Stochastic Integration for non-Martingales Stationary Increment Processes

Multi-color noise approach

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Outline

1. Introduction
   - Motivation
   - Fractional Brownian Motion
2. Main Result
   - Stochastic Processes Induced by Operators
   - The m-Noise Space and the Process B_m
   - The $\mathcal{S}_m$ Transform
   - Stochastic Integration with respect to B_m
3. Applications
   - Optimal Control

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Stochastic Processes and Colored noises

- Stochastic stationary noises with a non-white spectrum arises in application.
- Consider the stochastic differential equation

$$dX_t = G(X_t, t)\, dt + F(X_t, t)\, dB(t).$$

- If B is a Brownian motion, the notion of Itô integral can be used so the differential dB can be viewed as a stochastic process with a white spectrum.
- Such notion does not exists in general if we replace B by a general stationary increment Gaussian process.
- The aim of this talk is to give meaning to this notation by extending Itô's integration theory.

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Fractional Brownian Motion

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$$COV(t,s) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} + |t-s|^{2H}\right), \quad t,s \in \mathbb{R}.$$

  For $H \neq \frac{1}{2}$ it is not a semi-martingale.
- Stochastic calculus for fractional Brownian (fBm) has attracted much attention in the last two decades, especially due to apparent application in economics.
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Fractional Brownian Motion
Spectral Properties

- We have the following relation:

$$\frac{1}{2}\left(|t|^{2H}+|s|^{2H}+|t-s|^{2H}\right)=\int_{-\infty}^{\infty}\widehat{\mathbf{1}_{[0,t]}}\widehat{\mathbf{1}_{[0,s]}}^{*}m(\xi)d\xi,$$

  where
  - $\mathbf{1}_{[0,t]}$ is the indicator function of the interval $[0,t]$
  - $\hat{f}=\int_{-\infty}^{\infty}e^{-iu\xi}f(u)du$
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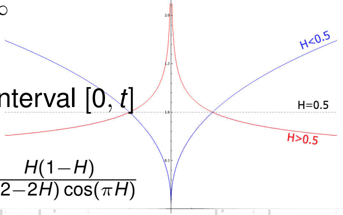

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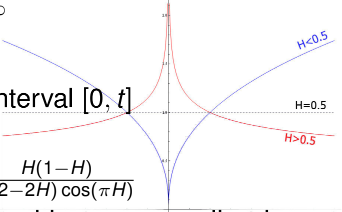

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Fractional Brownian Motion

Member of a Wide Family

- It suggests the the fBm is a member of a wide family of stationary increments Gaussian processes whose covariance function is of the form

$$COV_m(t,s) = \int_{-\infty}^{\infty} \widehat{\mathbf{1}_{[0,t]}} \widehat{\mathbf{1}_{[0,s]}}^{*} m(\xi) d\xi \qquad (1)$$

for a function $m(\xi)$ satisfies $\int_{-\infty}^{\infty} \frac{m(\xi)}{1+\xi^2} d\xi < \infty$.

Main Goal of this Talk

Extend the Itô integral for Brownian motion to this family of non-martingales stationary increments processes.

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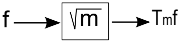

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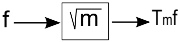

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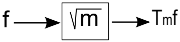

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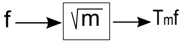

- This operator is in general unbounded.
- $\mathbf{1}_{[0,t]} \in dom T_m$ for each $t \geq 0$.
- The covariance function (1) can now be rewritten as

$$COV_m(t,s) = \int_{-\infty}^{\infty} \widehat{\mathbf{1}_{[0,t]}}\,\widehat{\mathbf{1}_{[0,s]}}^{*} m(\xi) d\xi = \left(T_m \mathbf{1}_{[0,t]}, T_m \mathbf{1}_{[0,s]} \right)_{L_2(\mathbb{R})}.$$

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- Application to optimal control theory.

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The m-Noise Space
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- $\mathscr{S}$ - Schwartz space of real rapidly decreasing functions.
- Ω is the dual of $\mathscr{S}$, the space of tempered distributions.
- $\mathcal{B}(\Omega)$ is the Borel σ-algebra.
- $\langle \omega, s \rangle = \langle \omega, s \rangle_{\Omega,\mathscr{S}}$, $s \in \mathscr{S}$ and $\omega \in \Omega$ will denote the bilinear pairing between $\mathscr{S}$ and Ω.

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Lemma

[Jorgensen] T_m as an operator from $\mathscr{S} \subset L_2(\mathbb{R})$, endowed with the Frèchet topology, into $L_2(\mathbb{R})$ is continuous.

