THE A,B,C OF PCF: A COMPANION TO PCF THEORY, PART I

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1. Introduction

This paper is intended to assist the reader learn, or even better, teach a course in pcf theory. Pcf theory can be described as the journey from the function $\beth$ — the second letter of the Hebrew alphabet — to the function $\aleph$, the first letter. For English speaking readers the fewer the Hebrew letters the better, of course; but during 1994-95 it seemed that for a group of 6 post doctoral students in Jerusalem learning pcf theory from Shelah required knowing all 22 Hebrew letters, for Shelah was lecturing in Hebrew.

This paper offers a less challenging alternative: learning pcf in a pretty relaxed pace, with all details provided in every proof, and in English.

Does pcf theory need introduction? This is Shelah's theory of reduced products of small sets of regular cardinals. The most well-known application of the theory is the bound $\aleph_{\omega_4}$ on the power set of a strong limit $\aleph_\omega$, but other applications to set theory, combinatorics, abelian groups, partition calculus and general topology exist already (and more are on the way).

The essence of pcf theory can be described in a few sentences. The theory exposes a robust skeleton of cardinal arithmetic and gives an algebraic description of this skeleton. Shelah's philosophy in this matter is plain: the exponent function is misleading when used to measure the collection of subsets of a singular cardinal.

The right way to measure the size of, say, $[\aleph_\omega]^{\aleph_0}$ is by $\mathrm{cf}\langle[\aleph_\omega]^{\aleph_0}, \subseteq\rangle$, the cofinality of the partial ordering of all countable subsets of $\aleph_\omega$ ordered by inclusion.

The usual $\aleph_\omega^{\aleph_0}$ is obtained from that number by multiplying by $2^{\aleph_0}$. While $2^{\aleph_0}$ is "wild" and cannot be bounded in ZFC, a great discovery of Shelah's is that $\mathrm{cf}\langle[\aleph_\omega]^{\aleph_0}, \subseteq\rangle$ is bounded in ZFC. This way of looking at $\aleph_\omega^{\aleph_0}$ separates chaos from structure: puts the chaotic exponent $2^{\aleph_0}$ aside, and clears the way for the study of the structure of $[\aleph_\omega]^{\aleph_0}$.

Shelah approximates $\mathrm{cf}\langle[\aleph_\omega]^{\aleph_0}, \subseteq\rangle$ by an interval of regular cardinals, whose first element is $\aleph_{\omega+1}$ and whose last element is $\mathrm{cf}\langle[\aleph_\omega]^{\aleph_0}, \subseteq\rangle$, and so that every regular cardinal λ in this interval is the *true cofinality* of a *reduced product* $\prod B_\lambda/J_{<\lambda}$ of a set $B_\lambda \subseteq \{\aleph_n : n < \omega\}$ modulo an ideal $J_{<\lambda}$ over ω.

When this reduction is done, the study of of $[\aleph_\omega]^{\aleph_0}$ can be continued algebraically within *pcf theory*: the theory of reduced products of small sets of regular cardinals.

This approach to cardinal arithmetic can be thought of as "algebraic set theory" in analogy to algebraic topology. The information provided by this view of cardinal arithmetic is enormous, and influences almost every branch of mathematics in which the notion of cardinality is important.

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For example, a recent application of pcf theory is taking a construction, due to M. E. Rudin, of a certain topological space of cardinality $\aleph_\omega^{\aleph_0}$ and re-enacting it on one on those approximations. The construction goes through and the result is space with the original properties whose cardinality is computable in ZFC (see [4]).

1.1. **The relation of this paper to Shelah's book.** Shelah's Cardinal arithmetic book (henceforth "the book") was published about a year ago and covers a large part of pcf theory and its applications. The book reflects the state of Shelah's research in pcf as it was in 1989 — the year Shelah "sealed" the book. The theory has advance considerably since — and, roughly, doubled its volume. An important advance in the development of the theory was the proof in ZFC of the existence of stationary sets in $I[\lambda]$. With this theorem the development of the basics of pcf is more transparent. The approach taken here is the one taken by Shelah in two pcf courses he taught in Jerusalem in 1991 and 1995, and which I tried to immitate in a course I gave at Carnegie-Mellon in 1994/95. The ideal $I[\lambda]$ is used every other page. Thus an initial effort is required to develop the properties of $I[\lambda]$ which may look disproportional to the simplifications it generates, but this is worth the effort, because the proofs are smoother, more transparent and more informative than earlier proofs in which $I[\lambda]$ is not used.

Answering an implicit question: yes, I recommend reading this paper before reading the book. You will benefit more from reading the book after already knowing that, for example, generators of pcf always exist.

All the theorems in this paper are Shelah's, unless otherwise stated. Not all the material in this paper is contained in the book, though.

Pointers to relevant places in the book and in other existing presentations of pcf theory will be found under the macro

`Where is this in the book?`

Many of the chapters in the book are not about pcf theory itself, but present applications of the theory. I believe that those parts of the book will be more accessible to a reader familiar with pcf theory, as presented here.

1.2. **Style of writing.** I decided to write *all* the details. I hope you are happy with this decision; but at the times you are bored with reading — don't blame me; it's Jim Baumgartner's fault. I tried to get Jim through a two-day crash-course in pcf during a meeting we both attended last summer, and after several hours Jim looked into my eyes and said grimmly: "This is not easy, Menachem. You keep saying that this is easy, but it's difficult stuff".

Not any more. When you read you will see.

The difference in the style of writing between this paper and the book lies in what literary critics call "the implied reader", this imaginary person, half way between the author and no-one, to whom the writing is addressed. Shelah's implied reader is very clever, sharp, has phenomenal memory, wants to know always the *most general* formulation of a theorem and, really, *already knows* pcf and needs only to be reminded of what she knows.

My implied reader is slower, wants to learn the important case first and generalizations only later, can learn one thing at a time, and can lose hours over a ζ mistaken for a ξ. She is pretty much like me.

For my reader's sake I made the rule to never use either of the following adverbs in this paper: "clearly","obviously", and "easily". Rules are, however, clearly made to be easily broken at obviously suitable circumstances.

1.3. **Additional material.** I included digressions into topics which are not needed for the development of pcf theory. For instance, additional club-guessing theorems and applications to the saturation of ideals. Those sections will be marked by the macro

`The rest of this Section is not needed for later sections`

and and may be skipped in a first reading. Some of the additional material is sketched in exercises.

1.4. **What is missing in this version.** At least three sections need to be added to this paper. One about the structure of pcf A; one about reconstruction a characteristic function of a model from pcf scales, and the applications to cardinal arithmetic; and one about smooth or transitive generators.

The first two are written and just need proof reading. Before I write the third I need to digest [13].

1.5. **Acknowledgments.** This is always a very pleasant section to write. I thank all the participants in the pcf course I taught at Carnegie-Mellon 1994/95: Mike Albert, Matt Bishop, Rami Grossberg, Olivier Lessman, Ric Statman, Boban Velickovic and Roberto Virga. I benefited from all the comments they have made ("you write too much on the board"; "you don't write enough on the board"; "you drop letters at the ends of words" and "your ξ looks like ζ"). But more than this, I feel they shared my excitement over some beautiful mathematics.

I thank Ferna Hartman for typing several handwritten notes, that later evolved into this text, and Roberto Virga for introducing me to AMSLaTeX. My enthusiasm about several AMS fonts is visible in several proofs which make a trully indulgent use of $\subsetneqq$ and similar symbols.

Special thanks are extended to Amir Leshem, Sami Lefsches and Charlie Beaton who allowed me to photocopy their notes from Shelah's pcf course in 1991 (after my own notes were lost). I am also thankful to Mirna Dzamonja and Andrzej Roslanowski — the first for the heroic effort of taking Hebrew notes from Shelah's lectures on pcf, the other for sending me copies of those notes.

Finally, I am more than grateful to Saharon Shelah for creating such wonderful mathematics, and for teaching it to me.

2. Dropping ordinals and guessing clubs

Definition 1. *For a set X of ordinals let* $\operatorname{acc}X$ *denote the set of accumulation points of X:* $\operatorname{acc}X = \{\alpha \in On : \alpha = \sup X \cap \alpha\}$. *Let* $\operatorname{nacc}X \overset{def}{=} X \setminus \operatorname{acc}X$. *Let* $\operatorname{cl}X = X \cup \operatorname{acc}X$.

We prepare some combinatorics to be used later on. We begin with some very simple operations on ordinals.

Definition 2.

- *Let $c \subseteq On$ be a set of ordinals, and let $\alpha \in On$ be an ordinal. Define* $\operatorname{drop}(\alpha, c) \overset{def}{=} \sup c \cap \alpha$ *if $c \cap \alpha \neq \emptyset$. If $c \cap \alpha = \emptyset$ then* $\operatorname{drop}(\alpha, c)$ *is not defined.*
- *If $X \subseteq On$ and and $c \subseteq On$ are sets of ordinals define* $\operatorname{Drop}(X, c) \overset{def}{=} \{\operatorname{drop}(\alpha, c) : \alpha \in X\}$.

Let us list a few simple facts about the operations of dropping ordinals into sets defined above:

Fact 3. *Suppose c and X are sets of ordinals and $\alpha \in On$.*

1. $\mathrm{drop}(\alpha, c) \leq \alpha$ *or* $\mathrm{drop}(\alpha, c)$ *is undefined. Equality holds if and only if* $\alpha \in \mathrm{acc}c$.
2. *If* $\mathrm{drop}(\alpha, c)$ *is defined then* $\mathrm{drop}(\alpha, c) \in \mathrm{cl}c$
3. $\mathrm{Drop}(X, c) \subseteq \mathrm{cl}c$
4. *If* $\alpha_1 \leq \alpha_2$ *are ordinals and* $\mathrm{drop}(\alpha_1, c)$ *is defined then* $\mathrm{drop}(\alpha_2, c)$ *is defined too and* $\mathrm{drop}(\alpha_1, c) \leq \mathrm{drop}(\alpha_2, c)$.
5. $\alpha \mapsto \mathrm{drop}(\alpha, c)$ *is a order homomorphism on an end segment of X and consequently* $\mathrm{otp}\,\mathrm{Drop}(X, c) \leq \mathrm{otp}X$.
6. *If $c_1 \subseteq c_2$ are sets of ordinals and $\alpha \in On$ and* $\mathrm{drop}(\alpha, c_1)$ *is defined then also* $\mathrm{drop}(\alpha, c_2)$ *is defined and* $\mathrm{drop}(\alpha, c_1) \leq \mathrm{drop}(\alpha, c_2)$;
7. *If $\langle c_i : i < i(*)\rangle$ is such that $c_i \subseteq On$ and $c_i \subseteq c_j$ for all $j \leq i < i(*)$ then for every $\alpha \in On$ there exists some $i(\alpha) < i(*)$ such that* $\mathrm{drop}(\alpha, c_i)$ *stabilizes at $i(\alpha)$, that is either* $\mathrm{drop}(\alpha, c_i)$ *is undefined or* $\mathrm{drop}(\alpha, c_i)$ *is constant for all $i(\alpha) \leq i < i(*)$.*
8. *If $\langle c_i : i < \kappa\rangle$ is sequence of sets of ordinals weakly decreasing in $\subseteq$, κ a regular cardinal and $X \subseteq On$, $|X| < \kappa$ then there exists some $i(X) < \kappa$ such that* $\mathrm{Drop}(X, c_i)$ *stabilizes at $i(X)$, that is* $\mathrm{Drop}(X, c_{i(X)}) = \mathrm{Drop}(X, c_i)$ *for all $i(X) \leq i < \kappa$.*
9. $\mathrm{acc}\,\mathrm{Drop}(X, c) = \mathrm{acc}X \cap \mathrm{acc}c$.

Proof. The first statement follows directly from the definition of $\mathrm{drop}(\alpha, c)$, and every other statement except the last one follows from the earlier ones and the wellfoundedness of the ordinals.

To see the last one suppose that $\beta \in \mathrm{acc}\,\mathrm{Drop}(X, c)$ and let $\langle \mathrm{drop}(\alpha_i, c) : i < i(*)\rangle$ be strictly increasing and cofinal in β with $\alpha_i \in X$ for all $i < i(*)$. For every $i < i(*)$ necessarily $\alpha_i \leq \mathrm{drop}(\alpha_{i+1}, c)$, or else $\mathrm{drop}(\alpha_i, c) \geq \mathrm{drop}(\alpha_{i+1}, c)$. Since $\mathrm{drop}(\alpha_i, c) \in \mathrm{cl}c$ for all i by 2, we conclude that $\beta \in \mathrm{acc}(\mathrm{acc}c) = \mathrm{acc}c$. Since $\mathrm{drop}(\alpha_i, c) \leq \alpha_i$ by 1 and $\alpha_i \leq \mathrm{drop}(\alpha_{i+1}, c) < \beta$ we conclude that $\beta \in \mathrm{acc}X$.

Conversely, suppose that $\beta \in \mathrm{acc}X \cap \mathrm{acc}c$. Find a set $\{\beta_i : i < i(*)\} \subseteq \beta \cap c$ unbounded in β. Since $\beta \in \mathrm{acc}X$, the ordinal $\alpha_i := \min\{\beta \cap X \setminus (\beta_i + 1)\}$ is defined for all $i < i(*)$. Therefore $\beta_i \leq \mathrm{drop}(\alpha_i, c) < \beta$. It follows that $\beta \in \mathrm{acc}\,\mathrm{Drop}(X, c)$. □

Now let us use those operations to prove the existence of "club guessing sequences".

Theorem 4. *Suppose that $\kappa^+ < \lambda$, that κ, λ are regular cardinals, and $S \subseteq \{\delta < \lambda : \mathrm{cf}\delta = \kappa\}$ is a given stationary subset of λ. Then there is a sequence $\overline{C} = \langle c_\delta : \delta \in S\rangle$ satisfying*

1. *$c_\delta \subseteq \delta$ is a closed subset of δ*
2. *For every club $E \subseteq \lambda$ the set $N(E) \stackrel{def}{=} \{\delta \in S : c_\delta \subseteq E \wedge \sup c_\delta = \delta \wedge \mathrm{otp}c_\delta = \kappa\}$ is a stationary subset of λ.*

Remark 5. *If $\langle a_\delta : \delta \in S\rangle$ satisfies $a_\delta \subseteq \delta$ and condition 2 in the Theorem then $\langle \mathrm{cl}c_\delta \cap \delta : \delta \in S\rangle$ satisfies both 1 and 2. Secondly, replacing c_δ by any cofinal subset does not spoil 2.*

Proof. Let us define by induction on $i < \kappa^+$ a sequence $\overline{C}_i = \langle c^i_\delta : \delta \in S\rangle$ and a club $E_i \subseteq \lambda$ as follows. Let $E_0 = \lambda$ and let $\overline{C}_0 = \langle c^0_\delta : \delta \in S\rangle$ be chosen such that for every $\delta \in S$ the set c^0_δ is a club of δ of order type κ.

Two of the induction hypotheses are: $c^i_\delta = \mathrm{Drop}(c^0_\delta, E_i)$ is a closed subset of δ and $j < i \Rightarrow E_i \subseteq E_j$. This holds for $i = 0$.

If $i < \kappa^+$ is limit and $E_j, \overline{C}_j$ are defined for all $j < i$, let $E_i := \cap_{j<i} E_j$. Since $i < \lambda$, this is indeed a club of λ. Let $c^i_\delta := \mathrm{Drop}(c^0_\delta, E_i)$ for all $\delta \in S$. By Fact 3.9 c^i_δ is a closed subset of δ, (and by Fact 3.5 its order type is $\leq \kappa$.)

Suppose that $i = j + 1 < \kappa^+$ and that $E_j, \overline{C}_j$ are defined. If there exists some club $E \subseteq \lambda$ such that for all $\delta \in E \cap S$ either c^j_δ is not a club of δ or $c_\delta \nsubseteq E$, then choose such E and let $E_i = E_j \cap E$.

If, however, there is no club $E \subseteq \lambda$ as required in the definition of E_i, then for every club $E \subseteq \lambda$ the set $N^j(E) := \{\delta \in S : \delta = \sup c^j_\delta \wedge c^i_\delta \subseteq E\}$, is stationary. To see this, suppose that $N^j(E) \cap E' = \emptyset$ for some club $E \subseteq \lambda$, namely $\delta \in S \cap E' \Rightarrow c^j_\delta \nsubseteq E$ for some club $E' \subseteq \lambda$. Then $E \cap E'$ is a club of λ which is as required for the definition of E_i.

Therefore, in the case that no E_{j+1} can be found, the proof is done: $c^j_\delta = \mathrm{Drop}(c^0_\delta, E_j)$ is a closed subset of δ by Fact 3(8) and its order type is $\leq \kappa$ by Fact 1.5, so if unbounded in δ it is a club of δ of order type κ. And, by the previous paragraphs, the prediction clause 2 in the theorem holds for $\langle c^j_\delta : \delta \in S\rangle$.

