinality at most λ.

1.9 Corollary: *If $\lambda > \aleph_1$ is regular then* $\operatorname{wcp} \mathcal{G}_\lambda \geq \max\{\lambda^+, 2^{\aleph_0}\}$.

This theorem will be extended to singular values of λ in the next section.

We turn now to the proof of the theorem. The homomorphism Φ will be factored through a reduced product of the inclusion relation over subsets of reals. We will prove the following stronger formulation:

1.10 Theorem: *Suppose that $\lambda > \aleph_1$ is regular. Then*

(0) there is a surjective homomorphism $\Phi : \langle \mathcal{G}_\lambda, \leq_w \rangle \to ([\mathbf{R}]^{\leq\lambda}, \subseteq)$

(1) Φ from (1) *can be chosen to be a composition $\psi\varphi$ where φ is a surjective homomorphism to a reduced power $\left([\mathbf{R}]^{\leq\lambda}, \subseteq\right)^\lambda / I$ for some normal ideal I over λ.*

Proof: : First let us notice that $([\mathbf{R}]^{\leq\lambda}, \subseteq)$ is a homomorphic image of $\left([\mathbf{R}]^{\leq\lambda}, \subseteq\right)^\lambda / I$ for every ideal I: Suppose that $\bar{A}$ is a representative of an equivalence class of $([\mathbf{R}]^{\leq\lambda})^\lambda/\mathbf{I}$. Define $\psi([\bar{A}]) := \{x \in \mathbf{R} : \{\delta < \lambda : \mathbf{x} \in \mathbf{A}(\delta)\} \notin \mathbf{I}\}$. In words, $\psi([\bar{A}])$ is the set of all reals that appear in a positive set of coordinates. It is routine to check that the definition of ψ does not depend on the choice of a representative and that ψ is a homomorphism.

Thus it suffices to prove that there is a surjective homomorphism $\varphi : \langle \mathcal{G}_\lambda, \leq_w \rangle \to \left([\mathbf{R}]^{\leq\lambda}, \subseteq\right)^\lambda / I$ for some normal ideal I over λ. This is in fact more than needed for (0). The set $\mathbf{R}$ can be replaced here by any set of equal cardinality. It is convenient for us to work with $\bar{\mathcal{P}}(\omega) = \mathcal{P}(\omega)/\mathrm{Fin}$.

We shall define a mapping $\varphi : (\mathcal{G}_\lambda, \leq) \to \left([\bar{\mathcal{P}}(\omega)]^{\leq\lambda}, \subseteq\right)^\lambda / I$ after specifying I. We shall show that φ is well defined, is a homomorphism and is surjective. For the definition of the mapping we fix a club guessing sequence $\overline{C} = \langle c_\delta : \delta \in S \rangle$, $S \subseteq \lambda$ stationary and otp $c_\delta = \omega$ for c_δ in $\bar{C}$. Now let $I = \mathrm{id}^b(\overline{C})$. For each $\delta \in S = S^\lambda_0$ let $\langle \alpha^\delta_n : n < \omega \rangle$ be the increasing enumeration of c_δ.

Given $G \in \mathcal{G}_\lambda$ we define $\varphi(G)$ after choosing two auxiliary parameters on G. First, we pick a well ordering $<$ of G of order type λ, namely a bijection h between the vertices of G and the ordinals below λ, and second, we fix a mapping r so that $r(v) = \emptyset$ for $v \in G$ in case there are no rays in $G[v]$ and $r(v)$ is a ray in $G[v]$ otherwise.

Let $G^{\alpha,<} = \{v \in G : h(v) < \alpha\}$ for $\alpha < \lambda$. When $<$ is unambiguous we write G^α for $G^{\alpha,<}$. For a vertex $v \in G$ and an ordinal $\delta \in S$ we define:

$$\varphi_{<,r}(v, \delta) = [\{n < \omega : r(v) \cap G^{\alpha_n} \subseteq r(v) \cap G^{\alpha_{n+1}}\}]_{\mathrm{Fin}} \tag{1}$$

Thus $\varphi_{<,r}(v, \delta)$ belongs to $\bar{\mathcal{P}}(\omega)$. The definition in (1) depends strongly on the choice of the well ordering $<$. In fact, for a vertex v whose neighbourhood $G[v]$ does contain a ray, $\varphi_{<,r}(v, \delta)$ can be made any prescribed element of $\bar{\mathcal{P}}(\omega)$ by a suitable choice of $<$. But replacing $r(v)$ by a tail-equivalent $r'(v)$ produces at most a finite change in $\{n < \omega : r(v) \cap G^{\alpha_n} \subseteq r(v) \cap G^{\alpha_{n+1}}\}$ and therefore does not change the definition (1).