Definition of the Probability Space

Bochner-Minlos Theorem

- It follows that $C_m(s) = e^{-\frac{1}{2}\|T_m s\|^2_{L_2(\mathbb{R})}}$ is a characteristic functional on $\mathscr{S}$.

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By the Bochner-Minlos theorem there is a unique probability measure P_m on Ω such that for all $s \in \mathscr{S}$,

$$C_m(s) = \exp\left\{-\frac{1}{2}\|T_m s\|^2_{L_2(\mathbb{R})}\right\} = \int_\Omega e^{i\langle\omega,s\rangle} dP_m(\omega) = \mathbb{E}\left[e^{i\langle\cdot,s\rangle}\right]$$

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- $\langle\omega, s\rangle$ is viewed as a random variable on Ω.
- The triplet $(\Omega, \mathcal{B}(\Omega), P_m)$ will be called the m-noise space.
- The case $T_m = id_{L_2(\mathbb{R})}$ $(m \equiv 1)$ will lead back to Hida's white noise space.

The Process B_m
Definition

- $\langle\omega, s\rangle$, $s \in \mathscr{S}$, is a zero mean Gaussian random variable with variance

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- The last relation can be extended to any $f \in dom(T_m)$, such that $\langle \omega, f \rangle$, $f \in dom(T_m)$ define a zero mean Gaussian random variable with variance

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$$\mathbb{E}\left[\langle \cdot, f \rangle^2\right] = \| T_m f \|^2_{L_2(\mathbb{R})}.$$

- For $t \geq 0$ we may define the stochastic process $B_m : \Omega \times [0, \infty] \longrightarrow \mathbb{R}$ by
$$B_m(t) := B_m(\omega, t) := \langle \omega, \mathbf{1}_{[0,t]} \rangle.$$

The Process B_m

Properties

- The process $\{B_m\}_{t\geq 0}$ is a zero mean Gaussian process with covariance function $\mathbb{E}\left[B_m(t)B_m(s)\right] = \left(T_m\mathbf{1}_{[0,t]}, T_m\mathbf{1}_{[0,s]}\right)_{L_2(\mathbb{R})}$.

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- $\frac{d}{dt}B_m$ (in the sense of distribution) has spectral density $m(\xi)$.
- In view of the previous isometry, it is natural to define for $f \in dom(T_m)$,

$$\int_0^t f(u)dB_m(u) = \langle \omega, \mathbf{1}_{[0,t]}f\rangle, \quad t \geq 0.$$

The Process B_m
Examples

Example (Standard Brownian Motion)

Take $m \equiv 1$, then $T_m = id_{L_2(\mathbb{R})}$ and

$$\mathbb{E}[B_m(t)B_m(s)] = \left(T_m \mathbf{1}_{[0,t]}, T_m \mathbf{1}_{[0,s]}\right) = \int_{-\infty}^{\infty} \mathbf{1}_{[0,t]} \mathbf{1}_{[0,s]}^* du = t \wedge s.$$

Example (Fractional Brownian Motion)

Take $m(\xi) = M(H)|\xi|^{1-2H}$, then

$$\mathbb{E}[B_m(t)B_m(s)] = \int_{-\infty}^{\infty} \widehat{\mathbf{1}_{[0,t]}} \widehat{\mathbf{1}_{[0,s]}}^* m(\xi) d\xi = \frac{|t|^{2H} + |s|^{2H} - |t-s|^{2H}}{2}.$$

Outline

An $\mathcal{S}$-Transform Approach for Stochastic Integration
Motivation

- We wish to define a Wick-Itô-Skorohod stochastic integral based on the process $\{B_m\}_{t\geq 0}$.
- Recall that the Itô-Hitsuda integral in the white noise space is defined by

$$\int_0^\Delta X(t)dB(t) \triangleq \int_0^\Delta X(t)\diamond \frac{d}{dt}B_m(t)dt,$$

where
  - $\{X(t)\}_{0\geq t\Delta}$ is a stochastic process
  - $\frac{d}{dt}B_m(t)$ is the time derivative(in the sense of distributions) of the Brownian motion.
  - $\diamond$ is the Wick product.
- We need a Wiener-Itô Chaos decomposition of the white noise space.

An $\mathcal{S}$-Transform Approach for Stochastic Integration
Motivation

Any $X \in L_2(\Omega, \mathcal{B}, P_m)$ can be represented as

$$X = \sum_{\alpha} f_\alpha H_\alpha(\omega).$$

Such basis for $L_2(\Omega, \mathcal{B}(\mathscr{S}'), P_m)$ depends explicitly on $m(\xi)$.