Assume, then, that E_{i+1} is defined for all $j < \kappa^+$ and we shall derive a contradiction. Let $E = \bigcap_{i<\kappa^+} E_i$. Since $\kappa^+ < \lambda$, this is a club of λ. Let $\delta \in \mathrm{acc} E \cap S$. The sequence $\langle c^i_\delta : i < \kappa^+\rangle$ must stabilize at some $i(\delta) < \kappa^+$ by Fact 3.8, since $\langle E_i : i < \kappa^+\rangle$ decreasing and $c^i_\delta = \mathrm{Drop}(c^0_\delta, E_i)$. Now since $\delta \in \mathrm{acc} c^0_\delta \cap \mathrm{acc} E_i$ for all $i < \kappa^+$, by Fact 3.9 we conclude that c^i_δ is a club of δ. Choose $j \geq i(\delta)$. Since $\delta \in E_{j+1}$, either c^j_δ is not a club of δ or $c_\delta \nsubseteq E_{j+1}$; but c^j_δ is a club of δ; therefore $c^j_\delta \nsubseteq E_{j+1}$. However $c^{i+1}_\delta \subseteq \mathrm{cl} E_{j+1} \cap \lambda = E_{j+1}$. So $c^i_\delta \neq c^{j+1}_\delta$, contrary to stabilization. $\square$

We make the following observation: the proof above gives a slightly stronger theorem:

Theorem 6. *Suppose $\kappa^+ < \lambda$ and κ, λ are regular cardinals, and $S \subseteq \{\delta < \lambda : \mathrm{cf}\delta = \kappa\}$ is a given stationary set. Then for every sequence $\langle c_\delta : \delta \in S\rangle$ satisfying that $c_\delta \subseteq \delta$ is a club of δ of order type κ and every club $E' \subseteq \lambda$ there exists a club $E \subseteq E'$ such that $\overline{C} := \langle \mathrm{Drop}(c_\delta, E) : \delta \in S\rangle$ is a club guessing sequence, namely satisfies the conditions of Theorem 4.*

What this theorem says is that every sequence of the right form, supported by a stationary set of λ, can be made into a club guessing sequence by dropping each of its members into a club E which is contained in a prescribed club $E' \subseteq \lambda$.

If $\kappa > \aleph_0$ the operation $\mathrm{Drop}(c^0_\delta, E_i)$ can be replaced by $c^0_i \cap E_i$ in the proof of Theorem 4. In this case the corresponding version of Theorem 7 is:

Theorem 7. *Suppose $\kappa^+ < \lambda$ and κ, λ are regular cardinals, and $S \subseteq \{\delta < \lambda : \mathrm{cf}\delta = \kappa\}$ is a given stationary set. Then for every sequence $\langle c_\delta : \delta \in S\rangle$ satisfying that $c_\delta \subseteq \delta$ is a club of δ of order type κ and every club $E' \subseteq \lambda$ there exists a club $E \subseteq E'$ such that $\overline{C} := \langle c_\delta \cap E : \delta \in S\rangle$ is a club guessing sequence, namely satisfies the conditions of Theorem 4.*

In other words, any sequence of the right form becomes a club guessing sequence when *relativized* to some club of λ. Theorem 7 of course does not work for $\kappa = \omega$.
`The rest of this Section is not needed for later sections`

We introduce now a second operation on ordinals, which is a generalization of drop.

difference between Drop and Fill can be described by the following quotation of a friend of mine, married to her third husband. She summed up her opinion about divorce as follows: "I don't believe any more in *replacing*; only in *adding*". Dropping a set of ordinals into, say, a club, *replaces* the members of the set by smaller ordinal from the club; Filling just joins new members to the set from the members of the club. The filling will be done, however, by dropping some sets — which we will fix — into the club.

Definition 8. *Let λ be regular uncountable, $\beta \in \lambda$ and and let $c \subseteq \lambda$.*

1. *Fix a sequence $\langle e_\beta : \beta \in \lambda \cap \mathrm{acc}\lambda \rangle$ such that e_β is a club of β and* $\mathrm{otp} e_b = \mathrm{cf}\beta$.
2. *let* $\mathrm{fill}(\beta, c) := \mathrm{Drop}(e_\beta, c)$ *if β is limit and else let* $\mathrm{fill}(\beta, c) := \emptyset$.

The operation fill depends on a choice of a sequence $\langle e_\beta : \beta \in \mathrm{acc}\lambda \rangle$, but since for our purposes here the actual choice of this sequence does not matter, we do not incorporate it into the notation.

We shall use fill to prove a club guessing theorem on cofinality κ of κ^+. The guessing in this case is weaker, but still useful (See [KS2] for an application). Let S^λ_κ denote the set of all elements of λ whose cofinality is κ.

Theorem 9. *Let $\kappa > \aleph_0$ be a regular cardinal and suppose $S \subseteq \{\delta < \kappa^+ : \mathrm{cf}\delta = \kappa\}$ is a given stationary set. There exists a sequence $\overline{C} = \langle c_\delta : \delta \in S \rangle$ such that*

1. *$c_\delta \subseteq \delta$ is a club of δ and $|c_\delta| = \kappa$*
2. *For every club $E \subseteq \kappa^+$ the set $N(E) = \{\delta \in S : \delta = \sup\{(\mathrm{nacc} c_\delta) \cap E \cap S^{\kappa^+}_\kappa\}\}$ is stationary.*

Any sequence $\overline{C}$ satisfying (0) will have the property that for every club $E \subseteq \lambda$, for stationarily many $\delta \in S$ unboundedly many points of c_δ enter the club E — just because $\kappa > \aleph_0$; so the crux of the theorem is in making those points be non-accumulation points of c_δ. Furthermore, it is required in the theorem that those non-accumulation points be of cofinality κ. It is this additional requirement which makes the proof work: I do not know how to prove the theorem without satisfying this requirement. What is being used in the proof is the fact that $S^{\kappa^+}_\kappa$ *does not reflect*: If $\beta < \kappa^+$ is a limit ordinal, then there is a club e_β of β such that $e_\beta \cap S^{\kappa^+}_\kappa = \emptyset$.

Proof. By removing a non-stationary set from S we may assume that $S \subseteq \mathrm{acc}\,\mathrm{acc}\kappa^+$ — namely that every ordinal in S is a limit of limit ordinals.

Let $c^0_\delta \subseteq \mathrm{acc}\kappa^+$ be a club of δ for all $\delta \in S$ and set $\overline{C}_0 = \langle c^0_\delta : \delta \in S \rangle$.

Let us define by induction on $n \leq \omega$ a club $E_n \subseteq \lambda$ and a sequence $\overline{C}_n := \langle c_\delta : \delta \in S \rangle$. We fix $\langle e_\beta : \beta < \kappa^+ \wedge \beta \in \mathrm{acc}\kappa^+ \rangle$ such that $e_\beta \subseteq \beta$ is a club of β and $\mathrm{otp} e_\beta = \mathrm{cf}\beta$ to be used in the definition of fill.

Let $E_0 = \mathrm{acc}\kappa^+$; $\overline{C}_0$ is already chosen. Suppose $E_n, \overline{C}_n$ are defined, and that c^n_δ is a club of δ with $|c_\delta| = \kappa$. If for every club $E \subseteq \kappa^+$ the set $\{\delta \in S : \delta = \sup\{\mathrm{nacc} c_\delta \cap E \cap S^{\kappa^+}_\kappa\}\}$ is stationary, then $\overline{C}_n$ is as required by the theorem, and we are done. Else, find a club $E \subseteq \lambda$ such that $\delta \in S \cap E \Rightarrow \sup\{(\mathrm{nacc} c_\delta) \cap E \cap S^{\kappa^+}_\kappa\} < \delta$ and let $E_{n+1} := E_n \cap E$.

Let $c^{n+1}_\delta = c^n_\delta \cup \bigcup\{\mathrm{fill}(\beta, E_{n+1}) \setminus \mathrm{drop}(\beta, c^n_\delta) : \beta \in \mathrm{nacc} c^n_\delta \setminus (E_{n+1} \cup S^{\kappa^+}_\kappa)\}$

Explanation: we are trying to correct c^n_δ to overcome the counterexample E_{n+1}. If a non-accumulation point β is of the wrong cofinality or does not enter E_{n+1}, we fill in the interval of c_δ^n just below β elements of E_{n+1} which are obtained by dropping e_β into E_{n+1}. If $\beta \in \operatorname{nacc} c_\delta^n$ is in E_{n+1} and has the right cofinality, we add nothing in the interval below it, and β remains a non-accumulation point also in c_δ^{n+1}.

Since c_δ^n is a club of δ, by Fact 3.9 also c_δ^{n+1} is a club of δ.

Suppose that E_n and $\overline{C}_n$ are defined for all $n < \omega$ and we shall derive a contradiction. Let $E' = \bigcap_n E_n$ and let $E = \operatorname{acc}(E' \cap S_\kappa^{\kappa^+})$. $E \subseteq \kappa^+$ is a club of κ^+. Fix $\delta \in S \cap E$.

Since $\delta \in E_{n+1}$ for all n, $\gamma_n := \sup\{(\operatorname{nacc} c_\delta) \cap E_n\} < \delta$. Let $\gamma = \sup\{\gamma_n : n < \omega\}$. Since $\operatorname{cf}\delta = \kappa > \aleph_0$, $\gamma < \delta$. Pick $\alpha \in E' \cap \delta$ such that $\alpha > \gamma$ and $\operatorname{cf}\alpha = \kappa$. Such an α exists because δ is a limit of $E' \cap S_\kappa^{\kappa^+}$.

Since $\alpha > \gamma_0$, $\alpha \notin \operatorname{nacc} c_\delta^0$. But also $\alpha \notin \operatorname{acc} c_\delta^0$, because $\operatorname{otp} c_\delta^0 = \kappa$, and therefore $\alpha \in \operatorname{acc} c_\delta^0 \Rightarrow \operatorname{cf}\alpha < \kappa$. So $\alpha \notin c_\delta^0$ altogether.

Suppose now that $\alpha \notin c_\delta^n$ and let $\beta_n := \min\{c_\delta^n \setminus \alpha\}$. Thus $\beta_n > \alpha$. Let $\alpha_n = \sup \alpha \cap c_\delta^n$. since c_δ^n is closed, $\alpha_n < \alpha$. We argue now that $\alpha \notin c_\delta^{n+1}$. The only way α can join c_δ^{n+1} is by belonging to $\operatorname{fill}(\beta_n, E_{n+1}) = \operatorname{Drop}(e_{\beta_n}, E_{n+1})$. Well, $\alpha \notin \operatorname{nacc} \operatorname{fill}(\beta_n, E_{n+1}) \cap (\alpha_n, \beta_n) \subseteq \operatorname{nacc} c_\delta^{n+1}$ because $\alpha > \gamma \geq \gamma_{n+1}$ and $\delta \in E_{n+2}$. But — and this is the point where the non-reflecting property of $S_\kappa^{\kappa^+}$ is used — also $\alpha \notin \operatorname{acc} \operatorname{Drop}(e_{\beta_n}, E_{n+1})$, because $\operatorname{acc} \operatorname{Drop}(e_{\beta_n}, E_{n+1}) \subseteq \operatorname{acc} e_{\beta_n}$ by Fact 3(8) and $\operatorname{acc} e_{\beta_n} \cap S_\kappa^{\kappa^+} = \emptyset$. Thus $\alpha \notin c_\delta^{n+1}$.

We have proved by induction on n, then, that $\alpha \notin c_\delta^n$ for all n. Let $\beta_n = \min c_\delta^n \setminus \alpha$. We obtain the desired contradiction by showing that the β_n-s are strictly decreasing. Since e_{β_n} is unbounded in β_n, we can choose $\gamma \in (\alpha, \beta_n)$. Certainly $\operatorname{drop}(\gamma, E_{n+1}) \geq \alpha$, and since $\alpha \notin A_{\alpha_n}$ (since $\alpha \notin c_\delta^{n+1}$) the inequality is sharp. So $c_\delta^{n+1} \cap (\alpha, \beta_n) \neq \emptyset$. Thus $\beta_{n+1} < \beta_n$.

We conclude, then, that some $\overline{C}_n$ is as required by the Theorem. □

The final club-guessing theorem we prove is a combination of the previous two: it combines guessing clubs in the stronger sense of Theorem 1 AND having all non-accumulation points have large cofinality, as in Theorem 2. A recent application of this club-guessing principle is the non-saturation of the non-stationary ideal over all regular cardinals $\lambda > \aleph_1$ (see [GS]) which is sketched below in an exercise.

Theorem 10. *Suppose that $\kappa < \theta < \lambda$ are regular cardinals, and that $S \subseteq S_\kappa^\lambda$ is stationary. There exists a sequence $\overline{C} = \langle c_\delta : \delta \in S \rangle$ such that*

1. *$c_\delta \subseteq \delta$ is a closed subset of δ*
2. *for every club $E \subseteq \lambda$ the set $\{\delta \in S : c_\delta \subseteq E \wedge \sup c_\delta = \delta \wedge \operatorname{otp} c_\delta = \kappa \wedge \beta \in \operatorname{nacc} c_\delta \Rightarrow \operatorname{cf}\beta \geq \theta\}$ is stationary*

Proof. Let us denote the set $\{\alpha < \lambda : \operatorname{cf}\alpha \geq \theta\}$ by $S_{\geq\theta}$, and $\{\alpha < \lambda : 1 < \operatorname{cf}\alpha < \theta\}$ by $S_{<\theta}$. Let $S \subseteq S_\kappa^\lambda$ be given. Using Theorem 1 we fix a sequence $\overline{C}_0 = \langle c_\delta : \delta \in S \rangle$ that satisfies conditions 1 and 2 there, which are condition 1 here and condition 2 here from which the clause "$\wedge\ \beta \in \operatorname{nacc}_\delta \Rightarrow \operatorname{cf}\beta \geq \theta$" is omitted. It remains to 'correct' $\overline{C}_0$ so that condition 2 here will hold in full.

We assume, by replacing each c_δ by $c_\delta \cap \operatorname{acc}\lambda$, if necessary, that $c_\delta \subseteq \operatorname{acc}\lambda$. There is no loss of generality in doing so, because for every club $E \subseteq \lambda$ the set $\{\delta \in S : c_\delta \subseteq E \cap \operatorname{acc}\lambda\}$ is stationary.

Fix a club e_β of β with $\operatorname{otp} e_\beta = \operatorname{cf}\beta$ for every limit $\beta < \lambda$, to be used as the required parameter in the operation fill.

We define below another operation, $\mathrm{Fill}(c_\delta, E)$, using a club E as a parameter, which is designed to take care of the cofinality requirement. Replacing every c_δ by $\mathrm{Fill}(c_\delta, E)$ satisfies that for almost all $\delta \in S$ the non-accumulation points of $\mathrm{Fill}(c_\delta, E)$ have cofinality $\geq \theta$; but the order type of $\mathrm{Fill}(c_\delta, E)$ may increase, and the guessing of clubs may be ruined. The proof goes about showing that if Fill is performed with a sufficiently 'thin' club, then the guessing property is not spoiled. The cofinality clause follows then from the guessing property, and the order-type demand is obtained by no more than minor cosmetics.

For a given club $E \subseteq \mathrm{acc}\lambda$ and $\delta \in S$ let us define $\mathrm{Fill}(c_\delta, E)$ in ω approximations as follows:

- $c_\delta^{E,0} = c_\delta$
- $c_\delta^{E,n+1} = c_\delta^{E,n} \cup \bigcup\{\mathrm{fill}(\beta, E) \setminus \mathrm{drop}(\beta, c_\delta^{E,n}) : \beta \in \mathrm{nacc}c_\delta^{E,n} \ \wedge \ \mathrm{cf}\beta < \theta\}$
- $\mathrm{Fill}(c_\delta, E) := \bigcup_{n<\omega} c_\delta^{E,n}$

When E is clear from context, or fixed, we shall write just c_δ^n for $c_\delta^{E,n}$. We list a few facts about $\mathrm{Fill}(c_\delta, E)$ for a fixed club $E \subseteq \mathrm{acc}\lambda$ and $c_\delta \subseteq \mathrm{acc}\lambda$:

Fact 11. 1. $\mathrm{fill}(\beta, E)$ *is defined for all* $\beta \in c_\delta^n$, *and all* $n < \omega$.
2. $c_\delta^n \setminus c_\delta^0 \subseteq E$ *for all* $n < \omega$ *and* $\delta \in S$. *Therefore If* $c_\delta \subseteq E$ *then* $\mathrm{Fill}(c_\delta, E) \subseteq E$.
3. c_δ^n *is closed for all* $\delta \in S$ *and all* $n < \omega$
4. *If* $\beta \in \mathrm{nacc}c_\delta^n \cap \mathrm{acc}E$ *for some* $n < \omega$, $\delta \in S$ *and* $\mathrm{cf}\beta < \theta$, *then* $\beta \in \mathrm{acc}c_\delta^{n+1}$

The first Fact is true for $c_\delta^0 = c_\delta$ because $c_\delta \subseteq \mathrm{acc}\lambda$. For $n+1$ use the second Fact and the fact that $E \subseteq \mathrm{acc}E$.

