Representations of Lie Algebras and Systems on Lie Groups

Eli Shamovich

Ben Gurion University of the Negev

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Lie Algebra Operator Vessels

- We fix a finite dimensional real Lie algebra $\mathfrak{g}$ and a linear map $\rho\colon \mathfrak{g} \to \mathcal{L}(\mathcal{H})$, for a Hilbert space $\mathcal{H}$, such that $\frac{1}{i}\rho$ is a representation.
- A $\mathfrak{g}$-operator vessel is a tuple:

$$\mathfrak{V} = (\mathcal{H}, \mathcal{E}, \rho, \Phi, \sigma, \gamma, \gamma_*) .$$

- Here $\mathcal{E}$ is the auxiliary Hilbert space, $\Phi \in \mathcal{L}(\mathcal{H}, \mathcal{E})$, σ is a linear map $\sigma\colon \mathfrak{g} \to \mathcal{L}(\mathcal{E})$ and γ and γ_* are linear maps $\gamma, \gamma_*\colon \mathfrak{g} \wedge \mathfrak{g} \to \mathcal{L}(\mathcal{E})$.

Lie Algebra Operator Vessels

- Furthermore the tuple satisfies the following conditions:
  - $\forall X, Y \in \mathfrak{g}\colon \gamma(X \wedge Y) - \gamma(X \wedge Y)^* = i\sigma([X, Y])$,
  - $\forall X, Y \in \mathfrak{g}\colon \gamma_*(X \wedge Y) - \gamma_*(X \wedge Y)^* = i\sigma([X, Y])$,
  - The colligation condition: $\forall X \in \mathfrak{g}\colon \frac{1}{i}(\rho(X) - \rho(X)^*) = \Phi^*\sigma\Phi$,
  - The input condition: $\forall X, Y \in \mathfrak{g}\colon \sigma(X)\Phi\rho(Y)^* - \sigma(Y)\Phi\rho(X)^* = \gamma(X \wedge Y)\Phi$.
  - The output condition: $\forall X, Y \in \mathfrak{g}\colon \sigma(X)\Phi\rho(Y) - \sigma(Y)\Phi\rho(X) = \gamma_*(X \wedge Y)\Phi$,
  - The linkage condition: $\forall X, Y \in \mathfrak{g}\colon \gamma_*(X \wedge Y) - \gamma(X \wedge Y) = i(\sigma(X)\Phi\Phi^*\sigma(Y) - \sigma(Y)\Phi\Phi^*\sigma(X))$.

Lie Algebra Operator Vessels

- Let us consider a simple example. If $\mathfrak{g} = \mathbb{R}^n$. Denote by $X_1, \ldots X_n$ the canonical basis. Set $A_j = \rho(X_j)$ for every $j = 1, \ldots, n$.
- Since $[X, Y] = 0$ for every $X, Y \in \mathfrak{g}$, from the first two vessel conditions we get that $\gamma(X) = \gamma(X)^*$ and $\gamma_*(X) = \gamma_*(X)^*$, for every $X \in \mathfrak{g}$.
- We thus recover a commutative n-operator vessel as defined in [Livsic et al, 1995]

Lie Algebra Operator Vessels

- The $ax + b$ is the only (up to isomorphism) non-commutative Lie group of the dimension 2.
- The $ax + b$ group can be realised as a subgroup of $GL_2(\mathbb{R})$, of matrices of the form:

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, a, b \in \mathbb{R}, a > 0$$

- The $ax + b$ associated Lie algebra can be realized as a subalgebra of $M_2(\mathbb{R})$ of the form:

$$\begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}, p, q \in \mathbb{R},$$

Lie Algebra Operator Vessels

- The fields $X_1 = a\dfrac{\partial}{\partial a}$ and $X_2 = a\dfrac{\partial}{\partial b}$ span the algebra.
- Let us make the identification $A_j = \rho(X_j)$, $\sigma_j = \sigma(X_j)$, $\gamma = \gamma(X_1 \wedge X_2)$ and $\gamma_* = \gamma_*(X_1 \wedge X_2)$.
- Then we have $[A_1, A_2] = \frac{1}{i} A_2$ and $\gamma - \gamma_* = i\sigma_2$.
- We will carry this example on and use the same notations.