Now let

$$\varphi_{<,r}(G, \delta) = \{\varphi_{<,r}(v, \delta) : v \in G\} \tag{2}$$

Since $|G| = \lambda$, we conclude that

$$\varphi_{<,r}(G, \delta) \in [\bar{\mathcal{P}}(\omega)]^{\le\lambda} \tag{3}$$

Finally, let

$$\varphi(G) = \varphi_{<,r}(G) = [\langle \varphi_{<,r}(G, \delta) : \delta \in S \rangle]_I \tag{4}$$

The sequence $\langle \varphi_{<,r}(G, \delta) : \delta \in S \rangle$ belongs to $\left([\bar{\mathcal{P}}(\omega)]^{\le\lambda}\right)^\lambda$, and we let $\varphi_{<,r}(G)$ be the equivalence class of this sequence in the reduced power mod I.

1.11 Proposition:

(a) The definition of $\varphi(G)$ does not depend on the choice of $<$ and r.

(b) φ is a homomorphism: if $G_1 \leq G_2$ then $\varphi(G_1) \subseteq_I \varphi(G_2)$.

(c) φ is surjective.

Proof: : We shall prove (a) and (b) simultaneously by proving

(d) If $G_1 \leq G_2$ are in $\mathcal{G}_\lambda$ and $<_1, <_2, r_1, r_2$ are any parameters as in (1) above for G_1, G_2 respectively, then $\varphi_{<_1,r_1}(G_1) \subseteq_I \varphi_{<_2,r_2}(G_2)$

Then (a) follows from (d) by putting $G_1 = G_2 = G$ and (b) follows from (a) and (d).

Let $<_1, <_2, r_1, r_2$ be given. We need to show that $\varphi_{<_1,r_1}(G_1) \subseteq_I \varphi_{<_2,r_2}(G_2)$, namely, that:

$$S \setminus \{\delta \in S : \varphi_{<_1,r_1}(G_1, \delta) \subseteq \varphi_{<_2,r_2}(G_2, \delta)\} \in I \tag{5}$$

Fixing an embedding from G_1 to G_2 we assume, without loss of generality, that G_1 is a subgraph of G_2. The following set is closed unbounded in λ by a standard back and forth argument:

$$E = \{\alpha < \lambda : G_2^\alpha \cap G_1 = G_1^\alpha\} \tag{6}$$

Therefore, by the definition of I, the set $N(E) = \{\delta \in S : c_\delta \subseteq E\}$ satisfies

$$S \setminus N(E) \in I \tag{7}$$

We shall show that for every $\delta \in N(E)$ we have for all $v \in G_1$ such that $r_1(v) \neq \emptyset$:

$$\varphi_{<_1,r_1}(v, \delta) = \varphi_{<_2,r_2}(v, \delta) \tag{8}$$

Let $v \in G_1$ be given. Clearly, $G_1[v] \subseteq G_2[v]$. Thus $r_1(x)$ and $r_2(x)$ are both rays in $G_2(x)$. Since $G_2 \in \mathcal{G}$, the rays $r_1(v), r_2(v)$ are tail-equivalent. Fix a common tail

$r(v)$ of $r_1(v), r_2(v)$. Using $r(v)$ instead of either $r_1(v)$ or $r_2(v)$ in the definition (1) for $\varphi_{<_i,r_i}(v,\delta)$ $(i = 1,2)$ makes no difference. So we may assume without loss of generality that $r_1(v) = r_2(v) = r(v)$ for all $v \in G_1$.

Suppose that $v \in G_1$ and that $c_\delta \subseteq^* E$. Then the sets $\{n < \omega : r(v) \cap G_1^{\alpha_n} \subseteq r(v) \cap G^{\alpha_{n+1}}\}$ and $\{n < \omega : r(v) \cap G_2^{\alpha_n} \subseteq r(v) \cap G^{\alpha_{n+1}}\}$ are equivalent modulo finite, because $G_1^{\alpha_n} = G_2^{\alpha_n}$ for an end segment of ω. Thus, the definition (2) above gives for every $\delta \in N(E)$:

$$\varphi_{<_1,r_1}(G_1,\delta) = \{\varphi_{<_1,r_1}(v,\delta) : v \in G_1\} = \{\varphi_{<_2,r_2}(v,\delta) : v \in G_1\} \subseteq \tag{9}$$

$$\subseteq \{\varphi_{<_2,r_2}(v,\delta) : v \in G_2\} = \varphi_{<_2,r_2}(G_2,\delta)$$

And now (5) follows by (7).

This proves (d) and hence (a) and (b).

To prove (c) fix a member $\bar{A} = \langle A_\alpha : \alpha < \lambda\rangle \in \left([\bar{\mathcal{P}}(\omega)]^{\leq\lambda}\right)^\lambda$. We shall construct a graph $G \in \mathcal{G}_\lambda$ so that $\varphi(G) = [\bar{A}]_I$.

