Operator Algebras Associated with Unitary Commutation Relations

Baruch Solel (jointly with S.C. Power)

Technion - Israel Institute of Technology

Function theory and operator theory: Infinite dimensional and free settings
BGU June 2011

Background: Non commutative disc algebras

Write E for $\mathbb{C}^n$ and consider the (full) Fock space of E:

$$\mathcal{F}(E) = \mathbb{C} \oplus E \oplus E^{\otimes 2} \oplus E^{\otimes 3} \oplus \cdots .$$

For $e \in E$, write L_e for the operator on $\mathcal{F}(E)$ defined by

$$L_e(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_k) = e \otimes \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_k$$

and $L_e c = ce$ for $c \in \mathbb{C}$.
Write $\mathcal{A}_n$ for the norm-closed algebra generated by all these operators. Equivalently, it is generated by the n shifts $L_{e_1}, L_{e_2}, \ldots, L_{e_n}$ where $\{e_1, \ldots, e_n\}$ is the standard basis for E.
This is Popescu's **non commutative disc algebra**.

The representations of $\mathcal{A}_n$

Note that $(L_{e_1}, \ldots, L_{e_n})$ is a row contraction (a contraction from $\mathcal{F}(E)^{(n)}$ to $\mathcal{F}(E)$; equivalently $\sum L_{e_i} L_{e_i}^* \leq I$). Thus $\mathcal{A}_n$ is **generated by a row contraction**.

Given a completely contractive representation π of $\mathcal{A}_n$ on H, if we write $T_i = \pi(L_{e_i})$ then $(T_1, \ldots, T_n)$ is a row contraction.

Conversely, given a row contraction in $B(H)$, there is a completely contractive representation π of $\mathcal{A}_n$ such that $T_i = \pi(L_{e_i})$ for all i.

These there facts above mean: $\mathcal{A}_n$ is the (unique) operator algebra **universal for row contractions**.

Classification

Theorem: $\mathcal{A}_n$ and $\mathcal{A}_m$ are (isometrically) isomorphic if and only if $m = n$.

Reason: The character space (one-dimensional representations) of $\mathcal{A}_n$ is $\overline{\mathbb{B}}_n$. If $\Phi : \mathcal{A}_n \to \mathcal{A}_m$ is an isomorphism $\alpha \mapsto \alpha \circ \Phi$ induces an homeomorphism from $\overline{\mathbb{B}}_m$ onto $\overline{\mathbb{B}}_n$. Thus $m = n$.

The Current Setup

A **product system** (of finite dimensional Hilbert spaces) over a semigroup S is a collection $\{H(s) : s \in S\}$ of f.d. Hilbert spaces together with isomorphisms $\{U_{s,t} : H(s) \otimes H(t) \to H(s+t)\}$ and these isomorphisms behave "associatively".

If $S = \mathbb{N}$ and we write E for $H(1)$, we get $H(m) = E^{\otimes m}$

Now: $S = \mathbb{N}^2$ and we write $H(1,0) = E$ and $H(0,1) = F$. Then $H(1,1) = U_{(0,1),(1,0)}(E \otimes F) = U_{(1,0),(0,1)}(F \otimes E)$ and $U := U^*_{(1,0),(0,1)} U_{(0,1),(1,0)}$ is an isomorphism of $E \otimes F$ and $F \otimes E$.

It is not hard to check that, from U we can reconstruct the product system (**with** $H(k,l) = E^{\otimes k} \otimes F^{\otimes l}$.)

Fixing the basis $\{e_i\}$ for E and $\{f_j\}$ for F, U is represented by an $nm \times nm$ unitary matrix u. We write

$$e_i \otimes f_j = \sum u_{(i,j),(k,l)} f_l \otimes e_k .$$

The product system is defined by (n, m, u).