The second fact is proved by induction on n, using Fact 1.3 and the definition of c_δ^{n+1}. The third fact follows by induction from Fact 3.9.

To prove Fact 4 suppose $\beta \in \mathrm{nacc}c_\delta^n \cap \mathrm{acc}E$ and that $\mathrm{cf}\beta < \theta$. Forming c_δ^{n+1} from c_δ^n the definition adds into the interval $(\mathrm{drop}(\beta, c_\delta^n), \beta)$ all points of $\mathrm{Drop}(e_\beta, E) \setminus \mathrm{drop}(\beta, c_\delta^n) + 1$. Since $\beta \in \mathrm{acc}E$, $\beta \in \mathrm{acc}\,\mathrm{Drop}(e_\beta, E)$ by Fact 3.9 and hence $\beta \in \mathrm{acc}c_\delta^{n+1}$.

Fact 4 is the important one. We are trying to 'get rid' of non-accumulation points whose cofinality is smaller than θ. And Fact (4) above tells us that if the 'bad' point $\beta \in \mathrm{nacc}c_\delta^n \cap S_{<\theta}$ happens to lie in $\mathrm{acc}E$, then it becomes an accumulation point of the next approximation c_δ^{n+1}.

Let us see next that if for some club $E \subseteq \lambda$ the sequence $\langle \mathrm{Fill}(c_\delta, E) : \delta \in S\rangle$ has club guessing property, then it also satisfies the requirement on cofinality of not accumulation points, and that this is sufficient for the theorem.

Claim 12. *Suppose there exists a club* $E \subseteq \lambda$ *such that* $\{\delta \in S : \mathrm{Fill}(c_\delta, E) \subseteq E'\}$ *is stationary for every club* $E' \subseteq \lambda$. *Then we can find a sequence* $\overline{C} = \langle c'_\delta : \delta \in S\rangle$ *as required in the Theorem.*

Proof of Claim. Suppose a club E is fixed such that $\{\delta \in S : \mathrm{Fill}(c_\delta, E) \subseteq E'\}$ is stationary for every club $E' \subseteq \lambda$. Denote the last set by $N(E')$. We first show that also $N'(E') := \{\delta \in N(E') : \beta \in \mathrm{nacc}c_\delta \Rightarrow \mathrm{cf}\beta \geq \theta\}$ is stationary.

To do this we make the following simple observation: since $N(E')$ is stationary for *every* club E', the set $N(E' \cap \mathrm{acc}E)$ is stationary. Let $\delta \in N(E' \cap \mathrm{acc}E) \subseteq N(E')$. By the definition of $N(E')$ we know that $\mathrm{Fill}(c_\delta, E) \subseteq (E' \cap \mathrm{acc}E)$.

We show now that all points of $\mathrm{Fill}(c_\delta, E)$ whose cofinality is smaller than θ are accumulation points. To do so pick any $\beta \in \mathrm{Fill}(c_\delta, E)$ with $\mathrm{cf}\beta < \theta$, and let n be the first such that $\beta \in c_\delta^n$. Now $\beta \in \mathrm{acc}E$, and if $\beta \in \mathrm{nacc}c_\delta^n$ then by Fact (3) above

$\beta \in \operatorname{acc} c_\delta^{n+1} \subseteq \operatorname{acc} c_\delta$. Thus, all non-accumulation points of $\operatorname{Fill}(c_\delta, E) \cap S_{<\theta}$ are of cofinality $\geq \theta$.

Finally we thin-out the sets $\operatorname{Fill}(c_\delta, E)$ so make them have the right order type, without spoiling the fact that their non-accumulation points have large cofinality: For every $\delta \in S$ such that $\beta \in \operatorname{Fill}(c_\delta, E) \Rightarrow \operatorname{cf}\beta \geq \theta$, find a set $A_\delta \subseteq \operatorname{nacc}\operatorname{Fill}(c_\delta, E)$ such that $\operatorname{otp} A_\delta = \kappa$ and $\sup A_\delta = \delta$ and define $c'_\delta := \operatorname{cl} A_\delta$. Let $c_\delta = \emptyset$ for all other $\delta \in S$. Let us show that $\overline{C}' = \langle c'_\delta : \delta \in S\rangle$ is as required by the Theorem. Suppose that $E' \subseteq \lambda$ is a given club. We have shown that the set $N'(E') = \{\delta \in S : \operatorname{Fill}(c_\delta, E) \subseteq E' \wedge \beta \in \operatorname{nacc}\operatorname{Fill}(c_\delta, E) \Rightarrow \operatorname{cf}\beta \geq \theta\}$ is stationary. For every $\delta \in N'(E')$ the set A_δ is contained in E', cofinal in δ, has order-type κ and is contained in $S_{\geq\theta}$. Since E' is closed, also $c'_\delta = \operatorname{cl} A_\delta \subseteq E'$. Finally, if $\beta \in \operatorname{nacc} c'_\delta$ then necessarily $\beta \in A_\delta$ and thus $\beta \in S_{\geq\theta}$, as required. □

We have yet to show that a club $E \subseteq \lambda$ as in the sufficient condition can be found. The proof uses the eventual stabilization of Fill, and is not much different than the proof of Theorem 4.

Claim 13. *There exists a club $E \subseteq \lambda$ such that for every club $E' \subseteq \lambda$ the set $\{\delta \in S : \operatorname{Fill}(c_\delta, E) \subseteq E'\}$ is stationary.*

Proof of Claim. Define by induction on $i < \theta$ a decreasing sequence $\langle E_i : i < \theta\rangle$ of clubs $E_i \subseteq \lambda$.

Let $E_0 = \operatorname{acc}\lambda$. If $i < \lambda$ is limit, let $E_i := \bigcap_{j<i} E_j$. Suppose that $i = j + 1 < \theta$ and E_j is defined. If $\langle \operatorname{Fill}(c_\delta, E_j) : \delta \in S\rangle$ satisfies the conclusion of the claim, then we are done. Else there is some club $E \subseteq \lambda$ such that $\delta \in E \cap S \Rightarrow \operatorname{Fill}(c_\delta, E_j) \nsubseteq E$. Let $E_i := E_j \cap E$.

Suppose that E_i is defined for all $i < \theta$, and pick $\delta \in \bigcap_{i<\theta} E_i$. This is possible because $\theta < \lambda$.

We show that $\operatorname{Fill}(c_\delta, E_i)$ is constant on some end-segment $[j, \theta)$. We write for simplicity $c_\delta^{i,n}$ for $c_\delta^{E_i,n}$ for $i < \theta$. We will show by induction on n that $c_\delta^{i,n}$ is constant on $[j(n), \theta)$ for some $j(n) < \theta$. This suffices, since we then set $j := \sup\{j(n) : n < \omega\}$, which, by regularity of θ is $< \theta$ and recall that $\operatorname{Fill}(c_\delta, E_i) = \bigcup c_\delta^{i,n}$.

For $n = 0$ there is little to prove: let $j(0) = 0$ and recall that $c_0^i = c_\delta$ for all $i < \theta$. Suppose that c_n^i is constant on $[j(n), \theta)$. For every $\beta \in c_\delta^{j(n),n} \cap S_{<\theta}$ there is some $j < \theta$ at which $\operatorname{fill}(\beta, E_i) = \operatorname{Drop}(e_\beta, E_i)$ stabilizes by Fact 3.8, because $|e_\beta| = \operatorname{cf}\beta < \theta$ and E_i is decreasing in i. By regularity of $\theta > |c_\delta^0|$ find $j(n) \leq j(n+1) < \theta$ such that $\operatorname{fill}(\beta, E_i)$ is constant on $[j(n+1), \theta)$ for all $\beta \in \operatorname{nacc} c_\delta^{j(n),n} \cap S^\lambda_{<\theta}$. If $i \in [j(n+1), \theta)$ then $c_\delta^{i,n} = c_\delta^{j(n+1),n}$ by the induction hypothesis. The definition of $c_\delta^{i,n+1}$ as $c_\delta^{i,n} \cup \bigcup\{\operatorname{fill}(\beta, E_i) \setminus \operatorname{drop}(\beta, c_\delta^{i,n}) : \beta \in \operatorname{nacc} c_\delta^{i,n} \wedge \operatorname{cf}\beta < \theta\}$ implies that $c_\delta^{i,n+1} = c_\delta^{j(n+1),n+1}$, because $c_\delta^{i,n} = c_\delta^{j(n+1),n}$ and $\operatorname{fill}(\beta, E_i) = \operatorname{fill}(\beta, E_{i(n+1)})$ for all $\beta \in \operatorname{nacc} c^{i,n}$.

Let $j(*) < \theta$ be fixed, then, so that $\operatorname{Fill}(c_\delta, E_i)$ is constant on $[j(*), \theta)$. Since $\delta \in E_{j(*)+1}$ it follows that $\operatorname{Fill}(c_\delta, E_i) \nsubseteq E_{j(*)+1}$. But $\operatorname{Fill}(c_\delta, E_{j(*)+1}) \subseteq E_{j(*)+1}$ by Fact (1) above because $c_\delta \subseteq E_{j(*)+1}$. So $\operatorname{Fill}(c_\delta, E_{j(*)}) \neq \operatorname{Fill}(c_\delta, E_{j(*)+1})$, contrary to stabilization. □

□

2.1. **Guessing ideals and applications.** We make a few more definitions regarding the club-guessing sequences we met above. Let $\overline{C} = \langle c_\delta : \delta \in S\rangle$ be any sequence of sets indexed by a stationary $S \subseteq \lambda$ for some regular λ. For each of the guessing requirement presented in Theorems 1-4 there corresponds naturally a *guessing ideal*. The ideal consists of all subsets of λ which 'fail to guess' some club $E \subseteq \lambda$. For example, the guessing requirement in Theorem 1 is that $\{\delta \in S : c_\delta \subseteq E\}$ is stationary. Therefore the ideal corresponding to this notion of guessing is $\{A \subseteq \lambda : \exists (E \subseteq \lambda)\,\text{club}\ [\delta \in A \cap S \cap E \Rightarrow c_\delta \not\subseteq E\}$. Saying that $\langle c_\delta : \delta \in S\rangle$ satisfies the conclusion of Theorem 1 is equivalent to saying that the guessing ideal is a proper ideal. It concentrates on S (that is, $\lambda \setminus S$ is in it) and extends the non-stationary ideal. It is λ-complete, but not always normal.

We can relax the guessing requirement, and thus shrink the ideal: if "to guess" means "to be included in E except for an initial segment" then guessing ideal is $\{A \subseteq \lambda : \exists E \subseteq \lambda\,\text{club}\,(\delta \in E \cap S \cap A \Rightarrow c_\delta \not\subseteq^* E\}$ and is sets is normal when the sequence under discussion satisfies the conclusion of Theorem 1. The symbol $A \subseteq^* B$ means that a proper end-segment of A is contained in B.

Shelah denotes by $\mathrm{id}^b(\overline{C})$ the guessing ideal for inclusion, by $\mathrm{id}^a(\overline{D})$ the guessing ideal for inclusion modulo initial segment and by $\mathrm{id}^p(\overline{C})$ the ideal of guessing in the sense of Theorem 3, namely having unboundedly many non-accumulation points enter the club.

`Where is this in the book?`

Club guessing is introduced in §1 of Chapter III, in terms of the guessing ideals. The existence of club guessing sequences is in §2. The operation defined here as "Drop" is denoted by gl there (for "glue"). Theorem 10 is not (as far as I know) in the book; it's in a different book: the non-structure theory book [8] p.189.

2.2. **Exercises.**

- Prove that $\mathrm{id}^a(\overline{C})$ is a normal ideal when $\overline{C}$ is the sequence from Theorem 4. Hint: If E_α witnesses that $A_\alpha \in \mathrm{id}^a(\overline{C})$ the $\bigcap_\alpha E_\alpha$ witnesses that $\bigcup_\alpha A_\alpha \in \mathrm{id}^a(\overline{C})$
- Prove that $\mathrm{Fill}(c_\delta, E)$ in the proof Theorem 10 is a closed set of ordinals.
- (Gitik-Shelah) Prove that if $\kappa^+ < \lambda$ and $\kappa > \aleph_0$ then the non-stationary ideal on S^λ_κ is not λ^+-saturated, namely that there are λ^+ many stationary subsets of S^λ_κ whose pairwise intersections are non-stationary.
  Hint:
  1. Show that there is *no* $\langle c_\delta : \delta \in S^\lambda_\kappa\rangle$ such that $c_\delta \subseteq \delta$, $|c_\delta| = \kappa$ and $\alpha \in c_\delta \Rightarrow \mathrm{cf}\alpha \geq \kappa^+\rangle$ with the property that for every club $E \subseteq \lambda$ the set $N(E) := \{\delta \in S^\lambda_\kappa : c_\delta \subseteq^* E\}$ contains a club intersected with S^λ_κ. Hint: Define a decreasing sequence of clubs $\langle E_n : n < \omega\rangle$ so that $E_{n+1} \subseteq \mathrm{acc} E_n \cap N(E_n)$ and consider the first point in the intersection of all clubs of cofinality κ.
  2. If you replace "contains a club intersected with S^λ_κ" by "stationary", the a sequence as above can be found on every stationary $S \subseteq S^\lambda_\kappa$ by Theorem 10.
  3. Use saturation to show that for every sequence $\overline{C}$ as in the previous paragraph, which is supported by S, there is a stationary $S' \subseteq S$ such that the restriction $C \restriction S'$ satisfies the condition in 1. (This is not a contradiction yet, because in 1. you use the fact that $S = S^\lambda_\kappa$). Hint: Find a decreasing chain of clubs $\langle E_i : i < \lambda^+\rangle$ and a chain of stationary

sets $S_i : i < \lambda^+\rangle$ such that $S_i = N(E_i)$ and the difference $S_i \setminus S_{i+1}$ is stationary. To define E_{i+1} assume that S_i is not as required. At limits let E_i be the diagonal intersection of the previous E_j-s and show that S_i is contained modulo the non-stationary ideal in every S_j for $j < i$. If the process continues λ^+ steps then saturation is violated.

4. use the previous paragraph to find a maximal antichain in λ/NS of stationary subsets of S^κ_λ, each carrying a sequence C exemplifying 2. By saturation assume the sets in the antichain are pairwise disjoint. Show that the union of all $\overline{C}$-s satisfies the condition forbidden by 1.

3. The ideal $I[\lambda]$

In this Section we introduce and develop the basic properties of the main combinatorial tool we shall be using in later sections. This is Theorem 18 below, that asserts the existence of a stationsry $S \subseteq S^\lambda_\kappa$ in $I[\lambda]$ for all regular k, λ such that $\kappa^+ < \lambda$.

Let $\lambda = cf\lambda$ be regular and uncountable cardinal. Let $I[\lambda]$ be an ideal over λ defined as follows:

Definition 14. *Let $S \subseteq \lambda$. Then $S \in I[\lambda]$ if and only if there exists a sequence $\bar{P} = \langle P_\alpha : \alpha < \lambda\rangle$ and a closed and unbounded set $E \subseteq \lambda$ such that:*

1. $P_\alpha \subseteq \mathcal{P}(\alpha)$ *and* $|P_\alpha| < \lambda$
2. *If $\delta \in E \cap S$ then δ is singular and there exists a set $c \subseteq \delta$ such that $\delta = \sup c$, $otp\, c < \delta$ and $\forall_{r<\delta}\ c \cap \gamma \in \bigcup_{\beta<\delta} P_\beta$.*

Fact 15. *$I[\lambda]$ is a normal ideal.*

Proof of Fact. Suppose that $S_\alpha \in I[\lambda]$ for $\alpha < \lambda$ and that $E_\alpha, \overline{P}_\alpha = \langle P^\alpha_\beta : \beta < \lambda\rangle$ witness $S_\alpha \in I[\lambda]$. There is no loss of generality in assuming that $\langle P^\alpha_\beta : \beta < \lambda\rangle$ is increasing.

Let E be the diagonal intersection of $\{E_\alpha : \alpha < \lambda\}$ and let $S = \{\alpha < \lambda : \exists\beta < \alpha\,[\zeta \in S_\beta]\}$ be the diagonal union of the S_α. Define $P_\alpha := \bigcup_{\beta<\alpha} P^\beta_\alpha$.

