Solution 3

Question 1.

$$a_k = \log(a_{k-1}) + 1 \le \log(a_{k-1}) + 2 = \log(4 * a_{k-1})) \le$$
$$\le \log(4 * (\log(4 * a_{k-2})))$$
$$\le \log(4 * \log(4 * \log(4 * a_{k-3})))$$
$$\dots$$
$$\le \log(4 * \log(4 * \dots \log(4 * a_0)))$$
$$= \log 4 + \log(\log(4 * \dots \log(4 * a_0))$$
$$= \log 4 + \log(\log 4) + \log\log\log(4 * \cdots \log(4 * a_0))$$
$$= \log 4 + \log(\log 4) + \log(\log(\log 4)) + \dots logloglog \dots \log a_0$$
$$<= 2 + 1 + 0 +.. 0 + 2 <= 5$$

The last element in the sum is $\log^{(k)} a_0 = \log^{(\log^* n + O(1))} a_0 = \log^{(\log^* n + O(1))} n$.
According the definition of $\log^* n$, the last element is smaller than 2.
Thus, $a_k \le 5 < 6$ ■

Question 2.

Let's apply Cole-Vishkin on the given graph.

Number of colors will decrease as follows:
$2^{2^{100}} \to 2 * 2^{100} \to 2 * 101 \to 2 * 8 \to 2 * 4 \to 2 * 3 = 6$
Thus, after 5 iterations the number of colors will be equal to 6.

Additional two iteration of shift-down will reduce the number of colors to ~~5~~ 3. ■

Question 3.

Correctness:
At the beginning each vertex has its own color and first iteration is same as shown in class, therefore after first iteration the coloring is correct. Starting from the second iteration, each vertex v will go over his sub-colors and sub-colors of his neighbors. Let's assume that v has a neighbor u and it is a first neighbor of v. Let's show that $\chi_1'(v) \neq \chi_1'(u)$.

The index that v chose for $\chi_1'(u1)$ is $i_1(v)$ and the index that u chose for his first neighbor is $i_1(u)$
Thus, $\chi_1'(v) = < binrep(i_1(v), \chi_1(v)[i_1(v)] > and\ \chi_1'(u) = < binrep(i_1(u), \chi_1(u)[i_1(u)] >$

If $i_1(v) \neq i_1(u)$, then their binary representations are different and as a result $\chi_1'(v) \neq \chi_1'(u)$.
Otherwise, according to the way v chose index $i_1(v)$, $\chi_1(v)[index] \neq \chi_1(u)[index]$ and as a result $\chi_j'(v) \neq \chi_j'(u)$.

In both cases the first fields of $\chi'(v)$ and $\chi'(u)$ were different, therefore the whole words are different too. ■

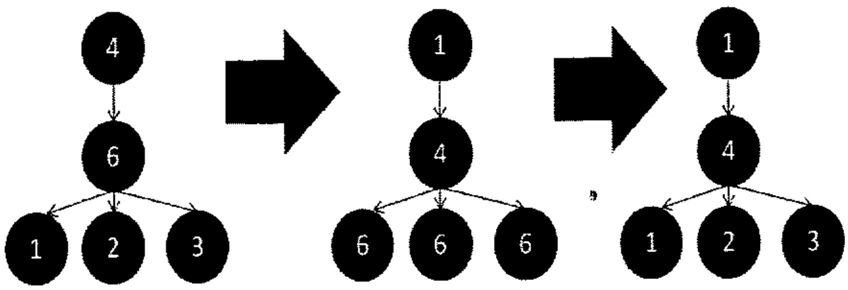
Run-time:
$\to \Delta \cdot \log(\Delta \cdot \log m)$
In the original algorithm, each iteration decreases size of the coloring from m to $\Delta * logm$, therefore after $\log^* m$ iterations we will get $\Delta * log\Delta$ bits for each vertex.
In the updated version, each iteration decreases size of the coloring from m to $\cancel{\Delta * (\log \frac{m}{\Delta})}$ $\Delta \cdot \log m$, therefore after $\log^* m$ iterations we will get $\Delta * 6$ bits for each vertex.
$\to \Delta \cdot \log\log m$ ..

Conclusion:
Updated algorithm converges faster than the original.

Question 4.

הסתכלו על החישוב שעשינו בשיעור לפתרון בכתה..

The algorithm is wrong. Assume the following graph G.



After one shift-down and color replacement the central vertex will remain without a color to choose from. ■