Enumerate $A_\delta = \langle X_{\delta,\alpha} : \alpha < \lambda\rangle$ for $\delta \in S$. By induction on $\alpha < \lambda$ we define an increasing and continuous union of graphs G_α such that:

(a) $|G_\alpha| < \lambda$

(b) $\beta < \alpha < \lambda$ implies that G_β is an induced subgraph of G_α and if α is a limit ordinal then $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$.

(c) For every $\gamma < \beta + 1 \leq \alpha$ and a vertex $v \in G_\gamma$ there is a vertex $u \in G_{\beta+1} \setminus G_\beta$ such that $G_{\beta+1}[u] = \{v\}$ $(= G_\alpha[v] \cap G_{\beta+1}$ because $G_{\beta+1}$ is an induced subgraphs of G_α by (b)).

(d) if $\alpha = \delta + 1$ and $\delta \in S$ then for all $\delta' \in S \cap (\delta + 1)$ and $\gamma \leq \delta$ there is a vertex $y(\delta',\gamma) \in G_\alpha \setminus G_\delta$ and a path $\langle z_n : n < \omega\rangle$ in G_δ such that:

(i) $G_\alpha[y(\delta',\gamma)] = \{z_n : n < \omega\}$ and $G_\alpha[z_n]$ contains no rays.

(ii) If the increasing enumeration of $c_{\delta'} \setminus X_{\delta',\gamma}$ is $\langle \alpha_{m(n)} : m < \omega\rangle$ then $z_n \in G_{\alpha_{m(n)+1}} \setminus G_{\alpha_{m(n)}}$.

(e) $G_\alpha \in \mathcal{G}$ and for every $\beta < \alpha$ and $v \in G_{\beta+1} \setminus G_\beta$ the set $G_\alpha[v] \setminus G_\beta$ is independent.

Suppose first that this construction is carried out, and let $G = \bigcup_{\alpha<\lambda} G_\alpha$. This is a graph of cardinality λ. By (e) it follows that G belongs to $\mathcal{G}_\lambda$: if $v \in G \setminus G_0$ let $\beta+1$ be the minimal so that $v \in G_{\beta+1}$; $G[v] = G_\beta[v] \cup G[v] \setminus G_\beta$. $G_\beta[v]$ contains at most one ray up to tail-equivalence because $G_{\beta+1} \in \mathcal{G}$ (condition (e)) and $G[v] \setminus G_\beta = \bigcup_{\beta<\alpha<\lambda} G_\alpha[v] \setminus G_\beta$ is an independent set by (e) and therefore contains no rays at all. If $u \in G_0$ then $G[u]$ is independent.

Fix any well ordering $<$ of G of order type λ. It is standard to check that for a closed unbounded set $E \subseteq \lambda$ it holds that $G_\alpha = G^{\alpha,<}$ for all $\alpha \in E$. We may restrict attention to this set of indices alone. Suppose now that $\delta' \in S \cap E$ and let $\gamma < \lambda$ be given. Find $\delta \in S$ so that $\delta > \max\{\delta', \gamma\}$. At stage $\delta + 1$ the vertex $y(\delta, \gamma)$ mentioned in (d) satisfies that $\varphi_{<,r}(y(\delta,\gamma)) = X_{\delta,\gamma}$ for all $\gamma < \delta$, where r is any function as in (1) above. This shows that $\varphi_{<,r}(G) = [\bar{A}]_I$.

Let us see that the induction can be carried out. For $\alpha = 0$ let G_α be a single vertex and at limits take unions. Let us check that conditions (a)–(e) hold for $\alpha = 0$ and hold for a limit α if they hold for all $\beta < \alpha$. At successor stages $\alpha + 1$ we distinguish two cases:

Case 1: $\alpha \notin S$. In this case $G_{\alpha+1}$ is obtained from G_α by adding, for each $v \in G_a$, new vertices $\{x_{v,\beta} : \beta < \alpha\}$ and adjoining each of them to v; thus $G_{\alpha+1}[x_{v,\beta}] = \{v\}$. Since we added $|\alpha| < \lambda$ new vertices (a) holds. (b) holds trivially. $G_{\alpha+1}$ was defined so that (c) holds and (d) holds vacuously. (e) holds as no two new vertices are joined by an edge.