Write

$$\mathcal{F}(n,m,u) = \sum_{k,l} H(k,l)$$

or, concretely,

$$\mathcal{F}(n,m,u) = \sum_{k,l} E^{\otimes k} \otimes F^{\otimes l}.$$

On this Fock space define "shifts":

Rightwards : $L_{e_i}\eta = e_i \otimes \eta$, $1 \leq i \leq n$.
Upwards : $L_{f_j}\eta = f_j \otimes \eta$, $1 \leq j \leq m$.

We get

$$L_{e_i} L_{f_j} = \sum_{k,l} u_{(i,j),(k,l)} L_{f_l} L_{e_k}. \qquad (1)$$

The unital semigroup generated by $\{I, L_e, L_f \ : \ e \in E, \ f \in F\}$ will be denoted $\mathbb{F}_u^+$ and the algebra it generates denoted $\mathbb{C}[\mathbb{F}_u^+]$. The norm closure of $\mathbb{C}[\mathbb{F}_u^+]$ will be written $\mathcal{A}_u$ and its closure in the weak* operator topology will be written $\mathcal{L}_u$.

The algebras

Remark: If u is a permutation matrix, the algebra $\mathcal{A}_u$ is the (non selfadjoint) algebra associated with a finite graph of rank 2 and a single vertex.

Representations

Every c.c. representation π of $\mathcal{A}_u$ gives rise to two row contractions $\tilde{T} = (T_1, \ldots, T_n)$ and $\tilde{S} = (S_1, \ldots, S_n)$ $(T_i = \pi(L_{e_i}),\ S_j = \pi(L_{f_j}))$ satisfying

$$T_i S_j = \sum_{k,l} u_{(i,j),(k,l)} S_l T_k.$$

But this is not sufficient. (Davidson et al.).

Define, for two u-commuting row contractions $\tilde{T}$, $\tilde{S}$ and $0 < s < 1$,

$$\Delta_s(T,S) = I - s^2(\tilde{T}\tilde{T}^* + \tilde{S}\tilde{S}^*) + s^4 \tilde{T}(I_{\mathbb{C}^n} \otimes \tilde{S}\tilde{S}^*)\tilde{T}^*$$

(viewing $\tilde{T}$ and $\tilde{S}$ as maps from $\mathbb{C}^n \otimes H$ to H).

Theorem

(Popescu, Skalski) Let $\tilde{T}$, $\tilde{S}$ be two u-commuting row contractions. If there is some $\rho \in (0,1)$ such that for all $s \in (\rho, 1)$, $\Delta_s(T,S) \geq 0$ then there is a (unique) c.c. representation π of $\mathcal{A}_u$ such that $\pi(L_{e_i}) = T_i$ and $\pi(L_{f_j}) = S_j$. In fact, $\mathcal{A}_u$ is "universal" for such pairs of row contractions (i.e.: It is generated by such a pair and every such pair defines a c.c. representation).

Corollary: The character space of $\mathcal{A}_u$ can be identified with

$$\Omega_u := \{(z,w) \in \overline{\mathbb{B}}_n \times \overline{\mathbb{B}}_m : z_i w_j = \sum_{k,l} u_{(i,j),(k,l)} z_l w_k\}.$$

$(\Delta_s(z,w) = I - s^2(\|z\|^2 + \|w\|^2) + s^4\|z\|^2\|w\|^2 =$
$(1 - s^2\|z\|^2)(1 - s^2\|w\|^2).)$

Also: The character space of $\mathcal{L}_u$ can be identified with

$$\{(z,w) \in \mathbb{B}_n \times \mathbb{B}_m : z_i w_j = \sum_{k,l} u_{(i,j),(k,l)} z_l w_k\}.$$

Classification

Notation:

$$V_u := \{(z,w) \in \mathbb{C}^n \times \mathbb{C}^m : z_i w_j = \sum_{k,l} u_{(i,j),(k,l)} z_l w_k\}.$$

So that $\Omega_u = V_u \cap \overline{\mathbb{B}}_n \times \overline{\mathbb{B}}_m$.