Suppose that $\delta \in E \cap S$. Then there is some $\beta < \alpha$ such that $\delta \in S_\beta$. Since $\delta \in E$ and $E = \{\alpha : \forall\beta < \alpha\,[\alpha \in E_\beta]\}$ it holds that $\delta \in E_\beta$. Thus δ is singular and there is some club $c \subseteq \delta$ with $c \cap \gamma \in \bigcup_{\epsilon<\delta} P^\beta_\epsilon$ for all $\gamma < \delta$. Since by the definition of P_ϵ it follows that $\bigcup_\epsilon P^\beta_\epsilon \subseteq \bigcup_{\epsilon<\delta} P_\epsilon$ we have that $c \cap \gamma \in \bigcup_{\epsilon<\delta} P_\epsilon$. This shows that $S \in I[\lambda]$. □

Fact 16. *For every regular uncountable λ the set $S^\lambda_0 = \{\alpha < \lambda : cf\lambda = \aleph_0\} \in I[\lambda]$*

Proof. Let $P_\alpha = [\alpha]^{<\aleph_0}$. If $\delta \in S^\lambda_0$ let $c \subseteq \delta$ be any cofinal subset of α with otp$c = \omega_0$. If $\gamma < \delta$ then $c \cap \gamma$ is finite and thus belongs to P_γ. □

Theorem 17. *If $\lambda = cf\lambda > \aleph_0$ is a regular uncountable cardinal, then*
$S^{\lambda^+}_{<\lambda} = \{\alpha < \lambda^+ : cf\alpha < \lambda\} \in I[\lambda^+]$.

Proof. Fact 16 above gives the theorem for $\lambda = \omega_1$, so we assume $\lambda > \omega_1$.

For every $\alpha < \lambda^+$ let $\left\langle a^\alpha_\zeta : \zeta < \lambda\right\rangle$ be a sequence of closed subsets of α such that:.

1. $\forall_\zeta < \lambda\ \ a^\alpha_\zeta \subseteq \alpha$ is closed and $\left|a^\alpha_\zeta\right| < \lambda$
2. $\zeta_1 < \zeta_2 \Rightarrow a^\alpha_{\zeta_1} \subseteq a^\alpha_{\zeta_2}$ and for limit ζ, $a^\alpha_\zeta = \bigcup_{\xi<\zeta} a^\alpha_\xi$
3. $\bigcup_{\zeta<\lambda} a^\alpha_\zeta = \alpha$

Let $P_\alpha = \left\{a^\beta_\zeta \cap \gamma : \beta \leq \alpha,\, \zeta < \lambda,\, \gamma \leq \alpha\right\}$. This definition implies that $P_\alpha \subseteq \mathcal{P}(\alpha)$ and $|P_a| \leq \lambda < \lambda^+$. Suppose that $\delta \in S^{\lambda^+}_{<\lambda}$. Without loss of generality, $\delta > \lambda$. We need to show that there is a club $c \subseteq \delta$ of δ with $c \cap \gamma \in \bigcup_{\beta<\delta} P_\beta$ for all $\gamma < \delta$. Let $c \subseteq \delta$ be any club of δ of order type $\kappa = cf\,\delta$ and so that $\min c > \lambda$. For every $\theta \in c$ the set $E_\theta = \left\{\zeta < \lambda : a^\delta_\zeta \cap \theta = a^\theta_\zeta\right\}$ is a club of λ (by the usual back and forth plus continuity argument). Therefore, $E = \bigcap_{\theta\in c} E_\theta$ is also a club of λ.

There must be some index ζ_0 for which $c \subseteq a^\delta_{\zeta_0}$. Find a point $\zeta(*) \in E$ such that $\zeta(*) > \zeta_0$. Now $a^\delta_{\zeta(*)}$ is a closed subset of δ, of cardinality $< \lambda$ and because it contains c as a subset, it is unbounded in δ. As $\delta > \lambda$, $otp\, a^\delta_{\zeta(*)} < \delta$. Let $\gamma < \delta$ be arbitrary. To show that $a^\delta_{\zeta(*)} \cap \gamma \in \bigcup_{\beta<\delta} P_\beta$ it suffices to show that $a^\delta_{\zeta(*)} \cap \theta \in P_\theta$ for some $\theta \in c$ which is greater than γ, because P_θ is closed under taking initial segments.

But, fixing such θ, we have $a^\delta_{\zeta(*)} \cap \theta = a^\theta_{\zeta(*)}$, as $\zeta(*) \in E \subseteq E_\theta$ and thus belongs to P_θ. $\square$

This is the main theorem of this Section:

Theorem 18. *If κ, λ are regular cardinals and $\kappa^+ < \lambda$ then there is a stationary set $S \subseteq S^\lambda_\kappa$ in $I[\lambda]$.*

Proof. If λ is a successor of regular, then the Theorem follows from the previous Theorem. The first $\lambda > \kappa^+$ for which the previous Theorem does not apply is $\kappa^{+\omega+1}$. So we may assume that $\kappa^{++} < \lambda$. We remark that in addition to λ successor of singular, there is another case which the previous theorem does not cover and this one does: the case λ regular limit, namely weakly inaccessible.

Fix a club guessing sequence $\overline{C} = \langle c_\alpha : \alpha \in S^{\kappa^{++}}_\kappa \rangle$ as in Theorem 1 in the previous section.

Description of the proof.

The proof will involve two elementary chains of models of $H(\chi)$, for some large enough regular χ. The first chain will be used to define $\langle P_i : i < \lambda\rangle$; the other, to prove that this choice works. The first chain is going to be an element of every member in the second. At some point of the proof, though, we shall need some set which is definable in the *second* chain to belong to some member of the *first* chain. We use the prediction, or "guessing, property of a club guessing sequence to obtain this.

Fix an elementary chain $\overline{M} := \langle M_i : i \leq \lambda\rangle$ of submodels of $\langle H(\chi), \in, \rangle$ (for a large enough regular χ) satisfying:

- $||M_i|| < \lambda$
- $\langle M_j : j \leq i\rangle \in M_{i+1}$, $i \in M_i$ and $i \subseteq M_i$
- $\overline{C}, \lambda \in M_0$ and $\kappa^{++} + 1 \subseteq M_0$

Let us define $P_i := M_i \cap \mathcal{P}(i)$. By condition (1) we see that $|P_i| < \lambda$.

Let $S \subseteq \lambda$ be the set

$$S := \left\{\alpha < \lambda : \mathrm{cf}\,\alpha = \kappa \wedge \exists c \,\text{club of}\, \alpha \big[(\forall \gamma < \alpha)(c \cap \gamma \in \bigcup_{i<\alpha} P_i)\big]\right\}$$

The sequence $\langle P_i : i < \lambda\rangle$ witnesses that $S \in I[\lambda]$ according to the definition of $I[\lambda]$. All that is left to be shown is that S is stationary.

Explanation If indeed there is a stationary $S \subseteq S^\lambda_\kappa$ in $I[\lambda]$ them M_0 should know about it and about some witness $\overline{P}$, by elementarity, and therefore P_i as chosen here must suffice.

We prove now that S is stationary. Fix a club $E \subseteq \lambda$ and we will show that E meets S. By shrinking E we may assume that $M_i \cap \lambda = i$ for all $i \in E$; this follows from the fact that $\{i < \lambda : M_i \cap \lambda = i\}$ is a club of λ.

Define a second elementary chain of models of $H(\chi)$, $\overline{N} := \langle N_\zeta : \zeta \leq \kappa^{++}\rangle$ satisfying:

- $\overline{M}, E, \lambda, \overline{C} \in N_0$ and $\kappa^{++} + 1 \subseteq N_0$
- $||N_\zeta|| = \kappa^{++}$
- $\langle N_\xi : \xi \leq \zeta\rangle \in N_{\zeta+1}$ for $\zeta < \kappa^{++}$

Define $f(\zeta) := \sup N_\zeta \cap \lambda$. The function $f : (\kappa^{++} + 1) \to \theta$ is increasing and continuous. Denote $\theta = f(\kappa^{++}$. By condition 3 in the choice of $\overline{N}$, we see that $f{\restriction}\zeta \in N_{\zeta+1}$ for every $\zeta < \kappa^{++}$.

We observe that for every $\zeta \leq \kappa^{++}$ the ordinal $f(\zeta)$ belongs to E. This is true by elementarity and the fact that $E \in N_\zeta$ for all ζ: if $\beta \in N_\zeta \cap \lambda$ is arbitrary, then $N_\zeta \models (\exists \gamma \in \lambda)[\gamma \in E \wedge \gamma > \beta]$. Therefore there exists $\beta < \gamma \in E \cap N$ is and thus E unbounded below $f(\zeta)$, implying $f(\zeta) \in E$. Thus $\operatorname{ran} f \subseteq E$.

Turn now to the chain $\overline{M}$ and work in $M_{\theta+1}$. Use the fact that $\theta \in M_{\theta+1}$ (condition (2) in the choice of $\overline{M}$) and the elementarity of $M_{\theta+1}$ and choose a function $g \in M_{\theta+1}$ such that $g : \kappa^{++} \to \theta$ increasing and continuous with $\theta = \sup \operatorname{ran} g$.

Since both f and g are increasing continuous on κ^{++} with ranges cofinally contained in θ, a standard argument allows us to fix a club$E \subseteq \kappa^{++}$ such that $f{\restriction}E = g{\restriction}E$.

Use the club guessing property of $\overline{C}$ to fix $\delta < S^{\kappa^{++}}_\kappa$ such that $c_\delta \subseteq E$, $\delta = \sup c_\delta$ and $\operatorname{otp} c_\delta = \kappa$. Define $c := f[c_\delta] = g[c_\delta]$.

Thus $c \subseteq f(\delta)$ is a club of $f(\delta)$ of order type κ. We already know that $f(\delta) \in E$; we will show that $f(\delta) \in S$ by showing that $c \cap \gamma \in \bigcup_{i<\theta_\delta} P_i$ for every $\gamma < f(\delta)$.

Let, then, $X = c_\delta \cap \zeta$ for some $\zeta \in c_\delta$ be an initial segment of c_δ and let $Y := f[X]$ be the corresponding initial segment of c. As c_δ and ζ belong to N_0, we have $X \in N_0 \subseteq N_{\zeta+1}$; and since $f{\restriction}\zeta \in N_{\zeta+1}$ by condition 3 in the choice of $\overline{N}$ we conclude that $Y = f[X] \in N_{\zeta+1}$.

If we knew that $Y \in M_i$ for *some* $i < \lambda$ we would be done: Suppose that there were some $i < \lambda$ such that

$$H(\chi) \models \exists i < \lambda\, [Y \in M_i)]$$

Since $Y, \overline{M}, \lambda \in M_{\zeta+1}$ and $N_{\zeta+1} \prec H(\chi)$

$$N_{\zeta+1} \models \exists i < \lambda[Y \in M_i]$$

by elementarity. Thus there is some $i < f(\zeta + 1) < f(\delta)$ such that $Y \in P_i$, as required.

But why is there such $i < \lambda$ at all? The reason is the club guessing sequence, which enables $\overline{M}$ to "predict" the set Y. Recall that $Y = f[X]$. The parameter X in this definition belongs to M_0. But f may not belong to any $M_i \in \overline{M}$. However, $f{\restriction}c_\delta = g{\restriction}c_\delta$ and so we can define Y using g instead. The function g belongs to

$M_{\theta+1}$, and the proof is now complete, as $Y = g[X]$ is definable in $M_{\theta+1}$ and hence $Y \in M_{\theta+1}$. $\square$

In the next Theorem we formulate a more convenient, equivalent definition of $I[\lambda]$.

Theorem 19. *$S \in I[\lambda]$ if and only if there is a club $E \subseteq \lambda$ and a sequence $\langle c_\alpha : \alpha < \lambda \rangle$ satisfying:*

(0) *$c_\alpha \subseteq \alpha$ and $\operatorname{otp} c_\alpha < \alpha$*
(1) *$\beta \in c_\alpha \Rightarrow \beta$ is a successor ordinal and $c_\alpha \cap \beta = c_\beta$*
(2) *If $\delta \in S \cap E$ then δ is singular, $\operatorname{otp} c_\delta = \operatorname{cf} \delta$ and $\delta = \sup c_\delta$.*

The condition (1) will be referred to as *coherence* of the sequence $\langle c_\alpha : \alpha < \lambda \rangle$.

Proof. Suppose such E and $\langle c_\alpha : \alpha < \lambda \rangle$ exist as above. Let $P_\alpha = \{c_\alpha\}$. If $\delta \in S \cap E$ then $c_\delta \subseteq \delta$ is unbounded in δ and $\operatorname{otp} c_\delta = \operatorname{cf} \delta$. If $\gamma < \delta$ then $c \cap \gamma = c \cap \epsilon$ where $\epsilon = \min[c_\delta - (\gamma + 1)]$ and $c_\delta \cap \epsilon = c_\epsilon \in P_\epsilon$. Thus $S \in I[\lambda]$

To prove the converse assume that $S \in I[\lambda]$ and fix a club $E \subseteq \lambda$ and a sequence $\langle P_\alpha : \alpha < \lambda \rangle$ that satisfy the conditions in definition 1 for S. We need to produce E and $\langle c_\alpha : \alpha < \lambda \rangle$ that satisfy conditions (0) - (2) for S.

We make first a few assumptions on P_α which can be made to hold without increasing $|P_\alpha|$, which is $< \lambda$. Assume that for some large enough regular χ and some $\alpha \in M \prec H(\chi)$, $P_\alpha = M \cap \mathcal{P}^{(\alpha)}$. Therefore if $x \in P_\alpha$ then also $otp\ x \in P_\alpha$; for every limit ordinal $\beta \in P_\alpha$ a club $e_\beta \subseteq \beta$ of $otp\ e_\beta = cf_\beta$ is also in P_α, and P_α is closed under set subtraction, union, intersection etc.

Next fix an increasing and continuous sequence $\langle \gamma_i : i < \lambda \rangle$ with $\gamma_i < \lambda$ limit ordinal for all $i < \lambda$ and $\gamma_{i+1} - \gamma_i \geq \left| \bigcup_{\alpha \leq \gamma_i} P_\alpha \right| + |\gamma_i| + \aleph_0$. By thinning $\langle \gamma_i : i < \lambda \rangle$ we may assume that $\gamma_i \in E$ and that if $\gamma_i \in S$ then γ_i is singular.

Let $F_i : \bigcup_{\alpha \leq \gamma_i} P_\alpha \times \gamma_i \to \{\zeta + 1 : \gamma_i < \zeta < \gamma_{i+1}\}$ be a 1-1 function. F codes every pair (x, β) for $x \in \bigcup_{\alpha \leq \gamma_i} P_\alpha$ and $\beta < \gamma_i$ by a successor ordinal in the interval (γ_i, γ_{i+1}).

We turn now to defining c_α for $\alpha < \lambda$.

Case 1. α is successor. As all γ_i's are limits, there is a last i such that $\gamma_i < \alpha$, and thus $\alpha < \gamma_{i+1}$. If $\alpha \notin \operatorname{ran} F_i$, then let $c_\alpha = \emptyset$.

Else, $\alpha = F_i(x, \beta)$ for some $x \in \bigcup_{\alpha \leq \gamma_i} P_\alpha$ and $\beta < \gamma_i$. If $\beta \geq \min x$ let $c_\alpha = \emptyset$ again.

The remaining case is $\beta < \min x$. If $x \in \bigcup_{\alpha \leq \gamma_{i'}} P_\alpha$ for some $i' < i$ let $c_\alpha = \emptyset$. Else:

Let

$$c_\alpha = \left\{ F_j(x \cap \zeta, \beta) : \begin{array}{ll} \text{(i)} & \zeta \in x \text{ and } otp\ x \cap \zeta \text{ belongs to } e_\beta \\ \text{(ii)} & j < i \text{ is the least s.t. } x \cap \zeta \in \bigcup_{\alpha \leq \gamma_j} P_\alpha \\ \text{(iii)} & \text{for all } \xi \in x \cap \zeta \text{ s.t. } otp\ (x \cap \xi) \in e_\beta \\ & \text{there is } j' < j \text{ s.t. } x \cap \xi \in \bigcup_{\alpha \leq \gamma_{j'}} P_\alpha \end{array} \right\}.$$

Case 2. α is the limit: *If possible*, find an unbounded $c_\alpha \subseteq \alpha$ of order type $\operatorname{cf}\alpha < \alpha$ with every $\beta \in c_\alpha$ a successor ordinal and $c_\alpha \cap \beta = c_\beta$. Otherwise, let $c_\alpha = \emptyset$.

We check the conditions on $\langle c_\alpha : \alpha < \lambda \rangle$.

Condition (0): $c_\alpha \subseteq \alpha$ and $otp\ c_\alpha < \alpha$. For limit α ondition (0) holds by the choice of c_α. If α is successor, then every closed subset of α has order type $< \alpha$.
Condition (1): Suppose that $\gamma \in c_\alpha$. Then $\gamma \in c_\alpha$ is a successor ordinal. Let $\theta \in c_\alpha \cap \gamma$. By the definition of c_α there is $\xi \in x$ with $otp\ x \cap \xi \in e_\beta$ and $j' < i$ least s.t. $x \cap \xi \in \bigcup_{\alpha \leq \gamma_{j'}} P_\alpha$ with $\theta = F_{j'}(x \cap \xi, \beta)$. (These are conditions (i), (ii) in the definition of c_α). By condition (iii), and as $\xi \in x \cap \zeta$, we have $j' < j$ such that $x \cap \xi \in \bigcup_{\alpha \leq \gamma_{j'}}, P_\alpha$.

