Assignment 4 Solution

1. First, observe that a γ-fraction of the vertex set V has degree at most $2a$. Next, we analyze the value γ. Let β, $0 < \beta < 1$, denote the fraction of vertices that have degree *greater* than $2a$. (In other words $\beta = 1 - \gamma$.) It holds that

$$\beta \cdot |V| \cdot (2a+1) \leq \sum_{v \in V} deg(v) = 2 \cdot |E|.$$

Hence

$$\beta \quad \leq \quad \frac{2}{2a+1} \cdot \frac{|E|}{|V|}.$$

Recall that $a \geq \frac{|E|}{|V|}$. Therefore

$$\beta \quad \leq \quad \frac{2a}{2a+1}.$$

Therefore, at least $1/(2a+1)$-fraction of the vertices in V have degree at most $2a$. In addition, the arboricity of any subgraph of G is at most a. Hence we can partition V (by a distributed algorithm) into subsets $H_1, H_2, ..., H_\ell$. For any $v \in H_i$, $1 \leq i \leq \ell$, it holds that the number of neighbors of v_i in $\bigcup_{j=i}^{\ell} H_j$ is at most $2a$.

Next, we evaluate ℓ. It holds that

$$\ell = O(\log_{((2a+1)/2a)} n) = O(\frac{\log n}{\log(1+1/2a)}) = O(a \cdot \log n).$$

(since $\log(1+1/2a) = O(1/a)$.)

Next, color in parallel all subgraphs $G(H_i)$, for $i = 1, 2, ..., \ell$, using the colors $\{1, 2, ..., 2a+1\}$. ($G(H_i)$ is the graph induced by H_i. Also, recall that the maximum degree of $G(H_i)$, for any $i \in \{1, 2, ..., \ell\}$, is at most $2a$.) Finally, perform $\ell - 1$ phases of recoloring. In phase i, $1 \leq i \leq \ell - 1$, perform recoloring of the subgraph $G(H_{\ell-i})$ to achieve a legal $(2a+1)$-coloring in $\bigcup_{j=\ell-i}^{\ell} H_j$. When the last phase terminates, the entire graph G is $(2a+1)$-colored legally.

The running time of computing the partition is $O(a \cdot \log n)$. the running time of the colorings of $G(H_i)$ is $O(a)$. The running time of a recoloring phase is $O(a)$ as well. There are $\ell - 1 = O(a \cdot \log n)$ phases. Therefore, the total running time is: $O(a^2 \log n)$.

2. a) The arboricity of a tree is $a = 1$. In any tree T, at least 1/3 of the vertices have degree at most 2. (Otherwise, it holds that

$$2/3 \cdot |V| \cdot 3 < \sum_{v \in V} deg(v) = 2 \cdot |E| \leq 2 \cdot (|V| - 1),$$

i.e., $2 \cdot |V| < 2 \cdot |V| - 2$. This is a contradiction.)

Consequently, we can partition the tree into subsets $H_1, H_2, ..., H_\ell$, such that the degree of each subgraph induced by a set H_i, $i \in \{1, 2, ..., \ell\}$ is at most 2. It holds that $\ell = O(\log n)$. Next, the subgraphs $G(H_1), G(H_2), ..., G(H_\ell)$ are colored in parallel, with 3 colors for each subgraph (in time $O(\log^* n)$). Finally, $\ell - 1$ recoloring phases are performed to achieve a 3-coloring of the entire graph. The overall running time is $O(\ell) = O(\log n)$.

b) If the maximum degree Δ of the input tree T is smaller than $\log n / \log \log n$, then compute a $(\Delta + 1)$-coloring directly using the standard algorithm that runs in $O(\Delta + \log^* n)$ time.
Otherwise:

Compute an H-partition $H_1, H_2, ..., H_\ell$ such that the maximum degree of the subsets is $(\log n)^{1/4}$. (Instead of subgraphs of maximum degree 2 that were computed in part (a) of the question.) Consequently, the number of sets is

$$\ell = O(\log_{(\log n)^{1/4}} n) = O(\frac{\log n}{\log(\log n)^{1/4}}) = O(\frac{\log n}{\log \log n}).$$

Next, color each subgraph with $O((\log n)^{1/4})$ colors (in time $O((\log n)^{1/4})$). Denote the resulting coloring by φ. Next, compute an additional coloring ψ as follows:

All vertices $u \in H_\ell$ select $\psi(u) = 1$.

For rounds $i = 1, 2, ..., \ell - 1$,

each vertex u of $H_{\ell-i}$ selects a color $\psi(u)$ that is different from the ψ-colors of all its neighbors in $\bigcup_{j=\ell-i+1}^{\ell} H_j$. The selected color is from the range $1, 2, ..., (\log n)^{1/4} + 1$. (Recall that the degree of this H-partition is $(\log n)^{1/4}$.)

This completes the description of computing ψ. Next, a legal $(\sqrt{\log n})$-coloring of the input tree T is obtained by selecting the color
$\varphi(v) \cdot ((\log n)^{1/4} + 2) + \psi(v)$ for each vertex $v \in V$. (This color can be seen

as an ordered pair $\langle \varphi(v), \psi(v) \rangle$. You can verify that the resulting coloring is indeed a legal $O(\sqrt{\log n})$ coloring of the input tree.)

Since we deal with the case $\Delta \geq \log n / \log\log n$, the computed coloring is in particular a $(\Delta+1)$-coloring of T.