The classification problem :

Let u be an $nm \times nm$ unitary matrix and v be an $n'm' \times n'm'$ unitary matrix. They give rise to $\mathcal{A}_u$ and $\mathcal{A}_v$.

The Aim: When are $\mathcal{A}_u$ and $\mathcal{A}_v$ isomorphic and what the isomorphisms $\Phi : \mathcal{A}_u \to \mathcal{A}_v$ look like?

The homeomorphism induced on the character space

Given an isomorphism Ψ of $\mathcal{A}_u$ onto $\mathcal{A}_v$, it defines a map from the character space of $\mathcal{A}_u$ to the character space of $\mathcal{A}_v$ (by $\phi \mapsto \phi \circ \Psi^{-1}$) and we get a map $\theta_\Psi : \Omega_u \to \Omega_v$. Clearly, θ_Ψ is a homeomorphism and, since $\Omega_u \cap (\mathbb{B}_n \times \mathbb{B}_m)$ is the interior of Ω_u, θ_Ψ maps $\Omega_u \cap (\mathbb{B}_n \times \mathbb{B}_m)$ onto $\Omega_v \cap (\mathbb{B}_n \times \mathbb{B}_m)$.

If Ψ is an isomorphism of $\mathcal{L}_u$ into $\mathcal{L}_v$ which is a homeomorphism with respect to the w^*-topologies, θ_Ψ is a homeomorphism of $\Omega_u \cap (\mathbb{B}_n \times \mathbb{B}_m)$ onto $\Omega_u \cap (\mathbb{B}_n \times \mathbb{B}_m)$.

An important role will be played by the **core**.

The core

For every (i,j), write $u_{(i,j)}$ for the $n \times m$ matrix whose k,l-entry is $u_{(i,j),(k,l)}$. Thus, the (i,j) row of u provides the n rows of $u_{(i,j)}$. Also write $E_{i,j}$ for the $n \times m$ matrix whose i,j-entry is 1 and all other entries are 0 and $C_{(i,j)}$ for the matrix $u_{(i,j)} - E_{i,j}$.

Lemma

With $C_{(i,j)}$ defined as above, we have

$$V_u = \{(z,w) \in \mathbb{C}^n \times \mathbb{C}^m \ : \ \forall i,j, \quad \langle C_{(i,j)} w, \bar{z} \rangle = 0 \}.$$

Definition

We define the core of Ω_u to be the set

$$\Omega_u^0 := \{(z,w) \in \bar{\mathbb{B}}_n \times \bar{\mathbb{B}}_m \ : \ \forall i,j, \ \ C_{(i,j)} w = 0, \ C_{(i,j)}^t z = 0 \}.$$

Characterization of the core :

Theorem

For $(z, w) \in \mathbb{B}_n \times \mathbb{B}_m$ *the following conditions are equivalent.*

(1) $(z, w) \in \Omega_u^0$.

(2) *There exists a completely isometric automorphism* $\Theta_{z,w}$ *of* $\mathcal{L}_u$ *that is a homeomorphism with respect to the* w^**-topologies and restricts to an automorphism of* $\mathcal{A}_u$*, such that* $\theta_{\Theta_{z,w}}(0,0) = (z, w)$.

(3) *There exists an algebraic automorphism* Ψ *of* $\mathcal{A}_u$ *such that* $\theta_\Psi(0,0) = (z, w)$.

The automorphism $\Theta_{z,w}$ can be written explicitly using Voiculescu's analysis of the automorphisms of $\mathcal{E}_n$.

About the proof of (3) $\Rightarrow$ (1)

It uses:

Lemma

A point $(z, w) \in \Omega_u$ lies in the core Ω_u^0 if and only if every $(\lambda, \mu) \in \mathbb{C}^n \times \mathbb{C}^m$ defines a homomorphism $\rho : \mathbb{C}[\mathbb{F}_u^+] \to T_2$ such that

$$\rho(L_{e_i}) = \begin{pmatrix} z_i & \lambda_i \\ 0 & z_i \end{pmatrix}$$

and

$$\rho(L_{f_j}) = \begin{pmatrix} w_j & \mu_j \\ 0 & w_j \end{pmatrix}$$

for all i, j.