Let $y := x \cap \zeta$. Now $\gamma = F_j(y, \beta)$ with $\beta < \min y$ and $\xi \in y$ with $otp\ y \cap \xi = otp\ x \cap \xi \in e_\beta$. Also, $j' < j$ is the least s.t. $y \cap \xi = x \cap \xi \in \bigcup_{\alpha \leq \gamma_{j'}} P_\alpha$ and $F_{j'}(y \cap \xi, \beta) = \theta$.

This settles (i), (ii) for c_γ with y substituted for x. Also, (iii) holds. We conclude that $\theta \in c_\gamma$. The converse is also true. So we have $c_\alpha \cap \gamma = c_\gamma$.

Condition (2): We have to say first what E' is. Let $E' = \{\gamma(i) : i \text{ is limit}\}$. Suppose that $\sigma \in S \cap E'$. We know that σ is singular (because $\{\gamma_i : i < \gamma\} \subseteq E$) and that there is a set $x \subseteq \delta$, $otp\ x < \sigma$ and $\forall_{\gamma < \sigma}\ x \cap \gamma \in \bigcup_{\alpha < \sigma} P_\alpha$.

We may assume that $x \in P_\sigma$. Let $\beta = otp\ x$. We have $\beta \in P_\sigma$ and by subtracting we assume that $\beta < \min x$.

Let $\delta = \sup \langle \gamma_i : i < i(*) \rangle$ so $\delta = \gamma_{i(*)}$.

$e_\beta \subseteq \beta$ and e_β has order type $cf\ x = cf\ \sigma$. We know that $e_\beta \in P_\beta$.

Define $y = \{\zeta \in x : otp\ (x \cap \zeta) \in e_\beta\}$. Then $otp\ y = otp\ e_\beta = cf\ \delta$ and y is club of δ.

Let $h(\zeta) = \min \left\{ j : x \cap \zeta \in \bigcup_{\alpha \leq \gamma_j} P_\alpha \right\}$ for $\zeta \in y$. So $h : y \to \{\gamma_i : i < \lambda\}$ and h is non-decreasing, as P_{γ_i} is closed under taking initial segments for all i. In addition, $\forall \zeta \in y\ h(\zeta) < \gamma_{i(*)} = \delta$.

Let $z = \{\zeta \in y : \forall \xi \in y \cap \zeta\ h(\xi) < h(\zeta)\}$.

Let $c = \left\{ F_{h(\zeta)}(x \cap \zeta, \beta) : \zeta \in z \right\}$.

The set c serves as a candidate to be c_δ. It is unbounded and of otp cfσ. All that we need to verify is that $\gamma \in c \Rightarrow \beta$ is successor and $c \cap \gamma = c_\gamma$. This involves checking that for $\alpha = F_{h(\zeta)}(x \cap \zeta, \beta)$ for $\zeta \in z$ then ζ and $j := h(\zeta)$, it holds that $c_\alpha = c \cap \alpha$. $\square$

`Where is this in the book?`

The ideal $I[\lambda]$ is defined in Definition 2.4(5), p.14. Theorem 17 is mentioned in Remark 2.4A as item (2) in the list of items that "will not be used and are included for the reader's amusemant". The important Theorem 18 is not in the book! It was discovered by Shelah in 1990, after the book was "sealed". The application of a stationary set in $I[\lambda]$ for obtaining least upper bounds and for finding generators for pcf (Sections 4 and 6 below) are in the book, though, where a stationary set in $I[\lambda]$ is treated as a *sufficient* condition for both; by now we know that the existence of stationary subsets in $I[\lambda]$ for λ succesor of singular or inaccessible is a Theorem of ZFC. .

4. Obtaining least upper bounds

Lub, lub, lub
All you need is lub
tat tatatata
All you need is lub, lub
Lub is all you need

In this Section we define and discuss the relations $<_I$, $\leq_i$ and $\lneqq_I$ over ordinal functions from an infinite set A, where I is an ideal over A. The important issues is when a sequence of ordinal functions on A which is increasing in $<_I$ has a least, and an exact, upper bound.

Let A be an infinite set, and let $I \subseteq \mathcal{P}(A)$ be an ideal over A. We denote by I^* the dual filter $\{X \subseteq A : A \setminus X \in\}$ and by I^+ we denote $\mathcal{P}(A) \setminus I$. We may ocasionaloly refer to sets in I as "measure zero sets", sets in I^* as "measure 1 sets" and sets in I^+ as "positive measure sets" or simply "positive sets".

Consider the relations $<_I$, $\leq_I$ and $\lneqq_I$ over all functions from A to the ordinals defined as follows:

- $f =_I g$ if and only if $\{a \in A : f(a) \neq g(a)\} \in I$
- $f \leq_I g$ if and only if $\{a \in A : f(a) > g(a)\} \in I$
- $f <_I g$ if and only if $\{a \in A : f(a) \geq g(a)\} \in I$
- $f \lneqq_I g$ if and only if $f \leq_I g$ and $\{a \in A : f(a) < g(a)\} \in I^+$.

We will also use the notation

- $f \leq g$ if and only if $\forall a \in A[f(a) \leq g(a)]$
- $f < g$ if and only if $\forall a \in A[f(a) < g(a)]$
- $f \lneqq g$ if and only if $f \leq g$ and $f \neq g$

Call a sequence $\langle f_\alpha : \alpha <\rangle$ of functions $f_\alpha : A \to \mathrm{On}$, for $\alpha < \lambda$, *increasing in* $<_I$ $(\leq_I, \lneqq_I)$ if and only if $\alpha < \beta < \lambda \Rightarrow f_\alpha <_I f_\beta$ $(f_\alpha \leq_I f_\beta, f_\alpha \lneqq_I f_\beta)$.

For the rest of the section fix an infinite set A and an ideal $I \subseteq \mathcal{P}(A)$.

Definition 20. *Suppose $\mathcal{F} = \{f_\alpha : \alpha < \alpha(*)\}$ is a set of ordinal functions on A.*

- *A function $g : A \to On$ is an* upper bound *of $\mathcal{F}$ if only if $f_\alpha \leq_I g$ for every $\alpha < \alpha(*)$.*
- *A function $g : A \to On$ is a* least upper bound *of $\mathcal{F}$ if and only if g is an upper bound of $\mathcal{F}$ and $g \leq_I g'$ for every upper bound g' of $\mathcal{F}$.*
- *A function $g : A \to On$ is an* exact upper bound *of $\mathcal{F}$ iff g is a least upper bound of $\mathcal{F}$ and for every $g' <_I g$ there is some $\alpha < \alpha(*)$ such that $g' <_I f_\alpha$.*

We wish to find sufficient conditions fo existence of a lub and an eub for a sequnce $\overline{f}$ which is increasing in $<_I$. The condition we shall provide is in Theorem 23 below. To state this condition we need the following definition of "obedience", which makes sense for every $\overline{C}$, but is useful when $\overline{C}$ has the coherence property.

Definition 21. *Let $\bar{C} = \langle c_\alpha : \alpha < \lambda\rangle$ be so that $c_\alpha \subseteq \alpha$. A sequence $\bar{f} = \langle f_\alpha : \alpha < \lambda\rangle$ of ordinal functions on A* obeys *$\bar{C}$ if and only if $\forall \alpha < \lambda \, \forall_\beta \in c_\alpha \, [f_\beta < f_\alpha]$.*

The next lemmat says that if a sequence $\overline{C}$ to which $\overline{f}$ obeys has coherence, then certain subsequences of $\overline{f}$ are increasing in $<$.

Lemma 22. *Suppose $\overline{f} = \langle f_\alpha : \alpha < \lambda\rangle$ obeys $\overline{C} = \langle c_\alpha : \alpha < \lambda\rangle$. If $\overline{C}$ satisfies that $\beta \in c_\alpha \Rightarrow c_\beta = c_\alpha \cap \beta$ then for every $\alpha < \lambda$ the sequence $\langle f_\beta : \beta \in c_\alpha\rangle$ is increasing in $<$.*

Proof. Let $\beta, \gamma \in c_\alpha$ for some $\alpha < \lambda$ and supose $\beta < \gamma$. Since $c_\alpha \cap \gamma = c_\gamma$ by coherence we have that $\beta \in c_\gamma$. By obedience of $\overline{f}$ we have $f_\beta < f_\gamma$. $\square$

The following Theorem provides a sufficient condition for the existence of an eub for an increasing sequence in $<_I$.

Theorem 23. *Suppose* $|A| \leq \kappa$ *and* $\kappa^+ < \lambda = \mathrm{cf}\lambda$. *Suppose* $S \subseteq S^\lambda_{\kappa^+} = \langle \delta < \lambda : \mathrm{cf}\delta = \kappa^+ \rangle$ *is stationary,* $S \in I[\lambda]$ *and that* $\overline{C}$ *witnesses* $S \in I[\lambda]$, *namely*

- $\overline{C} = \langle c_\alpha : \alpha < \lambda \wedge \mathrm{cf}\alpha \leq \kappa^+ \rangle$
- $c_\alpha \subseteq \alpha$ *and* $\mathrm{otp} c_\alpha < \alpha$
- $\beta \in c_\alpha \Rightarrow \beta$ *is successor and* $c_\alpha \cap \beta = c_\beta$
- $\mathrm{otp} c_\delta = \kappa^+$ *and* $\sup c_\delta = \delta$ *for all* $\delta \in S$

If a sequence of ordinal functions $\overline{f} = \langle f_\alpha : \alpha < \lambda \rangle$ *is increasing in* $<_I$ *and* obeys $\overline{C}$, *then* $\overline{f}$ *has an exact upper bound.*

Proof. First we prove the existence of a lub; then we shall show that every lub of $\overline{f}$ is an eub.

We find a lub by approximations. We start with an upper bound and decrease it whenever it is not a lub. At limit stages the obedience of $\overline{f}$ to $\overline{C}$ is used to produce a "smaller" upper bound from the the set of values of all previous upper bounds. The existence of lub will be obtained once we show that this process must terminate.

By induction on $\zeta < \kappa^+$ we define a sequence of upper bounds g_ζ to $\overline{f}$ so that:

$$\xi < \zeta < \kappa^+ \Rightarrow g_\zeta \lneqq_I g_\xi \tag{1}$$

Let $g_0(a) = \sup\{f_\alpha(a) + 1 : \alpha < \lambda\}$. For every $\alpha < \lambda$, $g_0 > f_\alpha$ so g_0 is an upper bound of $\overline{f}$.

For successor $\zeta + 1$ just choose, if possible, $g_{\zeta+1}$ satisfying (1) above. If this is not possible, g_ξ is a lub, as required, and the induction terminates.

Suppose that $\zeta < \kappa^+$ is limit. We shall show that if g_ξ is defined for all $\xi < \zeta$, then also g_ξ can be defined (and therefore the induction does not terminate at a limit stage $< \kappa^+$).

We need to define sets $S_\zeta(a)$, auxiliary functions h^ζ_α and an index $\alpha_\zeta < \lambda$ before defining g_ζ.

- For $a \in A$ let $S_\zeta(a) := \{g_\xi(a) : \xi < \zeta\}$.
- for $\alpha < \lambda$ let $h^\zeta_\alpha(a) := \min\{S_\zeta(a) \setminus f_\alpha(a)\}$.

Here are the facts we need about the functions h^ζ_α:

Fact 24. 1. $h^\zeta_\alpha \in \prod_{a\in A} S_\zeta(a)$ *and* $h^\zeta_\alpha \geq f_\alpha$
2. $f_\alpha(a) \leq f_\beta(a) \Rightarrow h^\zeta_\alpha(a) \leq h^\zeta_\beta(a)$ *for* $\alpha, \beta < \lambda$
3. *for a fixed* $\zeta < \kappa^+$ *the sequence* $\langle h^\zeta_\alpha : \alpha < \lambda \rangle$ *is increasing in* $\leq_I$ *and for every subsequence* $\langle f_\alpha : \alpha \in c \rangle$ *of* $\overline{f}$, $c \subseteq \lambda$, *which is increasing in* $<$ *the subsequence* $\langle h^\zeta_\alpha : \alpha \in c \rangle$ *is increasing in* $\leq$.
4. *if* $\alpha, \beta < \lambda$ *and* $h^\zeta_\alpha(a) < h^\zeta_\beta(a)$, *then* $h^\zeta_\alpha(a) < f^\zeta_\beta(a)$
5. *if* $\xi < \zeta < \kappa^+$ *and* $\alpha < \lambda$ *then* $h^\xi_\alpha \geq h^\zeta_\alpha$

The first item follows directly from the definition of h^ζ_α. For the second item fix $\alpha, \beta < \lambda$. If $f_\zeta(a) \leq f_\beta(a)$ then $h^\zeta_\alpha(a) = \min\{S_\zeta(a) \setminus f_\alpha(a)\} \leq \min\{S_\zeta(a) \setminus f_\beta(a)\} = h^\zeta_\beta$. The third item followd directly from the second. The fourth is immediate from the definitions. Finally, if $\xi < \zeta < \kappa^+$ then $S_\xi(a) \subseteq S_\zeta(a)$ for all $a \in A$ and therefore $h^\zeta_\alpha(a) = \min\{S_\zeta(a) \setminus f_\alpha(a)\} \leq \min\{S_\xi(a) \setminus f_\alpha(a)\} = h^\xi_\alpha$.

The existence of the index $\alpha_\zeta < \lambda$ is provided by the following:

Claim 25. *There is some $\alpha_\zeta < \lambda$ such that*

$$\alpha_\zeta \leq \alpha < \lambda \Rightarrow h^\zeta_\alpha =_I h^\zeta_{\alpha_\zeta}. \tag{2}$$

Proof of Claim. The sequence $\langle h^\zeta_\alpha : \alpha < \lambda\rangle$ is increasing in $\leq_I$ by 3 above. If the sequence $\overline{h} := \langle h^\zeta_\alpha : \alpha < \lambda\rangle$ does not stabilize modulo I, then $\forall\alpha < \lambda\exists\gamma < \lambda\left[h^\zeta_\alpha \lneqq h^\zeta_\gamma\right]$, and as $\overline{h}$ is increasing in $\leq_I$ there is a club $E \subseteq \lambda$ such that $\beta \in E \Rightarrow \forall\alpha < \beta\left[h^\zeta_\alpha \lneqq_I h^\zeta_\beta\right]$.

Use the fact that S is stationary to find $\delta \in S \cap \operatorname{acc} E$.

Now find a subset $c \subseteq c_\delta$ such that between any two members of c there is at least one point of E.

By Lemma 22 the sequence $\langle f_\alpha : \alpha \in c\rangle$ is increasing in $<$, and therefore, by Fact 24.3 the sequence $\langle h^\zeta_\alpha : \alpha \in c\rangle$ is increasing in $\leq$. Because between any two members of c there are points of E, the sequence $\langle h^\zeta_\alpha : \alpha \in c\rangle$ is increasing in $\lneqq_I$. Since $f \leq g \wedge f \lneqq_I g \Rightarrow f \lneqq g$ for all $f, g : A \to \mathrm{On}$, we conclude that $\langle h^\zeta_\alpha : \alpha \in c\rangle$ is increasing in $\lneqq$. This contradicts the fact that $|S_\alpha(a)| \leq \kappa$ for all $a \in A$: the sequence $\langle h^\zeta_\alpha(a) : \alpha \in c\rangle$ is increasing in $\leq$, and since $|S_\zeta(a)| \leq \kappa$ stabilizes at $\alpha_a < \kappa^+$. Putting $\alpha = \sup\{\alpha_a : \alpha \in A\} < \kappa^+$ we ontain $h^\zeta_\alpha = h^\zeta_{\alpha+1}$, contrary to $h^\zeta_\alpha \lneqq_I f^\zeta_{\alpha+1}$. □

Using Claim 25, fix, at a limit stage $\zeta < \kappa^+$ the ordinal $\alpha_\zeta < \lambda$ and define $g_\zeta := h^\zeta_{\zeta_\zeta}$. For every $\alpha_\zeta < \alpha < \lambda$ we have, of course, $g_\zeta = h^\zeta_\alpha$, so we may increase α_ζ at will.

Claim 26. *g_ζ is an upper bound of $\overline{f}$ and $g_\zeta \lneqq_I g_\xi$ for every $\xi < \zeta$.*

Proof of Claim. Let $\alpha < \lambda$ be given. By increasing α we may assume that $\alpha \geq \alpha_\zeta$. Then $f_\alpha \leq h^\zeta_\alpha =_I h^\zeta_{\alpha_\zeta}$. Thus g_ζ is a lub of $\overline{f}$.