Running time analysis: $O(\ell) = O(\log n/\log\log n)$ time for computing the H partition, $O((\log n)^{1/4})$ time for computing φ, $O(\ell)$ time for computing ψ. Total running time: $O(\log n/\log\log n)$.

Remark: For graphs with large Δ, one can improve the running time to $O(\log n/\log\Delta + \log^* n)$, by computing an H-partition with degree $\Delta^{1/4}$, and size $\ell = O(\log n/\log\Delta)$. The coloring φ should be computed using the algorithm of Linial, resulting in $O(\Delta^{1/2})$-colorings of the subgraphs. (The running time of the algorithm of Linial is $O(\log^* n)$, and it computes an $O(\Delta^2)$-coloring of a graph with maximum degree Δ.) Next, the coloring ψ is computed in the same way as described above, resulting in an $O(\Delta^{1/4})$-coloring. Finally, ψ and φ are combined into a unified legal $O(\Delta^{3/4})$-coloring of T, which is in particular a $(\Delta+1)$-coloring.

3. Let all vertices with degree at most $12 \cdot a$ join the set H_1. The number of remaining vertices that do not join H_1 is at most $1/6 \cdot |V|$. Otherwise, there are at least $1/6 \cdot |V|$ vertices with degree $12 \cdot a + 1$. Consequently, it holds that

$$2 \cdot |E| = 1/6 \cdot |V| \cdot 12 \cdot |E|/|V| < 1/6 \cdot |V| \cdot (12 \cdot a + 1) \leq \sum_{v \in V} deg(v) = 2 \cdot |E|.$$

This is a contradiction.

The maximum degree of $G(H_1)$ is $12 \cdot a$. Next, color $G(H_1)$ legally with $(12a+1)$ colors in $O(a + \log^* n)$ time. Consequently, at least $5/6$-fraction of the vertices of the graph are colored.

Remark: it is possible to improve the number of colors by selecting a value a' smaller than $12 \cdot a$, but greater than $2a$, and computing a partition $H_1, H_2, ..., H_\ell$ with degree a'. For a sufficiently large constant ℓ, this partition includes at least $5/6$-fraction of the vertices of the graph. Then it can be colored with $O(a)$ colors in $O(\ell \cdot a + \log^* n) = O(a + \log^* n)$ time, in the same way as in the solutions to questions 1 and 2.

4. a) Observe that the subgraph G' of $P_{n,d}$ for which the ratio between number of edges and vertices is maximal is the graph $P_{n,d}$ itself. Therefore,

the arboricity is determined by the ratio between the number of edges and vertices of $P_{n,d}$. Also, observe that $degen(P_{n,d}) = d$. Therefore, the number of edges in $P_{n,d}$ is

$$|E| = d \cdot (n - 1 - d) + d + (d - 1) + (d - 2) + (d - 3) + ... + 2 + 1.$$

(This number is calculated as follows: We start with $P_{n,d}$, and gradually remove its vertices. Each time a vertex is removed, we remove its adjacent edges that haven't been removed yet, and count these edges. First, we remove the vertices $v_n, v_{n-1}, ..., v_{d+2}$. Each such vertex has d adjacent edges upon removal. Then, we remove the vertices v_i, for $i := d + 1, d - 1, ..., 1$. Each such vertex has $i - 1$ adjacent edges upon removal. Observe that the number of vertices in the first phase is $n - (d + 2) + 1 = n - 1 - d$. Consequently, the equation stated above follows.)

Therefore, it holds that

$$|E| = d \cdot (n - 1) - d^2 + \frac{(1 + d) \cdot d}{2} = d \cdot (n - 1) - \frac{d^2}{2} + \frac{d}{2}$$

Therefore,

$$a = \left\lceil \frac{|E|}{n - 1} \right\rceil = \left\lceil d - \frac{d^2}{2(n - 1)} + \frac{d}{2(n - 1)} \right\rceil.$$

Since $d > \alpha\sqrt{2}\sqrt{n}$, it holds that

$$a \leq d - \frac{2\alpha^2 \cdot n}{2(n - 1)} + \frac{\alpha\sqrt{2}\sqrt{n}}{2(n - 1)} \leq d - (a^2 - \frac{\alpha}{\sqrt{2}}).$$

b) Let G' be the subgraph of G in which the ratio between the number of edges and vertices is maximal. Denote $d = degen(G')$. it holds that $d \leq degen(G)$. Let n' be the number of vertices in G'. By the definition of degeneracy, in G' there are $(n' - 1 - d)$ vertices $v_{n'}, v_{n'-1}, ..., v_{d+2}$, such that $v_{n'}$ has degree at most d, and for any $i \in d + 3, d + 4, ..., n'$, after the vertices $v_{n'}, v_{n'-1}, ..., v_i$ are removed, v_{i-1} has degree at most d. Once $v_{n'}, v_{n'-1}, ..., v_{d+2}$ are removed, there are $d + 1$ vertices left, denoted by $v_{d+1}, v_d, ..., v_1$. For any $i \in 2, 3, ..., d + 2$, once $v_{d+2}, v_{d+1}, ..., v_i$ are removed, the vertex v_{i-1} has degree at most $i - 2$. Consequently it holds that

$$|E'| \leq d \cdot (n' - 1 - d) + d + (d - 1) + (d - 2) + (d - 3) + ... + 2 + 1.$$

The remainder of the proof is analogous to part a).