Write $\rho_{z,w,\lambda,\mu}$ for it.

Given Ψ and (z, w) as in (3), for every $(\lambda, \mu) \in \mathbb{C}^n \times \mathbb{C}^m$, $\rho_{0,0,\lambda,\mu} \circ \Psi^{-1}$ is a homomorphism on $\mathbb{C}[\mathbb{F}_u^+]$ and, thus, it is of the form $\rho_{z,w,\lambda',\mu'}$ for some (unique) $(\lambda', \mu') = \psi(\lambda, \mu)$.

If (z, w) is not in Ω_u^0, then the set of all (λ, μ) for which there is $\rho_{z,w,\lambda,\mu}$ is a subspace of $\mathbb{C}^n \times \mathbb{C}^m$ of dimension strictly smaller than $n + m$ and, as is shown above, it contains the continuous image (under the injective map ψ) of $\mathbb{C}^n \times \mathbb{C}^m$. This is impossible.

About the proof of (1) $\Rightarrow$ (2)
We need to construct $\Theta_{z,w}$ s.t. $\theta_{\Theta_{z,w}}(0,0) = (z,w)$; that is

$$\alpha_{(0,0)}(\Theta_{z,w}^{-1}(X)) = \alpha_{(z,w)}(X).$$

It follows from:
Proposition Suppose $(z,w) \in \Omega_u^0 \cap (\mathbb{B}_n \times \mathbb{B}_m)$. Then there is a automorphism $\tilde{\Theta}_z$ of $\mathcal{A}_u$ that is unitarily implemented and such that, for every $X \in \mathcal{A}_u$,

$$\alpha_{(0,w)}(\tilde{\Theta}_z^{-1}(X)) = \alpha_{(z,w)}(X)$$

where $\alpha_{(z,w)}$ is the character associated with (z,w).

To construct $\tilde{\Theta}_z$, we first use Voiculescu's analysis.

Following Voiculescu, we have, associated with every $z \in \mathbb{B}_n$, an automorphism, denoted Θ_z of the Cuntz-Toeplitz algebra $\mathcal{E}_n$. It is defined by

$$\Theta_z(L_\zeta) = (x_0 I - L_\eta)^{-1}(L_{X_1\zeta} - \langle \zeta, \eta \rangle I) \tag{2}$$

where $L_\zeta = \sum \zeta_i L_i$ (where $\zeta \in \mathbb{C}^n$ and $\{L_i\}$ are the generators of $\mathcal{E}_n$) and where X_1, x_0 and η are associated with z as follows:

(i) $x_0 = (1 - \|z\|^2)^{-1/2}$,

(ii) $\eta = x_0 \bar{z}$, and

(iii) $X_1 = (I_{\mathbb{C}^n} + \eta\eta^*)^{1/2}$.

In fact, this automorphism is unitarily implemented.

Here, we also have

Lemma

Suppose $(z, w) \in \Omega_u^0 \cap (\mathbb{B}_n \times \mathbb{B}_m)$. *Let* $\Theta := \Theta_z$. *Then, for every* $1 \le i \le n$ *and* $1 \le j \le m$,

$$\Theta(L_{e_i})L_{f_j} = \sum_{k,l} u_{(i,j),(k,l)} L_{f_l} \Theta(L_{e_k}). \tag{3}$$