If $\xi < \zeta$ then $f_{\alpha_\zeta} <_I g_\xi$, because g_ξ is an upper bound and $\overline{f}$ increasing in $<_I$. This means that for measure 1 set of $a \in A$ we have $g_\xi(a) > f_{\alpha_\zeta}(a)$. Since $g_\xi(a) \in S_\zeta(a)$, for each such a we have $h^\zeta_{\alpha_\zeta}(a) \leq g_\xi(a)$. This establishes that $g_\zeta \leq_I g_\xi$ for all $\xi < \zeta$. Since ζ is limit and $g_\xi \gneqq g_{\xi+1}$ for $\xi < \zeta$, it follows the $g_\zeta \lneqq g_\xi$ for all $\xi < \zeta$. □

We prove the existence of lub by showing that the induction must stop before κ^+. We just showed the inductive definition of g_ζ does not break at limit $\zeta < \kappa^+$. Therefore it must break at some $\zeta + 1 < \kappa^+$; and this means g_ζ is a lub.

Claim 27. *For some $\zeta < \kappa^+$ the upper bound $g_{\zeta+1}$ is not defined.*

Proof of Claim. Suppose to the contrary that the induction goes through all $\zeta < \kappa^+$.

For every limit $\zeta < \kappa^+$ we have $g_\zeta = h^\zeta_{\alpha_\zeta} =_I h^\zeta_\alpha$ for all $\alpha \geq \alpha_\zeta$. By regularity of $\lambda > \kappa^+$ we find $\alpha(*) < \lambda$ such that $g_\zeta =_I h^\zeta_{\alpha(*)}$ for all limit $\zeta < \kappa^+$.

The sequence $\left\langle h^\zeta_{\alpha(*)} : \zeta \in \operatorname{acc}\kappa^+\right\rangle$ stabilizes at some $\zeta(*) \in \operatorname{acc}\kappa^+$ by 24 5 above. Let $\zeta > \zeta(*)$ be limit. So $g_{\zeta(*)} \gneqq g_\zeta = g_{\zeta(*)}$ — contradiciton to stabilization. □

We show next that the lub we found is an eub.

Claim 28. *Let g be a lub of $\overline{f}$. If $g' <_I g$ then $g' <_I f_\alpha$ for some $\alpha < \lambda$.*

Proof of Claim. Let $B_\alpha = \{a \in A : f_\alpha(a) > g'(a)\}$. For $\alpha < \beta < \lambda$ we have $B_\alpha \subseteq_I B_\beta$, because $f_\alpha <_I f_\beta$. If B_α/I does not stabilize, then there exists a club $E \subseteq \lambda$ so that $\alpha < \beta \in E \Rightarrow B_\alpha \subsetneqq_I B_\beta$. Fix $\delta \in S \cap \operatorname{acc} E$. Find cofinal $c \subseteq c_\delta$ with members of E between any two members of c.

Since between any two members of c there is a point of E, the sequence $\langle B_\alpha : \alpha \in c\rangle$ is increasing in $\subsetneqq_I$. But by Lemma 22 the sequence $\langle f_\alpha : \alpha \in c\rangle$ is increasing in $<$, which implies that $\langle B_\alpha : \alpha \in c\rangle$ is increasing in $\subseteq$. Since to increase in both $\subseteq$ and $\subsetneqq_I$ means to increase in $\subsetneqq$, we cnclude that $\langle B_\alpha : \alpha \in c\rangle$ is a strictly increasing sequence of subsets of A of length κ^+, which contradicts $|A| \le \kappa$.

We assume, then, that B_α/I stabilizes at $\alpha(*) < \lambda$. If $B_{\alpha(*)} \in I^*$, then $f_{\alpha(*)} >_I g'$, and we are done. Else, for all $\alpha \ge \alpha(*)$

$$f_\alpha \restriction A \setminus B_{\alpha(*)} \le_I g' \restriction A \setminus B_{\alpha(*)}$$

Define, then,

$$g'' = g' \restriction A \setminus B_{\alpha(*)} \cup g \restriction B_{\alpha(*)}.$$

g'' is an upper bound of $\overline{f}$, and, $g'' \lneqq g$ since $g' <_I g$ and $A \setminus B_{\alpha(*)} \in I^+$. This contradicts g being a *least* upper bound.

$\square$

$\square$

Let us discuss the role of $I[\lambda]$ in this proof. The obedience of $\overline{f}$ to $\overline{C}$ and the coherence of $\overline{C}$ implies, by Lemma 22, that $\overline{f}$ is increasing in $<$ "locally", or on subsequences of $\overline{f}$ indexed by $c_\delta \in \overline{C}$. We know that if $\overline{h}^\zeta$ does not stabilize modulo I, it increases in $\lneqq_I$ on a club of λ; we also know that it's impossible for any subsequence of $\overline{h}^\zeta$ of length κ^+ to increase in $\lneqq$, becuse the range of each $h^\zeta_\zeta \in \overline{h}^\zeta$ is in contained in $\prod S_\zeta(a)$.

Now since the set $S \in I[\lambda]$ is stationary, we can "trap" an accumulation point δ of E in S and by interweaving members of c_δ with members of E find a sequence of length κ^+ which increases in both relations. In other words, because the subseuences on which $\overline{h}$ increases in $<$ are indexed by c-s that converge to all ordinal in a *stationary* set, they intertwine with any potential club on which $\overline{h}^\alpha$ may increase.

The same argument is repeated in Claim 28: if the sequence of B_α does not stabilize modulo I it increases on a club. An accumulation point of the club is trapped in S; and a subsequence of length κ^+ is thus found wich increases in both $\subseteq$ and $\subsetneqq_I$.

The demand of obedience can be waived in Theorem 23 if we assume that $2^{|A|} < \lambda$ (see exercise below). The pigeon-hole principle can be used to prove the Lemma 25 and Claim 27.

A stationary set in $I[\lambda]$ liberates us, then, from assumptions about the size of $2^{|A|}$, provided the sequence at hand obeys a suitable $\overline{C}$.

`Where is this in the book?`

Compare with Lemma 2.3 in Burke-Magidor. See also Lemma 2.1 in Jech. The book mentions $I[\lambda]$ and a stationary set in $I[\lambda]$ as a sufficient condition for existence of eub (§2 of chapter I), but not as a theorem of ZFC.

Exercise 1. • *Prove that if $|A| = \kappa \ge \aleph_0$, I an ideal over A and $\langle f_\alpha : \alpha < \lambda\rangle$ is a sequence of ordinal frunctions on A which is increasing in $<_I$, then if $\lambda > 2^\kappa$ then $\overline{f}$ has an exact upper bound.*

Hint: Define a decreasing sequence of upper bounds of lenth κ^+. At limit $\zeta < \kappa^+$ prove Lemma 25 using the assumption $2^\kappa < \lambda$ and the pigeon-hole principle. Show that the induction cannot go on for κ^+ steps. Prove lemma 28 above using $2^\kappa < \lambda$ and the pigeon hole principle.

See also [2] for a proof using the Erdős-Rado Theorem.

5. PCF Theory

In this Section we begin the study of possible cofinalities of reduced products of small infinite sets of regular cardinals. We shall see that the set of all possible cofinalities of such products is well behaved, and is related to a sequence of ideals over the set.

Definition 29. *Let $\langle P, \leq \rangle$ be a quasi-ordered set, that is $\leq$ is a transitive and reflexive relation over P. Write $p < q$ for $p \leq q \wedge q \not\leq p$.*

1. *A subset $D \subseteq P$ is* cofinal *if and only if $\forall p \in P \exists d \in D\,[p \leq d]$*
2. *The* cofinality $\mathrm{cf}\, P$ *is the least cardinality of a cofinal subset $D \subseteq P$*
3. *A sequence $\overline{d} = \langle d_i : i < \lambda \rangle$ of elements of P is* increasing cofinal *if and only if $\overline{d}$ is increasing in $<$ and $\mathrm{ran}\,\overline{d}$ is cofinal*
4. P has true cofinality *if and only if there is an increasing cofinal $\overline{d}$. The* true cofinality $\mathrm{tcf}\, P$ *is defined when P has true cofinality and is the minimal length of an increasing cofinal $\overline{d}$.*

Every quasi ordered set has cofinality. The disjoint union of $\{\omega_n : n < \omega\}$, quasi ordered by the union of natural orderings on each ω_n, has cofinality $\aleph_\omega$ and does not have true cofinality. So the cofinality of P need not be regular, and P may have no true cofinality. However, if $\mathrm{tcf}\, P$ exists, then $\mathrm{tcf}\, P = \mathrm{cf}\, P$ and is regular (see exercise below).

Definition 30. *For a set A of regular cardinals let* $\mathrm{pcf}\, A = \{\mathrm{tcf} \prod A/I : I$ *is an ideal over*A *and* $\prod A/I$ *has true cofinality*$\}$.

The ordering $\leq_I$ was defined for all ordinal functions on A, and it applies in particular for all functions in the product $\prod A = \{f : f$ is a function on $A, \forall a \in A\,[f(a) \in a]\}$. Suppose that $\prod A/I$ has true cofinality and fix an increasing cofinal $\overline{f} = \langle f_\alpha : \alpha < \mathrm{tcf} \prod A \rangle$. For every ideal $I' \supseteq I$ over A the sequence $\overline{f}$ is increasing cofinal in $\prod A/I'$, and therefore $\mathrm{tcf} \prod A/I = \mathrm{tcf} \prod A/I'$. Since every ideal over A can be extended to a dual of an ultrafilter, we have proved the following fact:

Fact 31. $\mathrm{pcf}\, A = \{\mathrm{cf} \prod A/D : D$ *an ultrafilter over* $A\}$.

A product $\prod A/D$ where U is an ultrafilter is linearly ordered by Los' Theorem and always has true cofinality.

Why sets and not sequences, and why regular cardinals? For cofinality purposes it makes no difference replacing a cardinal by its cofinality; every ideal over a sequence fo cardinal either concentrae bla bla.

We remark that for every set of regular cardinals A we have $A \subseteq \mathrm{pcf}\, A$ via principal ultrafilters.

We begin our study of true cofinalities of products. The idea is to identify which subsets of A are "too small" to obtain a regular λ as true cofinality modulo any ideal. This is the contents of the very important definition below:

Definition 32. *Let $A \subseteq Reg$ and assume that $|A| < \min A$. For a regular λ define $J_{<\lambda}[A] = \{B \subseteq A : B \in D \Rightarrow \mathrm{cf} \prod A/D < \lambda$ for all ultrafilters D over $A\}$.*

To verify that $J_{<\lambda}$ is an ideal see exercise below.
Since $\mathrm{cf}\prod A/D$ is always regular, we have

Fact 33. *If μ is singular then $J_{<\mu^+} = J_{<\mu}$.*

The following Theorem provides information about all products modulo ideals that extend $J_{<\lambda}$, and has several important corollaries for pcf. The proof is a good warm-up for the next section, where the pcf Theorem is proved using $I[\lambda]$. The following proof makes an inessential use of $I[\lambda]$ and Theorem 23. For a proof of it without either 23 or $I[\lambda]$ see [MB] or Lemma 1.5 in chapter 1 of the book.

Theorem 34. *If $\min A > |A|$ then $\prod A/J_{<\lambda}$ is λ -directed.*

Proof. By induction on $\mu < \lambda$ we show that any set $F \subseteq \prod A$ with $|F| = \mu$ has an upper bound in $\prod A/J_{<\lambda}$.

Let $F \subseteq \prod A$ be given and suppose $|F| = \mu$ be given.

Case 1: μ singular.

Using the induction hypothesis, assume, eithout loss of generality, that $F = \langle f_\alpha : \alpha < \mu\rangle$ is increasing in $<_I$. Pick a cofinal sequence $\left\langle f_{\alpha_\zeta} : \zeta < cf\mu\right\rangle$.

As $cf\mu < \mu$, the induction hypothesis gives a bound $g \in \prod A/J < \lambda$ to $\left\langle f_{\alpha_\zeta} : \zeta < cf\mu\right\rangle$ which is also a bound to $\overline{f}$.

Case 2: μ is regular and $|\mu \cap A| < \aleph_0$. Let $g(a) = \sup\{f(a) : f \in F\}$ for all $a \geq \mu$ in A and $g(a) = 0$ for $a \in A \cap \mu$. Since $A \cap \mu$ finite, $A \cap \mu \in J_{<\mu}$ and $g \in \prod A$ is an upper bound of F.

Case 3: μ is regular and $A \cap \mu$ is infinite. Let $\kappa := |A|^+$. Since $|A| < \min A$, we have $\kappa^+ < \mu$. Use Theorem 18 to find $S \subseteq S^\mu_{\kappa^+}$ stationary, $S \in I[\mu]$. Fix $\overline{C} = \langle c_\alpha : \alpha < \mu \wedge \mathrm{cf}\alpha \leq \kappa^+\rangle$ that demonstrates that $S \in I[\mu]$. Enumerate F as $\langle f_\alpha : \alpha < \mu\rangle$. Using the induction hypothesis define by induction on $\alpha < \mu$ a sequence $\langle h_\alpha : \alpha < \mu\rangle$ satisfying

1. $h_\alpha >_I f_\beta$ and $h_\alpha >_I h_\beta$ for all $\beta < \alpha$
2. $\overline{h} = \langle h_\alpha : \alpha < \mu\rangle$ obeys $\overline{C}$

This involves taking, at stage α, an upper bound $g_\alpha \in \prod A$ of the set $\{h_\beta : \beta < \alpha\} \cup \{f_\beta : \beta < \alpha\}$ and then letting $h_\alpha(a) = \max(\{g_\alpha(a)\} \cup \{h_\beta(a) : \beta \in c_\alpha\})$ for all $a \in A \setminus \kappa^{++}$ and letting $h_\alpha(a) = \alpha$ for $a \in A \cap \kappa^{++}$.

Use Theorem 23 theorem to find an eub g in $\leq_{J_{<\lambda}}$ of $\overline{h}$ (as ordinal function). Because $h_\alpha(a) < a$ for all $\in A \setminus \kappa^{++} \in J_{<\lambda}$, we may assume, without loss of generality, that $g(a) \leq a$ for all $a \in A$. Let $B = \{a \in A : g(a) = a\}$.

Claim 35. $B \in J_{<\lambda}$

Proof of Claim. If $B \notin J_{<\lambda}$ then we can find and ultrafilter $D \ni B$ over A so that $D \cap J_{<\lambda} = \emptyset$. The dual of D extends $J_{<\lambda}$ and therefore $\overline{h} := \langle h_\alpha : \alpha < \mu\rangle$ is increasing also in $<_D$. For every $f \in F$ let f' be obtained from f by replacing $f\restriction(A \setminus B)$ by the constant function 0. As $B \in D$, $f =_D f'$. Also $f' <_{J_{<\lambda}} g$ and therefore, because g is an eub, there is some $\alpha < \mu$ so that $f' <_{J_{<\lambda}} h_\alpha$. As D extends $J^*_{<\lambda}$, $f =_D f' <_D h_\alpha$. Thus $\overline{h}$ is increasing cofinal in $\prod A/D$ demonstrating that $\mathrm{cf}\prod A/D = \mu < \lambda$. By definition of $J_{<\lambda}$ we conclude that $B \in J_{<\lambda}$. ☐

Now that we know $B \in J_{<\lambda}$ redefine g as 0 on B and it remains an eub in $\leq_{J_{<\lambda}}$ of $\overline{h}$, only now belongs to $\prod A$. By the definition of $\overline{h}$, g is an upper bound of F. ☐

We list now several corollaries of the λ-directedness Theorem above.

Corollary 36. *If D is an ultrafilter over A then* $\operatorname{cf}\prod A/D = \min\{\lambda : D \cap J_{<\lambda^+} \neq \emptyset\}$.

Proof. Suppose D is an ultrafilter over A and let λ be the least such that $D \cap J_{<\lambda^+}$ is not empty. By the definition of $J_{<\lambda^+}$ we have $\operatorname{cf}\prod A/D \leq \lambda$.

Conversely, D extends $J^*_{<\lambda}$ because $D \cap J_{<\lambda^+} = \emptyset$ and therefore $\leq_D$ extends $\leq_{J_{<\lambda}}$. Since by Theorem 34 the product $\prod A/J_{<\lambda}$ is λ^+-directed, also the product $\prod A/D$ is λ^+-directed, and must therefore have cofinality at least λ. $\square$

Corollary 37. *If μ is a limit cardinal then* $J_{<\mu} = \bigcup_{\lambda<\mu} J_{<\lambda}$

Proof. Let $J := \bigcup_{\lambda<\mu} J_{<\lambda}$ for some limit cardinal μ. The inclusion $J \subseteq J_{<\mu}$ is immediate by the definition of $J_{<\mu}$.