Lie Algebra Operator Vessels

- Every Lie algebra representation, $\tau\colon \mathfrak{g} \to \mathcal{L}(\mathcal{H})$, can be embedded in a Lie algebra operator vessel.
- Set $\rho = i\tau$, $\mathcal{E} = \overline{\sum_{X \in \mathfrak{g}} (\rho(X) - \rho(X)^*)\mathcal{H}}$ and $\Phi = P_{\mathcal{E}}$, the orthogonal projection onto $\mathcal{E}$.
- We then define:
  - $\sigma(X) = \frac{1}{i}(\rho(X) - \rho(X)^*)|_{\mathcal{E}}$,
  - $\gamma(X \wedge Y) = \frac{1}{i}((\rho(X)\rho(Y)^* - \rho(Y)\rho(X)^*) - \rho([X, Y])^*)|_{\mathcal{E}}$,
  - $\gamma_*(X \wedge Y) = \frac{1}{i}((\rho(Y)^*\rho(X) - \rho(X)^*\rho(Y)) + \rho([X, Y]))|_{\mathcal{E}}$.
- Note that in the case $\mathfrak{g} = \mathbb{R}^n$ we recover the construction of a vessel for an n-tuple of commuting operators.

Lie Algebra Operator Vessels

- Let $\mathfrak{G}$ be a connected, simply connected Lie group associated to $\mathfrak{g}$.
- The vessel defines a linear, time invariant, overdetermined system on $\mathfrak{G}$:

$$\begin{cases} iXf + \rho(X)f = \Phi^*\sigma(X)u, \\ y = u - i\Phi f \end{cases}$$

- Here $X \in \mathfrak{g}$ and $u, y \colon \mathfrak{G} \to \mathcal{E}$ and $f \colon \mathfrak{G} \to \mathcal{H}$ are smooth functions. We call u the input of the system, y the output and f the state.

Lie Algebra Operator Vessels

- Since the system is overdetermined, it admits a set of compatibility conditions in the input:

$$\forall X, Y \in \mathfrak{g}\colon \sigma(Y)Xu - \sigma(X)Yu + i\gamma(X \wedge Y)u = 0.$$

- One can show that if $u\colon \mathfrak{G} \to \mathcal{E}$ is a function satisfying the compatibility equations, then there exists $f\colon \mathfrak{G} \to \mathcal{H}$ that solves the system.
- Furthermore the output thus defined satisfies the output compatibility equations:

$$\forall X, Y \in \mathfrak{g}\colon \sigma(Y)Xy - \sigma(X)Yy + i\gamma_*(X \wedge Y)y = 0.$$

Lie Algebra Operator Vessels

- Let $\mathfrak{g} = \mathbb{R}^2$. Then $\mathfrak{G} = \mathbb{R}^2$ as well.
- Let $\sigma_1 = \sigma(X_1)$, $\sigma_2 = \sigma(X_2)$, $\gamma = \gamma(X_1 \wedge X_2)$ and $\gamma_* = \gamma_*(X_1 \wedge X_2)$.
- The system equations are:

$$\begin{cases} i\dfrac{\partial f}{\partial t_j} + A_j f = \Phi^* \sigma_j u, \\ y = u - i\Phi f \end{cases}$$

Lie Algebra Operator Vessels

- The input compatibility equation is then:

$$\sigma_2 \frac{\partial u}{\partial t_1} - \sigma_1 \frac{\partial u}{\partial t_2} + i\gamma u = 0.$$

- The output compatibility equation is:

$$\sigma_2 \frac{\partial u}{\partial t_1} - \sigma_1 \frac{\partial u}{\partial t_2} + i\gamma_* u = 0.$$

Lie Algebra Operator Vessels

- The $ax + b$ system of equations has the following form:

$$
\begin{aligned}
ia\frac{\partial f}{\partial a} + A_1 f &= \Phi^* \sigma_1 u, \\
ia\frac{\partial f}{\partial b} + A_2 f &= \Phi^* \sigma_2 u, \\
y &= u - i\Phi f.
\end{aligned}
$$

- The input compatibility equations are:

$$
a\sigma_2 \frac{\partial u}{\partial a} - a\sigma_1 \frac{\partial u}{\partial b} + i\gamma u = 0.
$$

Lie Group Representations

- Let $\mathfrak{G}$ be a Lie group countable at infinity. A continuous representation of $\mathfrak{G}$ on a Hilbert space $\mathcal{H}$ is a map $\pi\colon \mathfrak{G} \to \mathcal{L}(\mathcal{H})$, that satisfies:
  - π is a group homomorphism,
  - For every $\xi \in \mathcal{H}$, the function on $\mathfrak{G}$ defined by $x \mapsto \pi(x)\xi$ is continuous.
- A representation π is called unitary if $\pi(x)$ is a unitary in $\mathcal{L}(\mathcal{H})$, for every $x \in \mathfrak{G}$.
- A representation is called irreducible if there are no non-trivial π-invariant closed subspaces of $\mathcal{H}$. In other words if $V \subseteq \mathcal{H}$ is a closed subspace, then $\pi(\mathfrak{G})V \subseteq V$ implies that either $V = 0$ or $V = \mathcal{H}$.