Let U be the unitary operator implementing Θ. We can view $\mathcal{F}(n, m, u)$ as the sum

$$\mathcal{F}(n, m, u) = \sum_k F^{\otimes k} \otimes \mathcal{F}(E)$$

where $\mathcal{F}(E) = \mathbb{C} \oplus E \oplus (E \otimes E) \oplus \cdots$. We now let V be the unitary operator whose restriction to $F^{\otimes k} \otimes \mathcal{F}(E)$ is $I_k \otimes U$ (where I_k is the identity operator on $F^{\otimes k}$). It is easy to check that, for every f_j,

$$V L_{f_j} V^* = L_{f_j}.$$

Now, fix i. We can show, by induction, that, for every k and every $\xi \in F^{\otimes k} \otimes \mathcal{F}(E)$,

$$(I_k \otimes U)L_{e_i}\xi = \Theta(L_{e_i})(I_k \otimes U)\xi. \tag{4}$$

To prove it, we use the lemma above.
It now follows that the map $\tilde{\Theta}_z : X \to VXV^*$ defines a unitary endomorphism of $\mathcal{A}_u$. This automorphism satisfies the conditions of the previous proposition.

Definition

(i) An isomorphism $\Psi : \mathcal{A}_u \to \mathcal{A}_v$ is graded if it maps $span\{L_{e_i}, L_{f_j}\}$ into itself.

(ii) An isomorphism $\Psi : \mathcal{A}_u \to \mathcal{A}_v$ is said to be bigraded isomorphism if there are unitary matrices A $(n \times n)$ and B $(m \times m)$ such that

$$\Psi(L_{e_i}) = \sum_j a_{i,j} L_{e_j} \ , \quad \Psi(L_{f_k}) = \sum_l b_{k,l} L_{f_l} .$$

(iii) If $m = n$ and Ψ is a graded isomorphism such that

$$\Psi(L_{e_i}) = \sum_j a_{i,j} L_{f_j} \ , \quad \Psi(L_{f_k}) = \sum_l b_{k,l} L_{e_l}$$

for $n \times n$ unitary matrices A and B then we say that Ψ is a graded generator exchange isomorphism.

Notation:

We shall write $\Psi_{A,B}$ for the bigraded isomorphism (as in (ii)) and $\tilde{\Psi}_{A,B}$ for the graded generator exchange isomorphism.

* Both $\Psi_{A,B}$ and $\tilde{\Psi}_{A,B}$ are unitarily implemented.

Lemma

(i) *If $\Psi_{A,B}$ is a bigraded isomorphism then*

$$(A \otimes B)v = u(A \otimes B) \tag{5}$$

where $A \otimes B$ is the $mn \times mn$ matrix whose $(i,j),(k,l)$ entry is $a_{i,k}b_{j,l}$.

(ii) *If $m = n$ and $\tilde{\Psi}_{A,B}$ is a graded generator exchange isomorphism then*

$$(A \otimes B)\tilde{v} = u(A \otimes B) \tag{6}$$

where $\tilde{v}_{(i,j),(k,l)} = \bar{v}_{(l,k),(j,i)}$.

Given an isomorphism $\Psi : \mathcal{A}_u \to \mathcal{A}_v$ (where v may be $n'm' \times n'm'$) we get a homeomorphism $\theta_\Psi : \Omega_u \to \Omega_v$ and $\theta_\Psi(0,0) \in \Omega_v^0$.

Theorem

Let $\Psi : \mathcal{A}_u \to \mathcal{A}_v$ be an (algebraic) isomorphism. Then

$$\theta_\Psi(\Omega_u^0) = \Omega_v^0.$$

Theorem

Let Ψ be an algebraic isomorphism and let θ_Ψ be the associated map on Ω_u. Suppose $\theta_\Psi(0,0) = (0,0)$. Then we have the following.

(1) $\{n, m\} = \{n', m'\}$ *and we shall assume that* $n = n'$ *and* $m = m'$ *(interchanging* E *and* F *and changing* u *to* u^* *if necessary).*

(2) *There are unitary matrices* U $(n \times n)$ *and* V $(m \times m)$ *such that* $\theta_\Psi(z, w) = (Uz, Vw)$ *for* $(z, w) \in \Omega_u$. *(If* $n = m$ *it is also possible that* $\theta_\Psi(z, w) = (Vw, Uz)$*.)*

(3) *If* Ψ *is an isometric isomorphism, then* Ψ *is a bigraded isomorphism. (Or, if* $m = n$*, it may be a graded exchange isomorphism).*

The proof of (1) and (2) uses arguments similar to the ones used by S. Power in a previous work (based essentially on applying Schwarz's lemma for holomorphic maps on the unit disc).