Conversely, suppose to the contrary that $B \notin J$ and find an ultrafilter $D \ni B$ over A so that $D \cap J = \emptyset$. Since D extends $J_{<\lambda}$ for every $\lambda < \mu$, by Theorem 34 we have $\operatorname{cf}\prod A/D \geq \lambda$. Thus $\operatorname{cf}\prod A/D \geq \mu$ and $B \notin J_{<\mu}$. (Observe that if μ is regular limit the equality $\operatorname{cf}\prod A/D = \mu$ is not out of the question.) $\square$

Corollary 38. *For every $B \subseteq A$ there is a unique regular λ such that* $B \in J_{<\lambda^+} \setminus J_{<\lambda}$.

Proof. Since $\operatorname{cf}\prod A/D < |\prod A|^+$ for all ultrafilters D over A, for every $B \subseteq A$ there is some μ, and therefore a minimal μ, for which $B \in J_{<\mu}$. By the previous Corollary such μ is not limit for any $B \subseteq A$; so $\mu = \lambda^+$ is successor and $B \in J_{<\lambda^+} \setminus J_{<\lambda}$. Since $J_{<\mu}$ is increasing in μ, λ is unique. The cardinal λ must be regular by Fact 33. $\square$

Corollary 39. $\operatorname{pcf} A$ *has a last element.*

Proof. The previous corollary provides a unique regular λ for which $A \in J_{<\lambda^+} \setminus J_{<\lambda}$. Thus $\operatorname{cf}\prod A/D \leq \lambda$ for all ultrafilters D over A, and therefore $\lambda \geq \sup \operatorname{pcf} A$.

Conversely, let D be an ultrafilter over A so that $D \cap J_{<\lambda} \cap D = \emptyset$. Such D exists because $A \notin J_{<\lambda}$. The cofinality $\operatorname{cf}\prod A/D$ is at least λ because D extends $J^*_{<\lambda}$, and by the previous paragraphs is exactly λ, which means $\lambda \in \operatorname{pcf} A$.

Thus $\lambda = \max \operatorname{pcf} A$. $\square$

Corollary 40. *If $|A| < \min A$ then* $|\operatorname{pcf} A| \leq 2^{|A|}$.

Proof. By 5 there is a 1-1 correspondence between $\operatorname{pcf} A$ and the members of the increasing sequence of ideals $\langle J_{<\lambda} : \lambda \in \operatorname{pcf} A\}$. The length of any increasing sequence of ideals over A is at most $2^{|A|}$. $\square$

Let us observe the following about the operation pcf on sets of regular cardinals. We know that $A \subseteq pcf A$. It is seen (exercise belos) that $\operatorname{pcf} A \cup B = \operatorname{pcf} A \cup \operatorname{pcf} B$. If we knew that $\operatorname{pcf}\operatorname{pcf} A = \operatorname{pcf} A$ that this operation would satisfy the axioms of a topological *closure* operation. So let us ask:

Question 41. *Is* $\operatorname{pcf}\operatorname{pcf} A = \operatorname{pcf} A$ *for a set of regular cardinals A with* $\min A > |A|$*?*

This question is answered by a "yes" in a universe of set theory without inaccessible cardinals, as we shall see shortly. But the important issue is the following: if there is no inaccessible in $\operatorname{acc} pcf A$ then $\operatorname{pcf}\operatorname{pcf} A = \operatorname{pcf} A$.

Theorem 42. *Suppose* $B \subseteq \operatorname{pcf} A$ *and* $|B| < \min B$. *Then* $\operatorname{pcf} B \subseteq \operatorname{pcf} A$.

Corollary 43. *Suppose there is no inaccessible cardinal in* $\operatorname{acc}\operatorname{pcf} A$. *Then* $\operatorname{pcf}\operatorname{pcf} A = \operatorname{pcf} A$

Proof of Corollary. Let $\lambda_0 := \sup \operatorname{pcf} A$. Either λ_0 is not inaccessible or it is not an accumulation point of $\operatorname{pcf} A$. In either case, $\lambda_1 := \operatorname{cf}|\operatorname{pcf} A| < \lambda_0$. Let $B_0 = \operatorname{pcf} A \cap (\lambda_1, \lambda_0]$. Since $\min B_0 > |B_0|$ we may use Theorem 42 to conclude that $\operatorname{pcf} B_1 \subseteq \operatorname{pcf} A$.

Continue by induction to define $\lambda_{n+1} = \operatorname{cf}|\operatorname{pcf} A \cap (\lambda_n + 1)|$ and $B_n := \operatorname{pcf} A \cap (\lambda_{n+1}, \lambda_n]$. Because the sequence $\langle \lambda_n : n < \omega \rangle$ is strictly decreasing, it terminates after finitely many steps (namely, $B_{n+1} = \emptyset$ and $\lambda_{n+1} = 0$ for some n) and thus produces a partition of $\operatorname{pcf} A$ to finitely many parts $\{B_l : l \leq n\}$ so that $\operatorname{pcf} B_l \subseteq \operatorname{pcf} A$ for all $l \leq n$. Since every ultrafilter on $\operatorname{pcf} A$ concentrates on one of the B_l-s, we conclude that $\operatorname{pcf}\operatorname{pcf} A \subseteq \operatorname{pcf} A$. $\square$

In particular, if there are no inaccessibles in the universe then the operation of pcf satisfies the axioms of a topological closure operation.

Proof of the Theorem. Suppose $B \subseteq \operatorname{pcf} A$, $\min B > |B|$ and fix an ideal I_λ over A so that $\operatorname{tcf} \prod A/I_\lambda = \lambda$ for every $\lambda \in B$. Suppose that $\lambda(*) \in \operatorname{pcf} B$ and that $I(*)$ is an ideal over B so that $\operatorname{tcf} \prod B/I(*) = \lambda(*)$.

Let $I = \{X \subseteq A : \{\lambda \in B : X \notin I_\lambda\} \in I(*)\}$. We show that I is an ideal over A and $\operatorname{tcf} \prod A/I = \lambda(*)$.

The verification that I above is an ideal is immediate.

For every $\lambda \in B$ fix an increasing cofinal sequence $\overline{f}^\lambda = \langle f^\lambda_\alpha : \alpha < \lambda \rangle$ in $\prod A/I_\lambda$.

Define for every function $g \in \prod B$ a function $G(g) \in \prod A$ as follows: since $\prod A/J_{<|B|^+}$ is $|B|$-directed by Theorem 34, there is a bound $f \in \prod A/J_{<|B|^+}$ to the set $\left\{ f^\lambda_{g(\lambda)} : \lambda \in B \right\}$. Let $G(g)$ be such a bound.

Fix an increasing cofinal sequence $\langle g_\alpha : \alpha < \lambda(*) \rangle$ in $\prod B/I(*)$. We argue that for every $f \in \prod A$ there is some $\beta < \lambda(*)$ such that $f <_I G(g_\alpha)$ for all $\alpha \in (\beta, \lambda(*))$. Suppose that $f \in \prod A$ is given. Let $F(f) \in \prod B$ be defined by $F(f)(\lambda) = \min\{\alpha < \lambda : f <_{I_\lambda} f^\lambda_\alpha\}$ for $\lambda \in B$. This is well defined because $\langle f^\lambda_\alpha : \alpha < \lambda \rangle$ is increasing cofinal in $\prod A/I_\lambda$ for all $\lambda \in B$. Let $\beta < \lambda(*)$ be the least for which $F(f) <_{I(*)} g_\beta$. If $\alpha \in (\beta, \lambda(*))$ then $F(f) <_{I(*)} g_\alpha$ and thus $f <_{I_\lambda} f^\lambda_{g_\alpha(\lambda)}$ for all but a set in $I(*)$ of $\lambda \in B$. This means that $f <_I G(g_\alpha)$ by the definition of I.

The above implies that $\langle G(g_\alpha) : \alpha < \lambda(*) \rangle$ is cofinal in $\prod A/I$ and that for some club $E \subseteq \lambda(*)$ the sequence $\langle G(g_\alpha) : \alpha \in E \rangle$ is increasing in $<_I$. That establishes $\operatorname{tcf} \prod A/I = \lambda(*)$.

Thus $\operatorname{pcf} B \subseteq \operatorname{pcf} A$ and the proof is done. $\square$

`Where is this in the book?`The basic properties of pcf are in the first chapter of the book. The directedness of $J_{<\lambda}$ is Lemma 1.5 in chapter 1. The fact that $\operatorname{pcf}\operatorname{pcf} A = \operatorname{pcf} A$ when there are no inaccessible in $\operatorname{acc}\operatorname{pcf} A$ is proved using generators in Chapter 8, 3.5, and is implicit in Conclusion 1.12 of Chapter 1.

6. The existence of generators for pcf

In this section we will prove that for every $\lambda \in pcf A$, $A \subseteq \operatorname{Reg}$ with $\min A > |A|$, there is a set $B_\lambda \subseteq A$ that *generates* the ideal $J_{<\lambda^+}$ from $J_{<\lambda}$, that is $J_{<\lambda^+} = J_{<\lambda} + B_\lambda$, the minimal ideal extending $J_{<\lambda}$ which contains B_λ. Such B_λ are called *generators* for $\operatorname{pcf} A$. The generator B_λ is unique only up to $J_{<\lambda}$, of course. The theorem asserting the existence of generators for pcf (Theorem 53) is called *the pcf Theorem.*

`Where is this in the book?`

The book defines in Chapter 1 a $\lambda \in \operatorname{pcf} A$ for which a generator exists as *normal*. In this terminology, we are proving that every λ is normal. The pcf theorem is Theorem 2.6 in chapter 8 (p. 331), and is Theorem 7.9 in [1]. In [13] the pcf Theorem is re-proved under more general assumptions.

The following Lemma has particular importance:

Lemma 44. *If $\lambda \in \operatorname{pcf} A$ and $B \in J_{<\lambda^+} \setminus J_{<\lambda}$ then $\operatorname{tcf} \prod B/J_{<\lambda} = \lambda$.*

Proof. First let us observe that if $\lambda \cap A$ is finite there is little to prove, as $J_{<\lambda^+} = A \cap \lambda^+$ and $J_{<\lambda} = A \cap \lambda$, and thus $J_{<\lambda^+} \setminus J_{<\lambda} = \{\lambda\}$. So we assume that $\lambda > |A|^{++}$ and fix a stationary set $S \subseteq S^\lambda_{|A|^+}$ in $I[\lambda]$ together with a sequence $\overline{C}$ witnessing this.

Define $J := J_{<\lambda} \cup \{B \in J_{<\lambda^+} \setminus J_{<\lambda} : \operatorname{tcf} B_{J_{<\lambda}} = \lambda\}$.

Claim 45. *J is an ideal.*

Proof of Claim. Suppose that $B_1, B_2 \in J_{<\lambda^+} \setminus J_{<\lambda}$ are in J and fix increasing cofinal sequences $\overline{f}^1, \overline{f}^2$ in $\prod B_1/J_{<\lambda}, \prod B_2/J_{<\lambda}$ respectively to demonstrate this. Let $f_\alpha = \max\{f^1_\alpha, f^2_\alpha\}$. The sequence $\overline{f}$ is increasing in $\leq_{J_{<\lambda}}$ and cofinal on $B_1 \cup B_2$, and this is enough for $B_1 \cup B_2 \in J$. The cases in which one of B_1, B_2 belongs to $J_{<\lambda}$ are immediate. $\square$

If $J = J_{<\lambda^+}$ then we are done. Suppose that $B \in J_{<\lambda^+} \setminus J$ and we shall derive a contradiction.

Fix and ultrafilter $D \ni B$ over A so that $J \cap D = \emptyset$. By Corollary 36 we conclude that $\operatorname{cf} \prod A/D = \lambda$. Fix, then, an increasing cofinal sequence $\langle f_\alpha : \alpha < \lambda\rangle$ in $\prod A/D$. By induction on $\alpha < \lambda$ choose $h_\alpha \in \prod B$ so that $\langle h_\alpha : \alpha \in \lambda\rangle$ is increasing in $<_{J_{<\lambda}}$ and $h_\alpha >_{J_{<\lambda}} f_\beta$ for all $\beta < \alpha$. The definition is possible by λ-directedness modulo $J_{<\lambda}$ (Corollary 34). Furthermore, make $\overline{h}$ obey $\overline{C}$.

Let g be an eub of $\overline{h}$ so that $g(a) \leq a$ for all $a \in A$ and denote $B_g := \{a \in A : g(a) = a\}$. Since $\overline{h}$ is increasing cofinal below g by definition of an eub, we have that $\operatorname{tcf} \prod B_g/J_{<\lambda} = \lambda$, and therefore $B_g \in J$. On the other hand, if B_g were not in J, then $A \setminus B_g \in D$. Since $g{\restriction}(A \setminus B_g) \cup 0{\restriction}B_g \in \prod A$, we obtain that $\overline{h}$ and consequently $\overline{f}$ are bounded in $\prod A/D$ — contradiction. So $B_g \in D$. This contradicts $D \cap J = \emptyset$. $\square$

A generator is a maximal set with respect to $\subseteq_{J_{<\lambda}}$. To find it we will construct, in the proof of the pcf theorem below, an increasing sequence in $\subsetneqq_{J_{<\lambda}}$. To be able to use of $I[\lambda]$ for the purpose of mixing $\subsetneqq_{J_{<\lambda}}$ with $\subseteq$, we will use a special kind of increasing cofinal obedient sequences, which we call *humbly obedient.*

Definition 46. *Let $\lambda \in \operatorname{pcf} A \setminus |A|^{+++}$, and let $\overline{C}$ witness that a stationary $S \subseteq S^\lambda_{\kappa^+}$ is in $I[\lambda]$. A sequence $\overline{f} = \langle f_\alpha : \alpha \in \lambda\rangle$* humbly obeys *$\overline{C}$ if and only if $\overline{f}$ is increasing in $<_{J_{<\lambda}}$, $\overline{f}$ obeys $\overline{C}$ and $f_\delta = \sup\{f_\beta : \beta \in c_\delta\}$ for every $\delta \in S$.*

Explanation: To obey $\overline{C}$ means that $f_\alpha \geq \sup\{f_\beta : \beta \in c_\alpha\}$. To humbly obey means that for $\delta \in S$, $f_\delta = \sup\{f_\beta : \beta \in c_\delta\}$, namely f_δ is the *minimal* function allowed by obedience when $\delta \in S$. Notice that while to obey was defined for all sequences we reserve humble obedient for increasing sequences in $\prod A/J_{<\lambda}$ of length λ.

Humbly obedient sequences form a proper subclass of obedient sequences. Let us show that it is not an empty subclass by showing we can find for every increasing sequence $\overline{f}$ a humbly obedient $\overline{h}$ which bounds at least what $\overline{f}$ bounds.

Fix now $\lambda \in \operatorname{pcf} A \setminus |A|^{+++}$, S and $\overline{C}$ as in the definition for the rest of this section.

Fact 47. *For every sequence $\overline{f} = \langle f_\alpha : \alpha < \lambda \rangle$ which is increasing in $<_{J_{<\lambda}}$ there exists a humbly obedient sequence $\overline{h} = \langle h_\alpha : \alpha < \lambda \rangle$ such that $\overline{h}$ is increasing in $<_{J_{<\lambda}}$ and $f_\alpha \le h_\alpha$ for all $\alpha \in \lambda \setminus S$.*

Proof. By induction on $\alpha < \lambda$ define h_α. For $\alpha = \delta \in S$ define $h_\alpha = \sup\{h_\beta : \beta \in c_\delta\}$. For $\alpha \in \lambda \setminus S$ find, using λ-direc tedness, a bound $g_\alpha \in \prod A$ of $\{f_\beta : \beta < \alpha\} \cup \{h_\beta : \beta < \alpha\}$ and define $h_\alpha(a) = \sup\left(\{g_\alpha(a)\} \cup \{f_\alpha(a)\} \cup \{h_\beta(a) + 1 : \beta \in c_\alpha\}\right)$. $\square$

Let us define now a partial ordering on the class of humbly obedient sequences.

Definition 48. *If $\overline{f}^1, \overline{f}^2$ are humbly obedient, write $\overline{f}^1 \le \overline{f}^2$ if and only if $f^1_\alpha \le f^2_\alpha$ for all $\alpha < \lambda$.*

Fact 49. *If $\overline{f}^1, \overline{f}^2$ are humbly obedient then $\overline{f}^1 \le \overline{f}^2$ if and only if $f^1_\alpha \le f^2_\alpha$ for all $\alpha \in \lambda \setminus S$.*

Proof. One direction follows because $\lambda \setminus S \subseteq \lambda$. For the other direction suppose that $f^1_\alpha \le f^2_\alpha$ for all $\alpha \in \lambda \setminus S$ and fix a least $\delta \in S$ for which $f^1_\delta \not\le f^2_\delta$. Since for every $\beta \in c_\delta$ we have that $f^1_\beta \le f^2_\beta$ and $f^2_\delta = \sup\{f_\beta : \beta \in c_\delta\}$, $f^1_\delta = \sup\{f^1_\beta : \beta \in c_\delta\}$, contradiction follows. $\square$

Since every humbly obedient sequence $\overline{f}$ is in particular an increasing in $<_{J_{<\lambda}}$ obedient sequence of ordinal functions, it has, by Theorem 23, an exact upper bound. Choose eub $g_{\overline{f}}$ for every humbly obedient $\overline{f}$, such that, without loss of generality, $g_{\overline{f}}(a) \le a$ for all $a \in A$. Let us also denote $B_{\overline{f}} := \{a \in A : g_{\overline{f}}(a) = a\}$. Since $\overline{f}$ is cofinal below g, we obtain that $\operatorname{tcf} B_{\overline{f}}/J_{<\lambda} = \lambda$ and therefore $B_{\overline{f}} \in J_{<\lambda^+} \setminus J_{<\lambda}$ for every humbly obedient $\overline{f}$.

