

# Approximating Graph Metric by Spanning Trees (2005; Elkin, Emek, Spielman, Teng)

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**SYNONYMS:** Lower-Stretch Spanning Trees.

## 1 PROBLEM DEFINITION

Consider a weighted connected graph  $G = (V, E, \omega)$ , where  $\omega$  is a function from the edge set  $E$  of  $G$  into the set of positive reals. Given a spanning tree  $T$  of  $V$ , we define the distance in  $T$  between a pair of vertices  $u, v \in V$ ,  $dist_T(u, v)$ , to be the sum of the lengths of the edges on the unique path in  $T$  between  $u$  and  $v$ . We can then define the stretch of an edge  $(u, v) \in E$  to be

$$stretch_T(u, v) = \frac{dist_T(u, v)}{\omega(u, v)},$$

and the average stretch over all edges of  $E$  to be

$$avestr_T(E) = \frac{1}{|E|} \sum_{(u,v) \in E} stretch_T(u, v).$$

The average stretch of a graph  $G = (V, E, \omega)$  is defined as the smallest average stretch of a spanning tree  $T$  of  $G$ ,  $avestr_T(E)$ . The average stretch of a positive integer  $n$ ,  $avestr(n)$ , is the maximum average stretch of an  $n$ -vertex graph  $G$ . The problem is to analyze the asymptotic behavior of the function  $avestr(n)$ .

## 2 KEY RESULTS

The problem was introduced by Alon et al. [1], who showed that

$$\Omega(\log n) = avestr(n) = exp(O(\sqrt{\log n \cdot \log \log n})).$$

Elkin et al. [5] improved the upper bound and showed that

$$avestr(n) = O(\log^2 n \cdot \log \log n).$$

Dhamdhere et al. [4] further improved the upper bound of [5] to  $O(\log^2 n)$ .

## 3 APPLICATIONS

The main currently known application of low stretch spanning trees is for solving symmetric diagonally dominant linear systems of equations. Boman and Hendrickson [2] were the first to discover the surprising relationship between these two seemingly unrelated problems. They applied the spanning trees of [1] to design solvers that run in time  $m^{3/2} 2^{O(\sqrt{\log n \log \log n})} \log(1/\epsilon)$ . Spielman and

Teng [7] improved their results by showing how to use the spanning trees of [1] to solve diagonally-dominant linear systems in time

$$m2^{O(\sqrt{\log n \log \log n})} \log(1/\epsilon).$$

By applying the low-stretch spanning trees developed in [5], the time for solving these linear systems reduces to

$$m \log^{O(1)} n \log(1/\epsilon),$$

and to  $O(n(\log n \log \log n)^2 \log(1/\epsilon))$  when the systems are planar. Applying a recent reduction of Boman, Hendrickson and Vavasis [3], one obtains a  $O(n(\log n \log \log n)^2 \log(1/\epsilon))$  time algorithm for solving the linear systems that arise when applying the finite element method to solve two-dimensional elliptic partial differential equations.

## 4 OPEN PROBLEMS

The most evident open problem is to close the gap between the upper bound of  $O(\log^2 n)$  and the lower bound of  $\Omega(\log n)$  on  $avestr(n)$ . Another intriguing subject is the study of low-stretch spanning trees for various restricted families of graphs. Progress in this direction was recently achieved by Emek and Peleg [6].

## 5 CROSS REFERENCES

None is reported. Entry editors please feel free to add some.

## 6 RECOMMENDED READING

- [1] Noga Alon, Richard M. Karp, David Peleg, and Douglas West. A graph-theoretic game and its application to the  $k$ -server problem. *SIAM Journal on Computing*, 24(1):78–100, February 1995. Also available Technical Report TR-91-066, ICSI, Berkeley 1991.
- [2] Erik Boman and Bruce Hendrickson. On spanning tree preconditioners. Manuscript, Sandia National Lab., 2001.
- [3] Erik Boman, Bruce Hendrickson, and Stephen Vavasis. Solving elliptic finite element systems in near-linear time with support preconditioners. Manuscript, Sandia National Lab. and Cornell, <http://arXiv.org/abs/cs/0407022>.
- [4] Kedar Dhamdhere, Anupam Gupta and Harald Racke. Improved embeddings of graph metrics into random trees. In *Proceedings of Symp. on Discr. Algorithms, SODA'06*, pp. 61-69, 2006.
- [5] Michael Elkin, Yuval Emek, Daniel Spielman, and Shang-Hua Teng. Lower-Stretch Spanning Trees. In *Proceedings of ACM Symp. on Theory of Computing, STOC'05*, pp. 494-503, May 2005.
- [6] Yuval Emek and David Peleg. A tight upper bound on the probabilistic embedding of series-parallel graphs. In *Proceedings of Symp. on Discr. Algorithms, SODA'06*, pp. 1045-1053, 2006.
- [7] Daniel A. Spielman and Shang-Hua Teng. Nearly-linear time algorithms for graph partitioning, graph sparsification, and solving linear systems. In *Proceedings of the thirty-sixth annual ACM Symp. on Theory of Computing, STOC'04*, pages 81–90, 2004.