ASSIGNMENT 1 SOLUTIONS - DISTRIBUTED ALGORITHMS

(1) We will only concentrate on connected graphs (Note that for an unconnected graph G and for every vertex v in G, $Diam(G) = Rad(G, v) = \infty$).

(a) $\forall v \in V$, $Rad(G, v) = \max\{dist_G(v, u) \mid u \in V\} \leq \max\{dist_G(u, w) \mid u, w \in V\} = Diam(G) \leq^{1} \max\{dist_G(u, v) + dist_G(v, w) \mid u, w \in V\} = \max\{dist_G(u, v) \mid u \in V\} + \max\{dist_G(v, w) \mid w \in V\} = 2 \cdot Rad(G, v)$.

(b) As an example, let us observe the complete graph, K_n. Obviously, $Diam(K_n) = 1$, and $\forall v \in K_n$, $Rad(K_n, v) = 1$. Hence, $\forall v \in K_n$, $Diam(K_n) < 2 \cdot Rad(K, v)$.

(2) (a) We'll prove it by induction on the number of vertices in the graph $n = |V|$.

Proof. Basis: Each graph $G = (E, V)$ in which $|V| = n = 1$ is connected, regardless of the number of edges. Hence: $|E| \geq n - 1$.
Induction Step: Assume that for each connected graph $G = (E, V)$ in which $|V| < n$, $|E| \geq |V| - 1$.
Given a connected graph $G = (E, V)$ in which $|V| = n$. By removing a vertex $v \in V$ and $deg(v)$ edges: $\{(v, u) : u \in V, (u, v) \in E\}$ a set of graphs $G_1, G_2, ...G_k$ is created, where $k = deg(v)$. For each i, $G_i = (E_i, V_i)$ is connected and $|V_i| < n$, so the induction claim holds: $|E_i| \geq |V_i| - 1$.
Hence: $|E| = deg(v) + \sum_{i=1}^{deg(v)} |E_i| \geq deg(v) + \sum_{i=1}^{deg(v)} (|V_i| - 1) = deg(v) + \sum_{i=1}^{deg(v)} |V_i| - deg(v) = \sum_{i=1}^{deg(v)} |V_i| = |V - 1| = n - 1$ □

(b) We'll prove by induction on the number of vertices in the graph. (A reminder: a tree is a connected, acyclic graph).

Proof. Basis: A graph in which $|V| = 1$ and $|E| = 0$ is connected and contains no cycles.
Induction Step: Assume that every connected graph $G = (E, V)$ in which $|V| < n$ and $|E| = |V| - 1$ is a tree.
Given a connected graph $G = (E, V)$ in which $|E| = |V| - 1 = n - 1$.
Claim: There exists a vertex $v \in V$ for which $deg(v) = 1$.

Proof. Assume towards contradiction that $\forall v \in V : deg(v) \geq 2$. Hence: $|E| = \frac{1}{2} \sum_{v \in V} deg(v) \geq |V|$, in contradiction to $|E| = |V| - 1$. □

By choosing a vertex $v \in V$ for which $deg(v) = 1$, we can create the graph $G' = (E', V')$, where $E' = \{E - \{(v, w) : (v, w) \in E\}$ and $V' = V - \{v\}$.

[1] By triangle inequality

This graph is connected and holds $|E'| = |V'| - 1$, so the induction claim holds - G' is a tree (therefore contains no cycles). The graph G is composed of the graph G' plus the vertex v and the edge $\{(v,w) : (v,w) \in E\}$, hence is contains no cycles either. □

(3) We will prove by a complete induction on $Depth(T_v)$, that $\check{L}(v) = min\{dist(v,l_v) : l_v \in Leaves(T_v)\}$, as required.

Proof. Basis: $Depth(T_v) = 0$: v is a leaf ($Leaves(T_v) = v$) and thus, $\check{L}(v) := 0 = min\{dist(v,l_v) : l_v \in Leaves(T_v)\}$.

Induction Step: By induction hypothesis, the claim holds for all $u \in Ch(v)$.

$\check{L}(v) = 1 + min\{\check{L}(u) : u \in Ch(v)\} =$ (By induction hypothesis) $1 + min\{dist(u,l_u) : u \in Ch(v), l_u \in Leaves(T_u)\} = min\{dist(v,l) : u \in Ch(v), l \in Leaves(T_u)\} = min\{dist(v,l_v) : l_v \in Leaves(T_v)\}$. □

(4) Recall that $v \in T'$, i.e that v was never deleted.

(a) The statement $L_{T'}(v) \leq L_T(v)$ is correct. In fact, it can be strengthened as $L_{T'}(v) = L_T(v)$.

Proof. Indeed, $L_T(v) = dist(rt,v) = L_{T'}(v)$, as the (unique) path along which the distance is evaluated remains unchanged after the procedure applications (since v was not deleted). □

(b) The statement $\hat{L}_{T'}(v) \leq \hat{L}_T(v)$ is correct.

Proof. $\hat{L}_{T'}(v) := max\{dist_{T'}(v,l_v) : l_v \in Leaves(T'_v)\} =^2 max\{dist_T(v,l'_v) : l'_v \in Leaves(T'_v)\} \leq max\{dist_T(v,l_v) : l_v \in Leaves(T_v)\}$. □

(c) The statement $\check{L}_{T'}(v) \leq \check{L}_T(v)$ is incorrect, by the following counter example : Let $T = (V,E)$, $V = \{v, v_1, .., v_k, v_{k+1}\}$, $E = (v,v_1) \cup (v,v_{k+1}) \cup \bigcup_{i=1}^{k-1}(v_i,v_{i+1})$, where $k > 2$. Define $rt := v$, and let T' be the rooted tree T after removing v_{k+1}. Obviously, $\check{L}_T(v) = 1$, and $\check{L}_{T'}(v) = k$.

(5) (a) First, a spanning tree rooted at x, denoted by T_x, is constructed using the flooding algorithm taught in class. $Time(Flood) = \Theta(Diam(G))^3$, $Message(Flood) = \Theta(|E|)$.

Next, a convergecast on T_x is performed, during which a summing operation is performed, as follows: If v is a leaf in T_x, it immediately responds by sending Ack message to its parent with the content 1. If v is a non-leaf (intermediete) vertex in T_x, then it collects Ack's from all of its children and only then it sends an Ack message to its parent,

[2] Note that $Leaves(T'_v) \subseteq V(T_v)$

[3] Note that $\Theta(Diam(G)) = \Theta(|V|)$ unless otherwise specified

which is the sum of all Ack's of its children. We denote this process by Converge(+). $Time(Converge(+)) = Depth(T_x) = \Theta(Diam(G))$, $Message(Converge(+)) = n - 1$

Hence, the total message complexity is $\Theta(|E|)$, and the total time complexity is $\Theta(Diam(G))$.

(b) After performing the algorithm described in section (a), x knows the total number of vertices in the graph. Then, it broadcasts this number to all vertices in the graph using the constructed tree, T_x. Both message and time complexities of this stage exceeds the complexities of the algorithm described in section (a) only by a constant factor. Therefore, the total message complexity of the extended algorithm is $\Theta(|E|)$, and the time complexity is $\Theta(Diam(G))$.