We show next that every set $B \in J_{<\lambda^+} \setminus J_{<\lambda}$ is equivalent mod $J_{<\lambda}$ to a $B_{\overline{f}}$ for some humbly eobedient $\overline{f}$, and that if $\overline{f}^1 \le \overline{f}^2$ then $B_{\overline{f}^1} \subseteq_{J_{<\lambda}} B_{\overline{f}_2}$.

Lemma 50.
1. *Suppose $B \in J_{<\lambda^+} \setminus J_{<\lambda}$. Then there is a humbly obedient $\overline{f}$ such that $B =_{J_{<\lambda}} B_{\overline{f}}$.*
2. *If $\overline{f}^1 \le \overline{f}^2$ are humbly obedient sequences, then $B_{\overline{f}_1} \subseteq_{J_{<\lambda}} B_{\overline{f}^2}$*

Proof. Extend the ideal $J_{<\lambda}$ by the set $A \setminus B$. Thus we may ignore any coordinalte outside of B and work in either $\prod A/J_{<\lambda}$ or $\prod B/J_{<\lambda}$.

Using lemma 44 we fix an increasing cofinal seqeunce of length λ in $\prod B/J_{<\lambda}$ and replace it with a humbly obedient increasing cofinal $\overline{f}$, which exists by lemma 47. Let g be an eub of $\overline{f}$ and $B_{\overline{f}} = B_g = \{a \in A : g(a) = a\}$. Since we assume that $\overline{f}$ is constantly zero outside of B, we may assume that g is zero outside of B and therefore $B_g \subseteq B$.

Let now $C = B \setminus B_g = \{a \in B : g(a) \in a\}$. The sequence $\langle f_\alpha \restriction C : \alpha < \lambda \rangle$ is increasing cofinal in $\prod C/J_{<\lambda}$ and therefore $g \restriction C <_{J_{<\lambda}} f_\alpha$ for some $\alpha < \lambda$. Since $f_\alpha <_{J_{<\lambda}} g$ we conclude that $C \in J_{<\lambda}$.

Thus $B =_{J_{<\lambda}} B_g$.

To prove 2, observe that since $f^1_\alpha \le f^2_\alpha$ for all $\alpha < \lambda$ and $\overline{f}^1$ is cofinal $\prod B_{\overline{f}^1}/J_{<\lambda}$, also $\overline{f}^2 \restriction B_{\overline{f}^1}$ is cofinal in $\prod B_{\overline{f}^1}/J_{<\lambda}$, and therefore $B_{\overline{f}^1} \subseteq_{J_{<\lambda}} B_{\overline{f}^2}$. □

If a generator B_λ for $J_{<\lambda^+}$ over $J_{<\lambda}$ does exist, then $J_{<\lambda^+}/J_{<\lambda}$ is a boolean algebra, namely has a maximal element B_λ and is in particular κ-directed for all κ. The following lemma, which is needed to prove the pcf theorem, asserts a particular case of directedness: that $J_{<\lambda^+} \setminus J_{<\lambda}$ is $|A|^+$-directed. It says actually slightly more: that the relation $\le$ on humbly obedient sequences is $|A|^+$-directed.

Lemma 51. *Suppose that $\zeta(*) < |A|^+$.*

1. *If $\{\overline{f}^\zeta : z < \zeta(*)\}$ is a set of humbly obedient sequences, then there exists a humbly obedient sequence $\overline{f}$ so that $f_\zeta \le f$ for all $\zeta < \zeta(*)$.*
2. *If $\{B_\zeta : \zeta < \zeta(*)\} \subseteq J_{<\lambda^+}$ then there is a set $B \in J_{<\lambda^+}$ so that $B_\zeta \subseteq_{J_{<\lambda}} B$.*

Proof. Because every $B \in J_{<\lambda^+}$ is equivalent to $B_{\overline{f}}$ for some humbly obedient $\overline{f}$, and $\overline{f}_\zeta \le \overline{f}$ implies $B_{\overline{f}^\zeta} \subseteq_{J_{<\lambda}} B_{\overline{f}}$, and $B_{\overline{f}} \in J_{<\lambda^+} \setminus J_{<\lambda}$ for all humbly obedient $\overline{f}$, it is enough to prove the first item in the lemma.

Suppose that $\{\overline{f}^\zeta : \zeta < \zeta(*)\}$ are humbly obedient and $\zeta(*) < |A|^+$. Define by induction on $\alpha < \lambda$ an increasing in $<_{J_{<\lambda}}$, humbly obedient sequence $\langle f_\alpha : \alpha < \lambda \rangle$ such that $f^\zeta_\alpha \le_{J_{<\lambda}} f_\alpha$ for all $\zeta < \zeta(*)$, $\alpha < \lambda$. At the induction step you need to immitate the proof of 47 above, with the only change of taking an upper bound g_α over a union of $\zeta(*) \times \alpha$ many functions. □

The following lemma is the key to the proof of the pcf theorem:

Lemma 52. *If $\overline{f}^1 \le \overline{f}^2$ are humbly obedient sequences and $B := B_{\overline{f}^1} \in J_{<\lambda^+} \setminus J_{<\lambda}$ then there is a club $E \subseteq \lambda$ such that*

$$\delta \in S \cap E \Rightarrow f^1_\alpha \restriction B =_{J_{<\lambda}} f^2_\delta \restriction B \tag{3}$$

Proof. The sequence $\overline{f}^1$ is increasing cofinal in $J_{<\lambda}$ on B by the definition of B. Thus, for every $\alpha < \lambda$ there is some $\beta(\alpha) < \lambda$ such that $f^2_\alpha <_{J_{<\lambda}} f^1_\alpha$. Let $E \subseteq \lambda$ be a club closed under $\alpha \mapsto \beta(\alpha)$ (that is, $\delta \in E \wedge \alpha < \delta \Rightarrow \beta(\alpha) < \delta$). Suppose that $\delta \in E \cap S$. We show that $f^1_\delta \restriction B =_{J_{<\lambda}} f^2_\delta \restriction B$. We already know that $f^2_\delta \ge f^1_\delta$.

Conversely, let $C = \left\{a \in B : f^1_\delta(a) < f^2_\delta(a)\right\}$.

For every $a \in A$ we have $f^2_\delta(a) = \sup\left\{f^2_\alpha(a) : \alpha \in c_\delta\right\}$, and therefore for every $a \in C$ we can find an index $\alpha_a \in c_\delta$ such that $f^2_{\alpha_a}(a) > f^1_\delta(a)$. Such an index must exist because $f^2_\delta(a) = \sup\{f_\alpha(a) : a \in c_\delta\} > f^1_\delta(a)$. As $\mathrm{otp} c_\delta = |A|^+$, while $|C| \le |A|$ we can find $\gamma \in \mathrm{nacc} c_\delta$ which is greater than α_a for all $a \in C$. By coherence at γ it follows that $f^2_\gamma(a) > f^2_\alpha(a) > f^1_\delta(a)$ for all $a \in C$. But because $\delta \in E$ there is some $\beta < \delta$ for which $f^2_\gamma <_{J_{<\lambda}} f^1_\beta$. The set $C' := \{a \in A : f^2_\gamma(a) < f^1_\beta(a) < f^1_\delta(a)\}$ cannot meet C, since $a \in C' \cap c \Rightarrow f^2_\gamma(a) < f^1_\beta(a) < f^1_\delta(a) < f^2_\gamma(a)$, which is absurd. As $C' \cup A \setminus B$ is measure 1, it follows that $C \in J_{<\lambda}$ and $f^2_\delta \restriction =_{J<\lambda} f^1_\delta \restriction B$ v □

We have all the facts about humbly obedient needed to prove the pcf Theorem. Let us devote a few words to the description of what is going on in this proof, compared to the proof of the lub theorem (Theorem 23 above).

In the proof of the lub theorem we were able to mix the relations $\subseteq$, which appears locally, and $\subsetneq_I$, which occurs on a club, and thus obtain a contradiction from a sequence of lenght $|A|^+$ of subsets of A. That proof had a "one-dimensional" geometry.

This proof uses a similar idea, but is "two-dimensional", by which we mean that the increasing sequence of subsets of A of length $|A|^+$ is indexed by pairs of ordinals. The phenomenon that reflects on a club is not increasing in $\subseteq_{J_{<\lambda}}$, but equality modulo $J_{<\lambda}$

Theorem 53. *Suppose $A \subseteq Reg$ is infinite and $|A| < \min A$. Then for every $\lambda \in \operatorname{pcf} A$ there is a set $B_\lambda \subseteq A$ such that*

$$J_{<\lambda^+} = J_{<\lambda} + B_\lambda$$

Proof. Since for every $\lambda \in \operatorname{pcf} A$ for which $\lambda \cap A$ is finite the theorem does not say more than $\lambda^+ \cap A = \lambda \cap A \cup \{\lambda\}$, the Theorem is true for those λ.

Suppose, then, that $\lambda \in \operatorname{pcf} A \setminus |A|^{++}$ and fix a stationary $S \subseteq S^\lambda_{\kappa^+}$ in $I[\lambda]$ and a sequence $\overline{C}$ witnessing this. We shall find a generator for $J_{<\lambda^+}$ over $J_{<\lambda}$.

By induction on $\zeta < |A|^+$ we define humbly obedient sequences $\overline{f}^\zeta$ with $B_\zeta := B_{\overline{f}^\zeta}$ so that

- $\xi < \zeta < |A|^+ \Rightarrow \overline{f}^\xi \leq \overline{f}^\zeta$
- $\xi < \zeta < |A|^+ \Rightarrow B_\xi \subsetneq_{J_{<\lambda}} B_\zeta$

Suppose that $\zeta < |A|^+$ and that $\overline{f}^\xi$ and $B_\xi := B_{\overline{f}^\xi}$ are defined for all $\xi < \zeta$ and satisfy the conditions above. By $|A|^+$-directedness (Lemma 51 above) there is a set $B \in J_{<\lambda^+}$ such that $B_\xi \subseteq_{J_{<\lambda}} B$ for all $\xi < \zeta$. If B generates $J_{<\lambda^+}$, we are done. Else, there is a set B' so that $B \subsetneq_{J_{<\lambda}} B' \in J_{<\lambda^+}$. Use Lemma 50 to fix a humbly obedient sequence $\overline{f}$ so that $B' = B_{\overline{f}}$. Now let $\overline{f}_\zeta$ be provided by Lemma 51 so that $\overline{f} \leq \overline{f}^\zeta$ and $\overline{f}_\xi \leq \overline{f}^\zeta$ for all $\xi < \zeta$. Let $B_\zeta = B_{\overline{f}^\zeta}$ and it follows that $B_\xi \subsetneq_{J_{<\lambda}} B_\zeta$ for all $\xi < \zeta$.

Suppose to the contrary that $\overline{f}^\zeta$ and B_ζ are defined for all $\zeta < |A|^+$ and satisfy that $\overline{f}^\zeta$ is increasing in $\leq$ and B_ζ increasing in $\subsetneq_{J_{<\lambda}}$.

For every pair $\xi < \zeta < |A|^+$ there is, by Lemma 52, a club $E_{\xi,\zeta} \subseteq \lambda$ such that $\delta \in S \cap E_{\xi,\zeta} \Rightarrow f^\xi_\delta \restriction B_\xi =_{J_{<\lambda}} f^\zeta_\delta \restriction B_\xi$. Since $\overline{f}^\zeta$ is increasing cofinal on $B_\zeta \setminus B_\xi$ and $g_\xi \restriction B_\zeta \setminus B_\xi \in \prod B_\zeta \setminus B_\xi$ by definition of B_ξ, there is some $\alpha < \lambda$ for which $g_\xi \restriction (B_\zeta \setminus B_\xi) <_{J_{<\lambda}} f^\zeta_\alpha$. Thus, by subtracting α from $E_{\xi,\zeta}$ we may assume that $f^\xi_\delta \restriction B_\zeta \setminus B_\xi <_{J_{<\lambda}} f^\zeta_\delta$ for all $\delta < \lambda$ and $\zeta \in E$.

Let $E = \bigcap_{\xi<\zeta<|A|^+} E_{\xi,\zeta}$. Since $|A|^+ < \lambda$, the intersection E is a club of λ. Fix $\delta \in S \cap E$. For all $\xi < \zeta < |A|^+$ we have

$$f^\zeta_\delta \restriction B_\xi =_{J_{<\lambda}} f^\xi_\delta \restriction B_\xi \text{ and } f^\xi_\delta \restriction (B_\zeta \setminus B_\xi) <_{J_{<\lambda}} f^\zeta_\delta \restriction (B_\zeta \setminus B_\xi) \tag{4}$$

Picture?????

We do the last bit of the proof twice, for no real reason other than to emphasize the similarity to the proof of the lub theorem.

First run: For every $a \in A$,the sequence $\langle f^\zeta_\delta(a) : \zeta < |A|^+\rangle$ is increasing in $\leq$ because $\langle \overline{f}_\zeta : |z < |A|^+\rangle$ is increasing in $\leq$. Therefore, for every $a \in A$ there is a club $C_a \subseteq |A|^+$ on which $\langle f^\zeta_\delta(a) : \zeta < |A|^+\rangle$ is either constant or is strictly increasing (that is, increasing in $<$).

Let $C = \bigcap_{a\in A} C_a$. Since $|A| < |A|^+$, this intersection is a club of $|A|^+$.

Choose $\xi << \rho$ in C and $\zeta = \zeta(\xi) < \rho$. Since $\delta \in E \subseteq E_{\zeta,\rho}$ we know that $f^\zeta_\delta(a) = f^\rho_\delta(a)$ for all $a \in B_\zeta$ but a measure zero set. On the other hand, the set $\{\alpha \in B_\zeta \setminus B_\xi : f^\xi(a) < f^\zeta(a)\}$ is positive. That allows us to find $a \in B_\zeta \setminus B_\xi$ on which both relations occur: namely $f^\xi_\delta(a) < f^\zeta_\delta(a) = f^\rho_\delta(a)$. This is impossible, though, because $\xi, \zeta, \rho \in C \subseteq C_a$ and therefore $\xi < \zeta < \rho$ implies that either $f^\xi_\delta(a) = f^\zeta_\delta(a) = f^\rho_\delta(a)$ or $f^\xi_\delta(a) < f^\zeta_\delta(a) < f^\rho_\delta(a)$.

Second run: For every $\xi < \kappa$ define $X_{\zeta,\xi} : \left\{a \in A : f^\zeta_\delta(a) = f^\xi_\delta(a)\right\}$ for every $\zeta < |A|^+$. Since $\langle \overline{f}^\zeta_\delta : \zeta < |A|^+\rangle$ is increasing in $\leq$, the sequence $\langle X_{\zeta,\xi} : \zeta < |A|^+\rangle$ is decreasing in $\subseteq$ and stabilizes at some $\zeta(\xi)$. Let $C \subseteq |A|^+$ be a club of $|A|^+$ which is closed under $\xi \mapsto \zeta(\xi)$.

If $\xi < \zeta \leq \xi' < \zeta'$ are in C then $X_{\zeta,\xi} = X_{\zeta',\xi} \subseteq X_{\zeta',\xi'}$. The first equality is because $\zeta, \zeta' \in C$; the second inclusion is becaus $f^\xi_\delta \leq f^{\xi'}_\delta \leq f^{\zeta'}_\delta$. Thus, for all $\xi < \zeta \leq \xi' < \zeta'$ in C:

$$X_{\zeta',\xi'} \subseteq X_{\zeta,\xi} \tag{5}$$

On the other hand, if $\xi < \zeta \leq \xi' < \zeta'$ are in C then $B_{\xi'} \subseteq_{J_{<\lambda}} X_{\xi',\zeta'}$ by 4 and by the same also $B'_\xi \setminus B_\zeta \cap X_{\xi,\zeta} =_{J_{<\lambda}} \emptyset$. Hence:

$$X_{\zeta',\xi'} \neq_{J_{<\lambda}} X_{\zeta,\xi} \tag{6}$$

Combining 5 with 6 and denoting by $\xi' := \min\{C \setminus (\xi+1)\}$ we get that the sequence $\langle X_{\xi,\xi'} : \xi \in C\rangle$ is a strictly ioncreasing sequence of subsets of A of length $|A|^+$ - a contradiction. ☐

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