For (3) one shows first that $\Psi(L_{e_i}) = L_{Ue_i} + X$ where X is a sum of higher order terms. Then we apply it to ξ_0 (a wandering vector) to get

$$1 \geq \|\Psi^{-1}(L_{e_i})\xi_0\|^2 = \|L_{Ue_i}\xi_0\|^2 + \|X\xi_0\|^2 = 1 + \|X\xi_0\|^2.$$

Thus $X\xi_0 = 0$ implying $X = 0$.

Since every graded isomorphism Ψ satisfies $\theta_\Psi(0,0) = (0,0)$, we conclude the following.

Corollary

Every graded isometric isomorphism is either bigraded or, if $m = n$, it may be a graded generator exchange isomorphism.

Theorem

The following statements are equivalent for unitary matrices u, v in $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$.
(i) There is an isometric isomorphism $\Psi : \mathcal{A}_u \to \mathcal{A}_v$.
(ii) There is a graded isometric isomorphism from $\Psi : \mathcal{A}_u \to \mathcal{A}_v$.
(iii) The matrices u, v are product unitary equivalent or (in case $n = m$) the matrices $u, \tilde{v}$ are product unitary equivalent.

Proof.

Given Ψ in (i), let $(z, w) = \theta_\Psi(0,0)$. Then $(z, w) \in \Omega_v^0$. By the characterization of the core, there is a completely isometric automorphism Φ of $\mathcal{A}_v$ such that $\theta_\Phi(0,0) = (z, w)$ and, therefore, $\theta_{\Phi^{-1}\circ\Psi}(0,0) = (0,0)$. By the proposition above, $\Phi^{-1} \circ \Psi$ is a graded isometric isomorphism and (ii) holds.
It follows from previous results that (ii) implies (iii) and that (iii) implies (i). $\square$

Remark:

(i) Whenever $\mathcal{A}_u$ and $\mathcal{A}_v$ are isomorphic, we have $\{n, m\} = \{n', m'\}$.

(ii) If $\mathcal{A}_u$ and $\mathcal{A}_v$ are isometrically isomorphic, then so are $\mathcal{L}_u$ and $\mathcal{L}_v$. In fact, we can then find an isometric isomorphism of $\mathcal{A}_u$ and $\mathcal{A}_v$ that extends to an isometric isomorphism of $\mathcal{L}_u$ and $\mathcal{L}_v$ that is also a w^*-homeomorphism.

Theorem

For $n \neq m$ the isometric automorphisms of $\mathcal{A}_u$ are of the form $\Psi_{A,B}\Theta_{z,w}$ where $(z, w) \in \Omega_u^0$ and $(A \otimes B)u = u(A \otimes B)$. In case $n = m$ the isometric automorphisms include, in addition, those of the form $\tilde{\Psi}_{A,B}\Theta_{z,w}$ where $(A \otimes B)\tilde{u} = u(A \otimes B)$.

In order to work out examples, first note the following.

Lemma

(i) *If the core contains a vector* (z, w) *with* $z \neq 0$, *then* $dim(Ker(u - I)) \geq m$.

(ii) *If the core contains a vector* (z, w) *with* $w \neq 0$ *then* $dim(Ker(u - I)) \geq n$.

(iii) *If the core contains a vector* (z, w) *with* $z \neq 0$ *and* $w \neq 0$, *then* $dim(Ker(u - I)) \geq m + n - 1$.