Lie Group Representations

- A vector $\xi \in \mathcal{H}$ is called smooth (for π) if the map $x \mapsto \pi(x)\xi$ is smooth on $\mathfrak{G}$.
- The space of all smooth vectors will be denoted by $\mathcal{H}_\infty$. This space is also called the Garding space of π.
- The Garding lemma tells us that $\mathcal{H}_\infty$ is dense in $\mathcal{H}$ and is π invariant.
- Topologize $\mathcal{H}_\infty$ using the map $\xi \mapsto \pi(\cdot)\xi$. This map identifies the Garding space as a closed subspace of $C^\infty(\mathfrak{G}, \mathcal{H})$. This defines on $\mathcal{H}_\infty$ a Frechet space topology, which is generally finer then the topology induced from $\mathcal{H}$.

Lie Group Representations

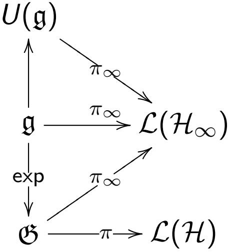
- Thus we get a representation of $\mathfrak{G}$ on the Frechet space $\mathcal{H}_\infty$. We will denote this representation by π_∞.
- Let $\mathfrak{g}$ be the Lie algebra of $\mathfrak{G}$. Let $\mathfrak{g}_\mathbb{C}$ be the complexification of $\mathfrak{g}$ and $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$.
- For every $\xi \in \mathcal{H}_\infty$ we have:

$$\forall \xi \in \mathcal{H}_\infty : \pi_\infty(X)\xi = \frac{d}{dt}|_{t=0}\pi(\exp(tX))\xi.$$

- We can then extend π_∞ to a representation of $U(\mathfrak{g})$ on $\mathcal{H}_\infty$. We will denote this representation by π_∞ as well.

Lie Group Representations

A picture to describe what we have done so far:



Nuclear Spaces

- Let E, F be locally convex Hausdorff topological vector space.
- A continuous linear mapping $T\colon E \to F$ is called nuclear if there exist an equicontinuous sequence $\{x'_k\} \subset E'$, a sequence $\{y_k\} \subset B \subset F$, with B complete and a sequence $\{\lambda_k\} \subset \mathbb{C}$, such that $\sum_k |\lambda_k| < \infty$ and for every $x \in E$, we have:

$$T(x) = \sum_k \lambda_k \langle x'_k, x\rangle y_k.$$

- A space E is called nuclear if every continuous mapping $T\colon E \to B$ for any Banach space B is nuclear.

Nuclear Spaces

- If E is nuclear, then for every other locally convex Hausdorff topological vector space F the canonical map $E \otimes_\pi F \to E \otimes_\epsilon F$ is an isomorphism.
- This implies that essentially there is a unique way to define the tensor product of a locally convex Hausdorff TVS with a nuclear space.
- We are concerned with Frechet spaces and Banach spaces mostly. We have the following facts about them:
  - A Frechet space E is nuclear if and only if its dual is.
  - A normable space (in particular a Banach one) is nuclear if and only if it is finite dimensional.

Nuclear Spaces

- The abstract Schwartz Kernel Theorem states:

Theorem

Let E and F be a locally convex complete Hausdorff topological space. If in addition E is Frechet and nuclear then $\mathcal{L}_b(E, F)$ is complete and we have:

$$E' \widehat{\otimes} F \cong \mathcal{L}_b(E, F)$$

Frequency Domain Theory

- Recall that in the commutative case, if we plug $e^{\langle \lambda, t \rangle} u_0$, for $\lambda \in \mathbb{C}^2$, into the compatibility condition, we obtain an algebraic equation:

$$(\lambda_1 \sigma_2 - \lambda_2 \sigma_1 + i\gamma) u_0 = 0.$$

- One can also obtain this using the Fourier Transform of the function u.
- There is an analogue of the Fourier transform in the Lie group case.

