Solution to Mood Aleph

1. Pair the color classes.
Color class 1 is paired with 2,
" — " — " 3 — " — " — " 4,
⋮
" — " — " $\mathcal{L}-1$ " — " — " — " $\mathcal{L}$.
(Assume for simplicity that $\mathcal{L}$ is even, and actually is a power of 2.)

Vs of color 2 that are adjacent to a vx of color 1 are knocked out. In parallel, vs of color 4 that are adjacent to a vx of color 3 are knocked out, etc. As a result, after this first phase we have a vx set $\mathcal{U}$ s.t. for every $v \in V \setminus \mathcal{U}$ there exists a neighbor $u \in \mathcal{U}$ from v.
This vx set $\mathcal{U}$ contains vs of $\mathcal{L}/2$ types.
The first type is colors 1 and 2,
the second type is colors 3 and 4, etc.

We then pair these vx sets $U_1, U_2, \ldots, U_{L/2}$ and again pair these sets.
In parallel, vs of U_1 knock out their neighbors of U_2, vs of U_3 knock out their neighbors of U_4, etc.
We then pair the resulting sets again, up until there is just one set left.

Analysis

The alg' obviously requires $O(\log L)$ rounds.

In the first phase we have L sets $U_1, U_2, \ldots, U_L$.

In the second ~~sets~~ phase we have $L/2$ sets, ..., in the i'th phase we have $L/2^i$ sets.
We prove by induction on i, $i = 1, 2, \ldots, \log L$, that each of the sets $U_1, U_2, \ldots, U_{L/2^i}$ on phase i is a 2-separated $(i-1)$-ruling set for ~~a~~ certain subsets $V_1, V_2, \ldots, V_{L/2^i}$

of vs. Specifically, V_1 is the set of vs originally colored by the first 2^i colors; V_2 is the set of vs originally colored by colors $2^i+1, 2^i+2, \ldots, 2^{i+1}$; etc.

<u>Base:</u> On the first phase $V_j = U_j$ for each $j \in [\ell]$. Obviously, U_j is 0-ruling for V_j. It is 2-separated (i.e., each pair of vs in it are at dist' ≥ 2 one from another), bcs these vs were all colored by color j.

<u>Step:</u> For i, $1 \leq i < \log \ell$, sps that each set U_j is a 2-separated $(i-1)$-ruling set for V_j, where V_j is defined as above. Here $j \in \{1, 2, \ldots, \ell/2^i\}$.

Let U_{12} be the set formed ~~for~~ by the alg' out of sets U_1 and U_2 (by knocking out all vs of U_2 that are adjacent to a vx of U_1).

Let $V_{12} = V_1 \cup V_2$. We show that $\mathcal{U}_{12}$ is a 2-separated i-ruling set for V_{12}. The proof for other sets (e.g., $\mathcal{U}_{34}$, $\mathcal{U}_{56}$, ...) is analogous.

Consider two vs $u, u' \in \mathcal{U}_{12}$.

If $u, u' \in \mathcal{U}_1$ or $u, u' \in \mathcal{U}_2$, then since by induction hypothesis both $\mathcal{U}_1$ and $\mathcal{U}_2$ are 2-sep'd, it follows that $dist_G(u, u') \geq 2$.

If $u \in \mathcal{U}_1$, $u' \in \mathcal{U}_2$ then $u' \in \mathcal{U}_{12}$ implies that there is no vx $z \in \mathcal{U}_1$ neighboring to u'. Hence $dist_G(u, u') \geq 2$, i.e., $\mathcal{U}_{12}$ is a 2-separated set.

Consider a vx $v \in V_{12} = V_1 \cup V_2$.

If $v \in V_1$ then there exists $u \in \mathcal{U}_1 \subseteq \mathcal{U}_{12}$ s.t. $dist_G(u, v) \leq i-1 < i$, as desired.

Otherwise, $v \in V_2$. Then there exists $u_2 \in \mathcal{U}_2$ s.t. $dist_G(u_2, v) \leq i-1$.

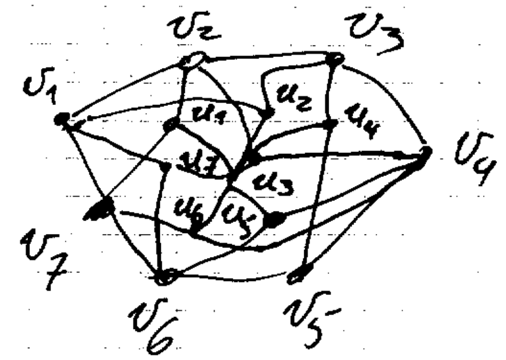
If $u_2 \in U_{12}$ then we are done.
Otherwise there exists a vx $u_1 \in U_1$ s.t. $(u_1, u_2) \in E(G)$. Hence

$$dist_G(u_1, v) \leq dist_G(u_1, u_2) + dist_G(u_2, v) \leq$$
$$\leq 1 + (i-1) = i.$$

QED

2. For each $i \in [\mathcal{L}]$, consider the set V_i of all vs colored by i.
This set induces a graph of maximum degree at most β.
Hence we can $(\beta+1)$-color each graph $G(V_i)$ in parallel in $O(\beta) + \log^* n$ time.
By combining these colorings into a single one we can get an $\mathcal{L}(\beta+1)$-coloring for the entire graph.
$\mathcal{L}(\beta+1) \leq \Delta + O(\sqrt{\Delta})$.
Within additional $O(\sqrt{\Delta})$ rounds the number of colors can be reduced to $(\Delta+1)$.
The overall time is $O(\beta + \sqrt{\Delta}) + \log^* n =$
$= O(\sqrt{\Delta}) + \log^* n$.

3)



$\{v_1, v_2, \ldots, v_7\}$ – the original vx set.

$\{u_1, u_2, \ldots, u_7, u\}$ – add'l new vs.

u is connected to $u_1, u_2, \ldots, u_7$, by a star. Each u_i is connected to v_{i-1} and v_{i+1}. (For $i = 1$, $i-1 = 7$, and for $i = 7$, $i+1 = 1$.)

The resulting graph has 15 vs. Its chromatic number is 4, and it is triangle-free.