Case 2: $\alpha = \delta \in S$. For every $v \in G_\alpha$ add new $\{x_{v,\beta} : \beta < \alpha\}$ exactly as in case 1. In addition add vertices $y(\delta', \beta)$ for $\delta' \le \delta$ and $\beta \le \alpha$. We specify the neighbours of $y(\delta', \beta)$: Let $\langle \gamma_n : n < \omega \rangle \subseteq C_{\delta'}$ be the increasing enumeration of $X_{\delta,\beta}$. By induction on $n < \omega$ choose vertices $z_n \in G_{\gamma_n}$ so that z_{n+1} is connected to z_n and $G_\delta[z_n]$ contains no rays. This is possible by condition (b) and the induction hypothesis, that implies that only the vertices $y(\delta', \beta)$ contain rays in their neighbourhoods. Then connect $y(\delta', \beta)$ to all z_n. The requirement that z_n has no rays in its neighbourhood is not needed before Section 4, where it is needed to show that the graphs constructed here lies in a proper subclass. For the purpose of this proof it can be ignored. Conditions (a)–(c),(e) hold as in the previous case. Condition (d) was just handled. $\triangle$

§2 Continuity, singulars and r-subgraphs

In this section we study the homomorphism Φ from Theorem 1.8 and show it has a certain continuity property.

As a result we will be able to prove that Theorem 1.9 holds also for many singular cardinals μ.

2.1 Definition: For $G_1, G_2 \in \mathcal{G}_\lambda$ say that G_1 is an r-subgraph of G_2 iff G_1 is a subgraph of G_2 and for all $v \in G_1$ if $G_2[v]$ contains a ray then $G_1[v]$ contains a ray. Equivalently, if r is a function on G_2 such that $r(x)$ is a ray in $G_2[x]$ if such a ray exists and $\emptyset$ otherwise, then $\forall x \in G_1\, (r(x) \subseteq^* G_1)$.

2.2 Claim: (Continuity) *If $G \in \mathcal{G}_\lambda$ and $G = \bigcup_{\alpha<\beta} G_\alpha$ for some $\beta < \lambda$ so that G_α is an r-subgraph of G for every $\alpha < \beta$ then*

$$\Phi(G) = \bigcup_{\alpha<\beta} \Phi(G_\alpha)$$

Proof: For every $\alpha < \beta$ the relation $\Phi(G_\alpha) \subseteq_I \Phi(G)$ follows because Φ is a homomorphism and $G_\alpha \leq G$.

On the other hand, suppose that $A \in \Phi(G)$, and we will show that for some $\alpha < \beta$ we have $A \in \Phi(G_\alpha)$.

Let $<, r$ and $<_\alpha, r_\alpha$ be chosen parameters for G and for every G_α for $\alpha < \beta$ respectively, as in the definition of φ. For every $\alpha < \beta$ there is a set $X_\alpha \in I$ such that for every $\delta \in \lambda \setminus X_\alpha$ and every $v \in G_\alpha$ for which $r(v) \neq \emptyset$ condition (8) in the proof of Proposition 1.11 holds:

$$\varphi_{<_\alpha, r_\alpha}(v, \delta) = \varphi_{<, r}(v, \delta) \tag{8}$$

Let $X = \bigcup_{\alpha<\beta} X_\alpha$. By λ-completeness of I we know that $X \in I$. Suppose $\delta \in \lambda \setminus X$ let $v \in G$.

Since $A \in \Phi(G)$, the set $Y := \{\delta < \lambda : A \in \varphi_{<,r}(G, \delta)\}$ is positive. For every $\delta \in Y$ pick $v_\delta \in G$ so that $\varphi_{<,r}(v_\delta, \delta) = A$.

For every $\delta \in Y$ there is some $\alpha(\delta) < \beta$ for which $v_\delta \in G_{\alpha(\delta)}$. Since I is λ-complete, there is a fixed $\alpha < \beta$ and a positive set $Y' \subseteq Y$ such that $\alpha(\delta) = \alpha$ for all $\delta \in Y'$.

Since $X \in I$, we may, without loss of generality, subtract this union from Y', and Y' would still be positive. But now it follows from (8) that for every $\delta \in Y'$ it holds that

$$\varphi_{<_\alpha, r_\alpha}(v_\delta, \delta) = \varphi_{<,r}(v_\delta, r) = A$$

Since Y' is positive this shows that $A \in \Phi(G_\alpha)$ and thus completes the proof.

2.3 Corollary: *If μ is singular and μ smaller than the first cardinal fixed point of second order, then* $\operatorname{cp} \mathcal{G}_\mu \geq \max\{\mu^+, 2^{\aleph_0}\}$.

Proof: We address only the term $2^{\aleph_0}$, the other following from 1.7. Assume, then, that μ is singular, smaller than the first fixed point of second order, and that $\mu < 2^{\aleph_0}$. To prove that $\operatorname{wcp} \mathcal{G}_\mu \geq 2^{\aleph_0}$ suppose $\kappa < 2^{\aleph_0}$ and that for every $\alpha < \kappa$ we are given $G_\alpha \in \mathcal{G}_\mu$, and we will exhibit a graph $G \in \mathcal{G}_\lambda$ for some regular $\lambda < \mu$ which is not embedded in any G_α for $\alpha < \kappa$.