Frequency Domain Theory

- We will now proceed to construct frequency domain theory for Lie algebra operator vessels.
- Let us assume that $\mathfrak{g}$ is the Lie algebra of a second countable, countable at infinity Lie group $\mathfrak{G}$ of type I.
- We will denote by $\widehat{\mathfrak{G}}$ the unitary dual of $\mathfrak{G}$. The collection of all equivalence classes of unitary irreducible representations.
- The idea is that $\pi \in \widehat{\mathfrak{G}}$ will be the frequency.

Frequency Domain Theory

- Applying the Fourier transform to a function f on $\mathfrak{G}$ we obtain an element of $\mathcal{L}(\mathcal{H}_\pi, \mathcal{H}_\pi \otimes \mathcal{E})$.
- For the following discussion to work we will restrict ourselves to the π-invariant subspace of smooth vectors, namely $\mathcal{H}_{\pi,\infty}$.
- We assume that $\mathcal{H}_{\pi,\infty}$ is nuclear and obtain from the Schwartz kernel theorem that:

$$\mathcal{L}(\mathcal{H}_{\pi,\infty}, \mathcal{H}_{\pi,\infty} \otimes \mathcal{E}) \cong \mathcal{H}'_{\pi,\infty} \otimes \mathcal{H}_{\pi,\infty} \otimes \mathcal{E}$$

Frequency Domain Theory

- The trajectories of the system are of the form:

$$u(g) = (\pi_\infty(g) \otimes I_{\mathcal{E}})\mathbf{u}_0$$
$$x(g) = (\pi_\infty(g) \otimes I_{\mathcal{H}})\mathbf{x}_0$$
$$y(g) = (\pi_\infty(g) \otimes I_{\mathcal{E}})\mathbf{y}_0$$

- Here $\mathbf{u}_0, \mathbf{y}_0 \in \mathcal{H}'_{\pi,\infty} \otimes \mathcal{H}_{\pi,\infty} \otimes \mathcal{E}$ and $\mathbf{x}_0 \in \mathcal{H}'_{\pi,\infty} \otimes \mathcal{H}_{\pi,\infty} \otimes \mathcal{H}$.
- The frequency domain system is modified accordingly:

$$iX_k\mathbf{x} + [I_{\mathcal{H}_{\pi,\infty}} \otimes A_k]\mathbf{x} = [I_{\mathcal{H}_{\pi,\infty}} \otimes \Phi^*\sigma_k]\mathbf{u}$$
$$\mathbf{y} = \mathbf{u} - i[I_{\mathcal{H}_{\pi,\infty}} \otimes \Phi]\mathbf{x}$$

Frequency Domain Theory

- The i/o compatibility conditions then become:

$$(\pi_\infty(X_k) \otimes \sigma_j - \pi_\infty(X_j) \otimes \sigma_k + iI_{\mathcal{H}_{\pi,\infty}} \otimes \gamma_{jk})\mathbf{u} = 0$$
$$(\pi_\infty(X_k) \otimes \sigma_j - \pi_\infty(X_j) \otimes \sigma_k + iI_{\mathcal{H}_{\pi,\infty}} \otimes \gamma_{*jk})\mathbf{y} = 0$$

- Note that the operators $\pi_\infty(X_k)$ are continuous on $\mathcal{H}_{\pi,\infty}$.
- Using the Fourier transform, one can translate between the frequency domain and time domain systems of a Lie algebra operator vessel.

Frequency Domain Theory

- We then define the joint transfer function of the vessel by:

  $$S(\pi) = I_{\mathcal{H}_{\pi,\infty}} \otimes I_{\mathcal{E}} - i \left(I_{\mathcal{H}_{\pi,\infty}} \otimes \Phi \right) R(X, \pi) \left(I_{\mathcal{H}_{\pi,\infty}} \otimes \Phi^* \sigma(X) \right).$$

  Where $R(X, \pi) = \left(i\pi_\infty(X) \otimes I_{\mathcal{H}} + I_{\mathcal{H}_{\pi,\infty}} \otimes \rho(X) \right)^{-1}$.
- We denote the space of solution of the input compatibility conditions by $\boldsymbol{\mathcal{E}}(\pi) \subset \mathcal{L}(\mathcal{H}_{\pi,\infty}, \mathcal{H}_{\pi,\infty} \otimes \mathcal{E})$. Similarly we denote by $\boldsymbol{\mathcal{E}}_*(\pi)$ the space of solutions of the output compatibility conditions.
- $S(\pi)$ maps the space $\boldsymbol{\mathcal{E}}(\pi)$ into $\boldsymbol{\mathcal{E}}_*(\pi)$.