In order to keep our construction as general as possible, we take an $\mathcal{S}$-transform approach for the Wick-Itô-Skhorhod integral.

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Lemma

$$(\mathcal{S}_m B_m(t))(s) = \left(T_m s, T_m \mathbf{1}_{[0,t]}\right)_{L_2(\mathbb{R})}$$

is everywhere differentiable with respect to t.

Outline

Definition of the Stochastic Integral

Definition

A stochastic process $X(t) : [0, \Delta] \longrightarrow L_2(\Omega, \mathcal{G}, P_m)$ will be called Wick-Itô integrable if there exists a random variable $\Phi \in L_2(\Omega, \mathcal{G}, P_m)$ such that

$$(\mathcal{S}_m\Phi)(s) = \int_0^\Delta (\mathcal{S}_m X(t))(s) \frac{d}{dt} (\mathcal{S}_m B_m(t))(s) dt.$$

In that case we define $\Phi(\Delta) = \int_0^\Delta X(t) dB_m(t)$.

Definition of the Stochastic Integral

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- For any polynomial $p \in \mathbb{R}[X]$, $p(B_m(t))$ is integrable.

- The Wick product of $X, Y \in L_2(\Omega, \mathcal{G}, P_m)$ can be defined by

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- If B_m is the fractoinal Brownian motion ($m(\xi) = |\xi|^{1-2H}$), our definition of the stochastic integral reduces to the one given in [Bender2003] which coincides with the Wick-Itô-Skorokhod integral defined in [Duncan,Hu 2000] and [Hu,Øksendal 2003].

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- where $f \in domT_m$ and $t \geq 0$, such that $\| T_m \mathbf{1}_{[0,t]} f \|^2$ is absolutely continuous in t.
- $F \in C^{1,2}([0,t], \mathbb{R})$ with $\frac{\partial}{\partial t} F(X_t)$, $\frac{\partial}{\partial x} F(X_t)$, $\frac{\partial^2}{\partial x^2} F(X_t)$ all in $L_1(\Omega \times [0,t])$.

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- The following holds in $L_2(\Omega, \mathcal{G}, P_T)$:

$$F(t,X_t) - F(0,0) = \int_0^t f(u)\frac{\partial}{\partial x}F(u,X(u))\,\mathrm{d}B_m(u)$$

$$+ \int_0^t \frac{\partial}{\partial u}F(u,X(u))\mathrm{d}u + \frac{1}{2}\int_0^t \frac{d}{du}\|T_m\mathbf{1}_{[0,u]}f\|^2\,\frac{\partial^2}{\partial x^2}F(u,X(u))\mathrm{d}u$$

Outline

Formulation of the Optimal Control Problem

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- Consider the scalar system subject to

$$\begin{cases} \mathrm{d}x_t = \left(A_t \mathrm{d}t + C_t \mathrm{d}B_m(t)\right) x_t + F_t u_t \mathrm{d}t \\ x_0 \in \mathbb{R} \quad \text{(deterministic)} \end{cases}$$

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- Using Itô's formula, one may verify that

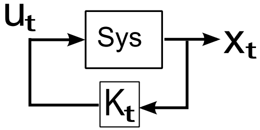
$$x_\Delta = x_0 \exp\left\{ \int_0^\Delta \left(A_t + F_t u_t\right) dt + \int_0^\Delta C_t dB_m(t) - \frac{1}{2} \| T_m \mathbf{1}_{[0,\Delta]} \|^2 \right\}$$

Formulation of the Optimal Control Problem
continue

- We present a quadratic cost functional

$$J\left(x_0, u_{(\cdot)}\right) := \mathbb{E}\left[\int_0^{\Delta}\left(Q_t x_t^2 + R_t u_t^2\right) dt + G x_{\Delta}^2\right].$$

where $R_{(\cdot)}, Q_{(\cdot)} : [0, \Delta] \to \mathbb{R}$, $R_t > 0$, $Q_t \geq 0\ \forall t \geq 0$ and $G \geq 0$.



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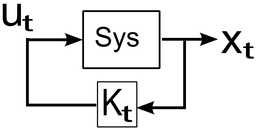
- We reduce ourselves to control signals of linear feedback type:



$$u_t = K_t \cdot x_t.$$

so the control dynamics reduces to

$$\begin{cases} \mathrm{d}x_t = \left[(A_t + F_t K_t)\,\mathrm{d}t + C_t \mathrm{d}B_m(t)\right] x_t \\ x_0 \in \mathbb{R} \quad \text{(deterministic)} \end{cases}$$

Formulation of the Optimal Control Problem
continue

- And the cost may be associated directly with the feedback gain $K_t : [0, \Delta] \longrightarrow \mathbb{R}$:

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Minimize the cost functional (2), for each given x_0, over the set of all linear feedback controls $K_{(\cdot)} : [0, \Delta] \longrightarrow \mathbb{R}$.