By the assumption on μ, and since $|G_\alpha| = \mu$ for $\alpha < \kappa$, for every $\alpha < 2^{\aleph_0}$ there exists a regular $\lambda < \mu$ and a family $\mathcal{F}_\alpha \subseteq [G_\alpha]^\lambda$ with $|\mathcal{F}_\alpha| = \mu$ such that for all $X \in [G_\alpha]^\lambda$ there is $F \in [\mathcal{F}_\alpha]^{<\lambda}$ so that $X \subseteq \bigcup F$ (see [KjS1],4.5). So while $\mathcal{F}_\alpha$ itself may not be dominating in $\langle [G_\alpha]^\lambda, \subseteq \rangle$, the set of all unions of $< \lambda$ members of G_α is dominating (but has cardinality larger than μ in this case). We may assume, by increasing each $A \in \mathcal{F}_\alpha$ to a larger set of the same cardinality, that each $A \in \mathcal{F}_\alpha$ spans an r-subgraph of G_α, and abusing notation we shall not distinguish between A and the subgraph it spans.

The cardinality of the following set is $\kappa \times \mu$:

$$D := \{\Phi(A) : A \in \bigcup_{\alpha < 2^{\aleph_0}} \mathcal{F}_\alpha\}$$

As $\kappa \times \mu < 2^{\aleph_0}$ and $|\Phi(A)| = \lambda$ for $\Phi(A) \in D$, we have $|\bigcup D| < 2^{\aleph_0}$. Thus we can find a set $B \in [R]^\lambda$ such that $B \not\subseteq \bigcup D$ (we can actually choose B to be disjoint of the union of D)

Using the surjectivity of Φ, fix a graph $G \in \mathcal{G}_\lambda$ with $\Phi(G) = B$.

Suppose to the contrary that G is isomorphic to a subgraph of G_α for some $\alpha < 2^{\aleph_0}$. Without loss of generality G is a subgraph of, say, G_0. By the covering property of $\mathcal{F}_\alpha$ we can find a subset $F \in [\mathcal{F}_0]^{<\lambda}$ such that $G \leq \bigcup F$. Because every member of F is an r-subgraph of G_0, every member of F is also an r-subgraph of (the induced subgraph of G_α spanned by) $\bigcup F$. Therefore $\Phi(\bigcup F) = \bigcup_{A\in F} \varphi(A)$ by continuity Claim 2.2 above. Thus

$$B = \Phi(G) \subseteq \Phi(\bigcup F) = \bigcup_{A\in F} \Phi(A) \subseteq \bigcup D$$

a contradiction to the choice of $B \not\subseteq \bigcup D$. $\triangle$

Discussion. Corollary 2.3 has two improvements over Corollary 1.9: The first is that it extends the result to singular cardinals. The second is that also it says something stronger than setting a lower bound for complexity: it says that for every set of fewer than $2^{\aleph_0}$ members in $\mathcal{G}_\mu$ there is a *smaller* graph $G \in \mathcal{G}_\lambda$ for some regular $\lambda < \mu$. The second statement holds for all regular $\lambda > \aleph_2$ below the first fixed point of second order as well, by the same proof.

Generalization to higher cardinals

We state now the most general form of representation we know for $\mathcal{G}$. This involves replacing club guessing using c_δ-s of order type ω with club guessing using c_δ-s of order type μ, for some μ of cofinality $\aleph_0$.

Replacing ω and $\mathcal{P}(\omega)$ with μ and $[\mu]^{\aleph_0}$ respectively in the proof of Theorem 1.10, there is only one difficulty: in establishing condition (d) in the proof, the argument that $c_\delta \subseteq^* E$ implies that $\varphi_{<_1,r}(v,\delta) = \varphi_{<_2,r}(v,\delta)$ breaks down, because a proper initial segment of c_δ may contain all the members of the subset of c_δ chosen by v in equation (1) in the proof.

This difficulty vanishes if we demand that $C_\delta \subseteq E$, namely work with $I = \mathrm{id}^a(\bar{C})$ rather than $I = \mathrm{id}^b(\bar{C})$.

Since normality of I was not used so far (only λ-completeness), all the results so far hold when replacing $I = \mathrm{id}^b(\bar{C})$ by the λ-complete $I = \mathrm{id}^a(\bar{C})$.

Thus we have proved:

2.4 Theorem: *Suppose that λ is regular. Then for every μ satisfying $\mu^+ < \lambda$ there is a surjective homomorphism $\Phi_\mu : \langle \mathcal{G}_\lambda, \leq_w\rangle \to \langle [\mu^{\aleph_0}]^{\leq l}, \subseteq\rangle$.*

2.5 Corollary: *For every regular uncountable $\lambda > \aleph_1$ we have* $\operatorname{cp}\mathcal{G}_\lambda \geq \sup(\{\lambda^+\} \cup \mu^{\aleph_0} : \mu^+ < \lambda\})$. △

Thus, for example, one can have GCH up to $\aleph_\omega$ with $\aleph_\omega^{\aleph_0}$ large (see [Ma]), say $\aleph_{\omega+\omega+1}$, and in this model the complexity of $\mathcal{G}_{\aleph_{\omega+2}}$ is larger than $\aleph_{\omega+3}$.