Frequency Domain Theory

- The next example is the $ax + b$ group. This is the smallest non-commutative group. It is the group of affine transformations of the plane.
- The Plancherel measure on the dual is the counting measure on two points representing the only two classes of infinite-dimensional irreducible representations, π_+, a representation on $L^2(\mathbb{R}_{>0}, ds/s)$, and π_-, a representation on $L^2(\mathbb{R}_{<0}, ds/s)$.
- Then using the Schwartz Kernel Theorem again we obtain that the input compatibility conditions are differential equations for vector-valued distributions on $\mathbb{R}^2$.

Frequency Domain

- After a complexification process we obtain the following PDEs:

$$\frac{1}{2}\sigma_2 K(z,w) + z\sigma_2 \frac{\partial}{\partial z} K(z,w) - 2\pi i z \sigma_1 K(z,w) + i\gamma K(z,w) = 0$$
$$\frac{1}{2}\sigma_2 K(z,w) + z\sigma_2 \frac{\partial}{\partial z} K(z,w) - 2\pi i z \sigma_1 K(z,w) + i\gamma_* K(z,w) = 0$$

- The joint transfer function is then:

$$S(\pi) = I_{\mathcal{E}} - i\Phi(A_2 - 2\pi t I_{\mathcal{H}})^{-1}\Phi^* \sigma_2$$

This is an isomonodromic mapping that maps solutions of the above PDE's one to another:

Taylor Joint Spectrum

- We will describe the spectrum of a Lie algebra representation following [Taylor, 1972].
- Let $\mathfrak{g}$ be a Lie algebra of dimension n, $\mathfrak{g}_{\mathbb{C}}$ its complexification and $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$.
- Every representation of $\mathfrak{g}$ on a locally convex space X can be extended to a representation of $\mathfrak{g}_{\mathbb{C}}$ and $U(\mathfrak{g})$ on X. Thus one gets a structure of a left $U(\mathfrak{g})$ module on X.
- On the other hand every left $U(\mathfrak{g})$ module defines a representation of $\mathfrak{g}$.

Taylor Joint Spectrum

- Consider the augmented Koszul complex, that is a free resolution of $U(\mathfrak{g})$ as a topological bimodule over itself:

$$0 \longleftarrow U(\mathfrak{g}) \longleftarrow U(\mathfrak{g})\widehat{\otimes}U(\mathfrak{g}) \longleftarrow$$

$$\longleftarrow U(\mathfrak{g})\widehat{\otimes}U(\mathfrak{g}) \otimes \mathfrak{g} \longleftarrow U(\mathfrak{g})\widehat{\otimes}U(\mathfrak{g}) \otimes \wedge^2\mathfrak{g} \longleftarrow$$

$$\longleftarrow \cdots \longleftarrow U(\mathfrak{g})\widehat{\otimes}U(\mathfrak{g}) \otimes \wedge^n\mathfrak{g}$$

Taylor Joint Spectrum

- Let M be a $U(\mathfrak{g})$ topological bimodule. One can then compute the Hochschild homology of M, $H_k(U(\mathfrak{g}), M)$, by tensoring M (in the category of $U(\mathfrak{g})$ topological bimodules) with the above Koszul complex.
- Now we define the resolvent set of $\mathfrak{g}$ to be the topological $U(\mathfrak{g})$-bimodules, such that the $H_k(U(\mathfrak{g}), M)$ vanish for all $k > 0$.
- How is this related to Lie algebra operator vessels?

Taylor Joint Spectrum

- Let $\mathfrak{G}$ be a Lie group as in the previous section. Let $\pi \in \widehat{\mathfrak{G}}$, be an irreducible unitary representation of $\mathfrak{G}$ on $\mathcal{H}_\pi$. Finally let $\mathcal{H}_{\pi,\infty}$ be the space of smooth vectors of π.
- Let ρ be a representation of $\mathfrak{g}$ on a Hilbert space $\mathcal{H}$. it then extends uniquely to a representation of $\mathfrak{g}_\mathbb{C}$ on $\mathcal{H}$, that we will also call ρ. Consider $M_\pi = \mathcal{H}_{\pi,\infty} \otimes \mathcal{H}$ as a $U(\mathfrak{g})$ bimodule with the obvious action.
- **Question:** Assume that the imaginary part of $\rho(X)$ is compact, for every $X \in \mathfrak{g}$. Is it true then that the resolvent in the definition of the joint transfer function of the vessel defined by ρ exists if and only if M_π is in the resolvent set of $U(\mathfrak{g})$?

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Thank You!