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- This control problem was formulated and solved in the case of fractional Brwonian motion by Hu and Yu Zhou 2005, and appears in [Biagini,Hu,Øksendal,Zhang 2008].

Solution
Riccati Equation

Theorem

If $\frac{d}{dt}\|T_m\mathbf{1}_{[0,t]}C_{(\cdot)}\|^2$ *is bounded in* $(0,\Delta)$*, then the optimal linear feedback gain* $\tilde{K}_t$ *is given by*

$$\tilde{K}_t = -\frac{F_t}{R_t}p_t. \tag{3}$$

where $\{p_t, t \in [0,\Delta]\}$ *is the unique positive solution of the Riccati equation*

$$\begin{cases} \dot{p}_t + 2p_t\left[A_t + \frac{d}{dt}\|T_m\mathbf{1}_{[0,t]}C_{(\cdot)}\|^2\right] + Q_t - \frac{F_t^2}{R_t}p_t^2 = 0 \\ p_\Delta = G \end{cases} \tag{4}$$

Solution
Idea of Proof

Proof.

Using Itô's formula with:

$x_t = x_0 \exp\left[\int_0^t c_u dB_m(u) + \int_0^t (A_u + F_u K_u)\, du - \frac{1}{2}\| T_m(\mathbf{1}_t C)\|^2\right]$, leads to

$$p_\Delta x_\Delta^2 = p_0 x_0^2 + 2\int_0^\Delta x_t^2 C_t p_t dBm(t)$$
$$+ \int_0^\Delta x_t^2 \left[\dot{p}_t + 2p_t(A_t + F_t K_t) + 2p_t \frac{d}{dt}\| T_m \mathbf{1}_t\|^2\right] dt.$$

Taking the expectation of both sides and substituting the Riccati equation (4) yields

$J(x_0, K(\cdot)) = p_0 x_0^2 + \mathbb{E}\int_0^\Delta \left(K_t + \frac{B_t}{R_t} p_t\right)^2 dt,$

of which the result follows. □

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  $\frac{A^2}{C^2} = SNR$, $x_0 = 5$, $F = 0.3$
  in the state-space model which results in

$$\begin{cases} dx_t = \left(A + \frac{1}{2}0.3K_t\right)x_t dt + x_t C dB_m(t), & \left(SNR = \frac{A^2}{C^2}\right) \\ x_0 = 5. \end{cases}$$

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- We take B_m that corresponds to the spectral density

$$m(\xi) = \alpha |\xi|^{1-2H} + \beta sinc^2 \left(\Delta(\xi - 2\pi f_0)\right),$$

with $\Delta = 20$, $f_0 = 2$, $H = 0.6$, $\alpha = 0.05$ and $\beta = 80$.

Simulation

$m(\xi) = \alpha|\xi|^{0.6} + \beta sinc^2\left(\Delta(\xi - 2\pi f_0)\right)$

We compare the cost function

$$J_{(x_0, K_{(\cdot)})} = \mathbb{E}\left[\int_0^\Delta (1 + 2K_t)\, x_t^2 dt + 2x_\Delta^2\right],$$

for the two controllers $K_{Opt}(\cdot)$ and $K_{Nai}(\cdot)$ and their corresponding state-space trajectories. Where:

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- $K_{Opt}(\cdot)$ is the optimal controller from Theorem 7 for a system perturbated by dB_m.
- $K_{Nai}(\cdot)$ is the optimal controller designed for a system perturbated by the time derivative of a Brownian motion, so it assumes $m(\xi) \equiv 1$.

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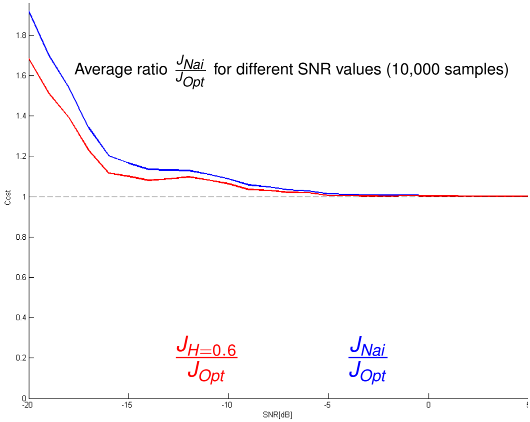

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- We have formulated and solved a stochastic optimal control problem in this new setting.

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