§3 Horizontal complexity

The complexity $\operatorname{cp}\mathcal{G}_\lambda$ measures the "depth" or "height" of the quasi ordering $\langle \mathcal{G}_\lambda, \leq \rangle$. Another legitimate way to measure how complicated a quasi-ordering is, is by estimating its "width", namely the supremum of cardinalities of anti-chains. In this context it means calculating the possible number of pair-wise non-embeddable graphs in the class.

We use the representation Theorem from Section 2 to prove that in the class $\mathcal{G}$ under study this number is alway the maximal possible in every regular $\lambda > \aleph_1$. This result used no cardinal arithmetic assumptions. The idea is to use the homomorphism to pull back antichains from the range. Since the range of Φ may be too small (if, say, $\lambda > 2^{\aleph_0}$), we have to use φ.

The existence of antichains in the range of φ follows from the completeness of I for all successor λ and by Solovay's theorem for inaccessibles. We shall be using the existence of λ pair-wise disjoint positive sets with respect to $\operatorname{id}^a(\overline{C})$. By Solovay's theorem we can find λ pairwise disjoint stationary subsets of S_0^λ and by Shelah's proof each of them supports a club guessing sequence. The union of all sequences is a $\overline{C}$ that satisfies the required property.

3.1 Theorem: *For every regular $\lambda > \aleph_1$ there are 2^λ pair-wise incomparable elements in $\mathcal{G}_\lambda$ with respect to $\leq_w$. In other words, there are 2^λ graphs in $\mathcal{G}_\lambda$ with no one of them weakly embeddable in another.*

Furthermore, those graphs can be chosen to be in $\Phi^{-1}(A)$ for a any given subset $A \in [\mathbf{R}]^{\mathbf{2}}$.

Proof: Fix a sequence $\langle S_\alpha : \alpha < \lambda \rangle$ of pairwise disjoint positive sets with respect to $I = \operatorname{id}^a(\overline{C})$, $\overline{C}$ a club guessing sequence. Let Φ be the homomorphism from Theorem 1.10 defined using I.

Let $A = \{X, Y\}$ where $X, Y \subseteq \bar{\mathcal{P}}(\omega)$ are two distinct, non-empty sets.

Fix a collection $L \subseteq \mathcal{P}(\lambda)$ of size 2^λ with the property that for every two distinct members $\eta_1, \eta_2 \in L$ it holds that $\eta_1 \not\subseteq \eta_2$ and $\eta_2 \not\subseteq \eta_1$ (L can be chosen, for example, to be an independent family over λ). For every $\eta \in L$ we shall find a graph G_η with $\Phi(G_\eta) = A$ and such that none of those graphs is embeddable in another.

Suppose $\eta \in L$ is given. Using the surjectivity of φ find G_η such that $\varphi(G)(\delta) = X$ iff $\delta \in \bigcup_{\alpha \in \eta} S_\alpha$ and $A_\delta = Y$ otherwise. (We neglect coordinates δ which are not in S).

For every $\eta \subseteq \lambda$ it holds that $\Phi(G_\eta) = A$, because each S_α is positive and both η and $\lambda \setminus \eta$ are not empty. Suppose now that $\eta_1, \eta_2 \in L$ are distinct and we shall show that $G_{\eta_1} \not\leq_w G_{\eta_2}$. Since φ is a homomorphism, it is enough to show that $\varphi(G_{\eta_1}) \not\subseteq_I \varphi(G_{\eta_2})$. Suppose then, to the contrary, that for some set $H \in I$ it holds that $\delta \in S \setminus H$ implies that $\varphi(G_{\eta_1})(\delta) \subseteq \varphi(G_{\eta_2})(\delta)$. Let $\alpha \in \eta_1 \setminus \eta_2$ be picked. Since S_α is positive and $H \in I$ the set $S_\alpha \setminus H$ is positive. For every $\alpha \in S_\alpha \setminus H$ it holds that $\varphi(G_{\eta_1})(\delta) = \{X\}$ and $\varphi(G_{\eta_2}) = \{Y\}$. Thus $\varphi(G_{\eta_1}) \not\subseteq_I \varphi(G_{\eta_2})$. $\triangle$

Discussion The property $\Phi(G) = A$ can be regarded as a strong homogeneity property: modulo I, all rays in neighbourhoods of elements of G converge to their supremum (when the graphs is well ordered) in exactly one of two possible convergence rates. Yet, the graphs G_η chosen above are pairwise incomparable.