Examples

Example: $n = m = 2$
Consider the different possibilities for $d = dim(Ker(u - I))$.
Case I: $d = 0$
For every $(z, w) \in \bar{\mathbb{B}}_2 \times \bar{\mathbb{B}}_2$, $(z, w) \in \Omega_u$ if and only if the vector $(z_1 w_1, z_1 w_2, z_2 w_1, z_2 w_2)^t$ lies in $Ker(u - I)$. Thus, in case I, Ω_u is the minimal possible and is equal to

$$\Omega_{min} := (\bar{\mathbb{B}}_2 \times \{0\}) \cup (\{0\} \times \bar{\mathbb{B}}_2).$$

It follows from the lemma that, in this case,

$$\Omega_u^0 = \{(0, 0)\}.$$

It then follows that every isometric automorphism of $\mathcal{A}_u$ is graded and the isometric automorphisms of $\mathcal{A}_u$ are given by pairs (A, B) of unitary matrices such that $A \otimes B$ either commutes with u or intertwines u and $\tilde{u}$.

Case II: $d = 1$

When $d = 1$ it still follows from the lemma above that

$$\Omega_u^0 = \{(0,0)\}$$

but now it is possible for Ω_u to be larger than Ω_{min}. In fact, if the non zero vector $(a,b,c,d)^t$ spanning $Ker(u - I)$ satisfies $ad \neq bc$ then $\Omega_u = \Omega_{min}$ but if $ad = bc$ then the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is of rank one and can be written as $(z_1, z_2)^t(w_1, w_2)$. Thus, $(z, w) \in V_u$ and Ω_u contains some (z, w) with non zero z and w. Since $\Omega_u^0 = \{(0,0)\}$, it is still true that isometric isomorphisms and automorphisms of these algebras are all graded.

Case III: $d = 2$

When $d = 2$ it is possible that Ω_u^0 will contain non zero vectors (z, w) but, as the lemma shows, it does not contain a vector with both $z \neq 0$ and $w \neq 0$. All other possibilities may occur. For example write u_1, u_2 and u_3 for the three diagonal matrices:

$$u_1 = diag(1, -1, -1, 1)$$

,

$$u_2 = diag(1, -1, 1, -1)$$

and

$$u_3 = diag(1, 1, -1, -1).$$

Using the definition of the core, we easily see that

$$\Omega_{u_1}^0 = \{(0, 0)\}$$

,

$$\Omega_{u_2}^0 = \{(0, 0, w_1, 0) : \ |w_1| \leq 1\}$$

and

$$\Omega_{u_3}^0 = \{(z_1, 0, 0, 0) : \ |z_1| \leq 1\}.$$

Thus, the only isometric automorphisms of $\mathcal{A}_{u_1}$ are graded, the isometric automorphisms of $\mathcal{A}_{u_2}$ are formed by composing graded automorphisms with automorphisms of the type $\Theta_{z,w}$ (with $z = (0,0)$ and $w = (w_1, 0)$). Similarly, for the automorphisms of $\mathcal{A}_{u_3}$.

Case IV: $d = 3$

Corollary

Every matrix u with $\dim(Ker(u - I)) = 3$ is product unitary equivalent to a unique matrix of the form $u(a, \lambda)$

$$u(a,\lambda) = \begin{pmatrix} (\lambda-1)a^2+1 & 0 & 0 & (\lambda-1)a(1-a^2)^{1/2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ (\lambda-1)a(1-a^2)^{1/2} & 0 & 0 & \lambda+(1-\lambda)a^2 \end{pmatrix}.$$

(with $0 \le a \le 1/\sqrt{2}$, $|\lambda| = 1$ and $\lambda \neq 1$).