§4 A decomposition Theorem

Forbidding a few more countable configurations enables a decomposition theorem: we prove that such a graph is r-indecomposable iff its image is a singleton.

4.1 Definition: Suppose that $\lambda > \aleph_1$ is regular and that $\Phi : \mathcal{G}_\lambda \to [\mathbf{R}]^{\leq\lambda}$ is the homomorphism from Theorem 1.10. Say that $G \in \mathcal{G}_\lambda$ is (r,λ)-indecomposable iff for every $\beta < \lambda$ and r-subgraphs $\langle G_\alpha : \alpha < \beta \rangle$ so that $G = \bigcup_{\alpha<\beta} G_\alpha$, there is some $\alpha < \beta$ such that $\Phi(G) = \Phi(G_\alpha)$.

4.2 Claim: *Suppose that $\lambda > \aleph_1$ is regular and that $\Phi : \mathcal{G}_\lambda \to [\mathbf{R}]^{\leq\lambda}$ is the homomorphism from Theorem 1.10. If $G \in \mathcal{G}_\lambda$ and $|\Phi(G)| = 1$ then G is (r,λ)-indecomposable.*

Proof: : Suppose $\Phi(G) = \{a\}$. Fix a positive set $Y \subseteq \lambda$ and vertices $v_\delta \in G$ for $\delta \in Y$ so that $\varphi(v, \delta) = a$. By completeness of I we can find a fixed $\alpha < \beta$ and a positive $Y' \subseteq Y$

such that $v_\delta \in G_\alpha$ for all $\delta \in Y'$. Since G_α is an r-subgraph, $r(v_\delta) \subseteq^* G_\alpha$ for every $\delta \in Y'$. On a measure 1 set equation (8) above holds and tells us that $\varphi(v_\delta, d)$ can be computed in either G_α or in G. Intersecting Y' with this measure 1 set yields a positive measure $Y'' \subseteq Y$ so that $\varphi(v_\delta, d) = \{a\}$ for $\delta \in Y''$. This shows that $\Phi(G) \subseteq \Phi(G_\alpha)$. The other inclusion holds because $G_\alpha \leq G$. $\triangle$

We may ask if the converse is also true: namely that $\Phi(G)$ can be represented as a union of fewer than λ r-subgraphs G_α, each with $\varphi(G_\alpha)$ different from, hence properly included in, $\Phi(g)$ whenever $|\Phi(G)| > 1$. While this is not true for $\mathcal{G}$, forbidding an additional set of countable configurations gives a large subclass of $\mathcal{G}$ for which it is true.

4.3 Definition:

(0) Let Γ^* be the set of all countable graphs G in which there is a vertex $v \in G$ with a ray $R \subseteq G[v]$ and a vertex $u \in R$ with a ray $S \subseteq G[u]$.

(1) Let $\mathcal{G}^*$ be the subclass of $\mathcal{G}$ resulting by forbidding all graphs in Γ^*.

4.4 Claim: *If λ and Φ are as above, then $\Phi|\mathcal{G}^*_\lambda$ is surjective. Hence the Theorems and Corollaries proved above for $\mathcal{G}$ hold also for $\mathcal{G}^*$.*

Proof: The graphs constructed in the proof of 1.10 to demonstrate surjectivity are all in $\mathcal{G}^*$ by condition (d)(i) in the inductive construction.

4.5 Claim: *Suppose λ and Φ are as above. Suppose that $G \in \mathcal{G}^*$ and that $|\Phi(G)| > 1$. Then there are two r-subgraphs G_1, G_2 of G such that $\Phi(G) \not\subseteq \Phi(G_i)$ for $i = 1, 2$.*

Proof: Let $a, b \in \Phi(G)$ be two distinct members of $\Phi(G)$. For every $\delta < \lambda$ let $\langle v_{\delta,a,i} : i < j(\delta, a)\rangle$ be an enumeration of all vertices $v \in G$ with $\varphi(v, \delta) = a$. Similarly, let $\langle v_{\delta,b,i} : i < j(b, \delta)\rangle$ enumerate all vertices in G with $\varphi(v, \delta) = b$. Let r be a function on G such that $r(v)$ is a ray in $G[v]$ if such a ray exists and is $\emptyset$ otherwise. Let $A := \{v_{\delta,a,i} : \delta < \lambda,\ i < j(\delta, \alpha)\}$ and let $G_1 = A \cup \{r(v) : v \in A\}$. Since $r(u) = \emptyset$ for every vertex $u \in r(v)$, for $v \in A$ — because Γ^* is omitted — we conclude that G_1 is an r-subgraphs of G. Also, since $r(u) \neq \emptyset$ for every $u \in B := \{v_{\delta,b,i} : \delta < \lambda,\ i < j(\delta, b)\}$, it follows that $G_1 \cap B = \emptyset$.

Let $C := G \setminus G_1$ and let $G_2 = C \cup \{r(v) : v \in C\}$. Again, $r(u) \cap A = \emptyset$ for $u \in C$.

By restricting attention to a measure 1 subset of λ we may assume that $\varphi(v, \delta)$ for a vertex $v \in G_1$ (or in G_2) remains unchanged when computed in G. Thus we see that

$\Phi(G_1) = \{a\}$. On the other hand, $\varphi(v, \delta) \neq a$ for all $u \in G_2$, because all vertices $v \in G$ for which $\varphi(v, \delta) = a$ for some $\delta < \lambda$ are in A, and $A \cap G_2 = \emptyset$. This proves the claim. $\triangle$

Both claims show that in $\mathcal{G}^*_\lambda$ a graph is (r, λ)-indecomposable if and only if its image under Φ is a singleton.

References

[S] S. Shelah, **Cardinal Arithmetic**, Oxford University press, Oxford 1994.

[D1] R. Diestel, *On the problem of finding small subdivision and homomorphism bases for classes of countable graphs*, Discrete Math. 55 (1985) 21–33.

[D2] R. Diestel, *The end structure of a graph: recent results and open problems* Discrete Math. 100 (1992) 313–327

[DHV] R. Diestel, R. Halin and W. Vogler, *Some remarks on universal graphs*, Combinatorica 5 (1985) 283–293

[E] W. B. Easton, *Powers of regular cardinals*, Annals of Mathematical logic. 1 (1970) 139–178.

[FW] M. Foreman and H. Woodin, *The generalized continuum hypothesis can fail everywhere*, Annals of Mathematics 133 (1991) 1–36.

[GS] M. Gitik and S. Shelah, *Less saturated ideals*, preprint, Logic Eprints

[GrS] R. Grossberg and S. Shelah, *On universal locally finite groups*, Israel Journal of Math. 44 (1983) 289–302

[HK] A. Hajnal and P. Komjath, *Embedding graphs into colored graphs* Trans. Amer. Math. Soc 307 (1988) 395-409.

[J] T. Jech, *Singular cardinal problem: Shelah's theorem on* $2^{\aleph_\omega}$, Bull. London Math. Soc. 24 (1992) 127-139

[Je] R. B. Jensen, *The fine structure of the constructible universe*, Annals of Math. Logic 4 (1972) 229–308

[Jo] B. Jónsson, *Universal relational systems*, Math. Scand. 4(1956) 193–208

[k] M .Kojman, *On universal graphs without cliques or without large bipartite graphs*, submitted

[KjS1] M. kojman and S. Shelah, *nonexistence of universal orders in many cardinals* Journal of Symb. Logic 57 (1992) 875–891

[KjS2] M. kojman and S. Shelah, *The universality spectrum of stable unsuprestable theories* Annals Pure Appl. Logic 58 (1992) 57–72

[KjS3] M. kojman and S. Shelah, *Universal abelian groups* Israel J. Math, in press.

[KP] P. Komjath and J. Pach, *Universal Graphs without large bipartite subgraphs*, Mathematika 31 (1984) 282–290

[KP1] P. Komjath and J. Pach, *Universal elements and the complexity of certain classes of infinite graphs*, Discrete Math. 95 (1991) 255–270

[KS] P. Komjath and S. Shelah, *Universal graphs without large cliques*, preprint

[L] R. Laver, *On Fraisse's order type conjecture*, Annals of Mathematics 93 (1971) 89–111

[M] A. Macintyre, *Existentially closed structures and Jensen's principle* $\diamondsuit$, Israel Journal of Math. 25 (1976) 202–210

[Ma] M. Magidor, *On the singular cardinal problem II*, Annals of Mathematics 106 (1977), 517–547

[Me] Alan. H. Mekler, *Universal structures in power* $\aleph_1$, Journal of Symbolic Logic 55(2) (1990) 466–477.

[MV] M. Morley and R. L. Vaught, *Homogeneous universal models*, Math. Scand. 11 (1962) 35–57

[R] R. Rado, *Universal graphs and universal functions*, Acta. Arith. 9. (1964) 331-340

[S1] S. Shelah, *Independence results*, Journal of Symbolic Logic 45 (3) (1980) 563–573

[S2] S. Shelah, *On universal graphs without instances of CH*, Annals of Pure. Appl. Logic 26 (1984) 75–87.

[S3] S. Shelah, *Universal graphs without instances of CH: revisitied*, Israel J. Math 70 (1990) 69–81

[S4] S. Shelah, *Universal in $< \lambda$-stable abelian groups*, preprint

[S5] S. Shelah, *Non-existence of universals for classes like reduced torsion free abelian groups under non necessarily pure embeddings*, in preparation

[So] R. M. Solovay, *Real-valued measurable cardinals* Axiomatic set theory, Amer. Math. Soc. Proc. Symposia Pure Math. 13, part I, 397–428