Theorem

If $a = 0, |\lambda| = 1, \lambda \neq 1$, *then* $\Omega_{u(0,\lambda)}$ *is the union*

$$\{(z_1, z_2, w_1, 0) : z \in \mathbb{B}_2; |w_1| \leq 1\} \cup$$

$$\{(z_1, 0, w_1, w_2) : w \in \mathbb{B}_2; |z_1| \leq 1\},$$

and

$$\Omega^0_{u(0,\lambda)} = \{(z_1, 0, w_1, 0) : |z_1| \leq 1; |w_1| \leq 1\}.$$

If $a \neq 0$ *then*

$$\Omega_{u(a,\lambda)} = \{(z, w) \in \overline{\mathbb{B}}_2 \times \overline{\mathbb{B}}_2 : az_1 w_1 + (1 - a^2)^{1/2} z_2 w_2 = 0\}$$

and

$$\Omega^0_{u(a,\lambda)} = \{(0, 0)\}.$$

Theorem

Let $0 \leq a, b \leq 1/\sqrt{2}, \quad |\lambda| = |\mu| = 1, \quad \lambda, \mu \neq 1$*. Then*

(1) $\mathcal{A}_{u(a,\lambda)}$ *and* $\mathcal{A}_{u(b,\mu)}$ *are isometrically isomorphic if and only if* $a = b$ *and* λ *equals either* μ *or* $\bar{\mu}$*.*

(2) *When* $a \neq 0$ *the isometric automorphisms of* $\mathcal{A}_{u(0,\lambda)}$ *are all bigraded*

(3) *If* $a = 0$ *then there are isometric isomorphisms that are not graded*

Case V: $d = 4$

This is the case where $u = I$. We have $\Omega_u = \Omega_u^0 = \bar{\mathbb{B}}_n \times \bar{\mathbb{B}}_m$ and the isometric automorphisms are obtained by composing graded automorphisms and the automorphisms $\Theta_{z,w}$.

Subproduct systems over $\mathbb{N}^2$ (M. Gurevich)

Subproduct Systems $\{X(n,m) = E^{\otimes n} \otimes F^{\otimes m}\}$ is a product system ($X(s+t) \cong X(s) \otimes X(t)$ for $s, t \in \mathbb{N}^2$). Consider $\{Y(n,m) \subseteq E^{\otimes n} \otimes F^{\otimes m}\}$. It is a subpruduct system if we assume only $Y(s+t) \subseteq Y(s) \otimes Y(t)$. As we shall see, this introduces new polynomial relations.

The algebra $\mathcal{A}_Y$ is now the norm-closed algebra generated by $S_{e_i} := PL_{e_i}P$ where P is the projection of $\mathcal{F}(n,m,u)$ onto $\mathcal{F}(Y) := \sum Y(k,l)$. In fact

$$\mathcal{A}_Y = P\mathcal{A}_u P.$$

Question: **(open)** Is $\mathcal{A}_Y$ a quotient (as an operator algebra) of $\mathcal{A}_u$?

Sample results:

To describe the character space of $\mathcal{A}_Y$, define

$$p_{i,j}(z,w) = z_i w_j - \sum_{k,l} u_{(i,j),(k,l)} w_l z_k.$$

Also, for $x \in E^{\otimes i} \otimes F^{\otimes j} \ominus Y(i,j)$, we can define an associated polynomial q^x. If $x = e_1 \otimes f_3 - e_2 \otimes f_1$, $q^x(z,w) = z_1 w_3 - z_2 w_1$ such that, if J is the ideal generated by all these polynomials,

Theorem

The character space of $\mathcal{A}_Y$ can be identified with

$$\{(z,w) \in \overline{\mathbb{B}_n} \times \overline{\mathbb{B}_m} : p(z,w) = 0, \forall p \in J\}.$$

Theorem

If Y and Z are subproduct systems with the same (n, m) and $\phi : \mathcal{A}_Y \to \mathcal{A}_Z$ is an isometric isomorphism that preserves the character associated to 0. Then (under some condition) there is a unitary operator $V : \mathcal{F}_Y \to \mathcal{F}_Z$ such that $\phi(T) = V^*TV$ for all $T \in \mathcal{A}_Y$.

Theorem

Suppose Y and Z are subproduct systems and there is a bounded isomorphism $\phi : \mathcal{A}_Y \to \mathcal{A}_Z$. Then $n_Y + m_Y = n_Z + m_Z$.