ARITHMETIC AND GEOMETRY

(A BEHAVIORAL APPROACH TO GEOMETRIC CONTROL)

Paul A. Fuhrmann, Ben-Gurion University of the Negev

and

Uwe Helmke, University of Wuerzburg

ABSTRACT

The talk will focus on the algebra of polynomial matrices on the one hand and the geometry of subspaces on the other. The context is linear algebra and linear system theory.

We shall use functional models and realization theory as a bridge between the two.

To limit the scope, we shall concentrate on the EQUIVALENCE vs. SIMILARITY PARADIGM.

Connecting link: REALIZATION THEORY

EQUIVALENCE vs. SIMILARITY

UNIMODULAR EQUIVALENCE

$A, B \in \mathbb{F}^{n \times n}$

$A \simeq B \Leftrightarrow U(z)(zI - A) = (zI - B)V(z)$

$U(z), V(z)$ **unimodular**

INVARIANTS: FINITE INVARIANT FACTORS

STRICT EQUIVALENCE

$A, B \in \mathbb{F}^{n \times n}$

$A \simeq B \Leftrightarrow P(zI - A) = (zI - B)Q$

P, Q **nonsingular**, $(P = Q)$

In this case: unimodular equivalence = strict equivalence = similarity

REMARK

STRICT EQUIVALENCE $\Rightarrow$ UNIMODULAR EQUIVALENCE

IN GENERAL, UNIMODULAR EQUIVALENCE HAS LESS INVARIANTS THAN STRICT EQUIVALENCE

A SHORT HISTORY

WEIERSTRASS [1867], Regular pencils

KRONECKER [1890], Singular pencils

BRUNOVSKY [1970], Feedback canonical form

MORSE [1973], Morse group

FUHRMANN [1976], The shift realization

ALING AND SCHUMACHER [1984], Direct sum decomposition

WILLEMS [1986,1989,1991], Behaviors

EQUIVALENCE AND SIMILARITY

1. **Strict (Unimodular) equivalence (arithmetic) of monic pencils vs. similarity (geometry).**
2. **Weierstrass:**
   Strict equivalence of regular matrix pencils (arithmetic) vs. similarity, incl. at ∞, (geometry).
3. **Brunovsky:**
   Strict equivalence of input pencils vs. Feedback equivalence (geometric control).
4. **Kronecker, Morse:**
   Strict equivalence of singular matrix pencils (arithmetic) vs. Morse group equivalence (geometric control). Kalman's state space isomorphism theorem a special case.

CHANGE OF PARADIGMS

- **Rota [1951] proved that every strict contraction in a Hilbert space is isomorphic to the restriction of the backward shift to one of its invariant subspaces, i.e. the backward shift is a universal operator. This is easily extendable to an algebraic setting. This represents a paradigm shift from arithmetic (operators) to geometry (subspaces).**
- **Beurling [1948] characterized all invariant subspaces of the (backward) shift in H^2. This was extended by Lax and Halmos to the vector case. Parametrizing invariant subspaces makes Rota's theorem practical.**

CHANGE OF PARADIGMS

- **The algebraic analogs are $\mathbb{F}[z]$-submodules of the spaces $\mathbb{F}[z]^m$ and $z^{-1}\mathbb{F}[[z^{-1}]]^m$. Since these spaces are intrinsically infinite dimensional, they, the respective shifts and the corresponding invariant subspaces, did not make an apppearance in Linear Algebra texts.**

  Examples of this are: Maclane and Birkhoff, Lang, Hoffmann and Kunze as well as Halmos, Dym.

CHANGE OF PARADIGMS

- **Influenced by operator theory, polynomial and rational models were introduced in Fuhrmann [1976] and applied to the realization problem. This provided a bridge between abstract module theory (Kalman), polynomial algebra (Rosenbrock [1970]) and state space methods.**
- **Willems [1986,1989,1991] introduced behaviors, (a class of backward shift invariant subspaces) into linear systems theory. This represents a paradigm shift from I/O maps, transfer functions or state representations (arithmetic), to geometry (behaviors). For behaviors, the arithmetic counterpart is given by kernel representations.**

FROM POLYNOMIALS TO MODEL SPACES

Polynomial Arithmetic:

$D(z) \in \mathbb{F}[z]^{m\times m}$, $\det D(z) \neq 0$

Geometry:

Polynomial Model:

$$\begin{cases} \pi_D : \mathbb{F}[z]^m \longrightarrow \mathbb{F}[z]^m \\ \pi_D f = D\pi_- D^{-1} f \\ \boxed{X_D = \operatorname{Im} \pi_D \simeq \mathbb{F}[z]^m / D(z)\mathbb{F}[z]^m} \end{cases}$$

Rational Model:

$$\begin{cases} D(\sigma) : z^{-1}\mathbb{F}[[z^{-1}]]^m \longrightarrow z^{-1}\mathbb{F}[[z^{-1}]]^m \\ D(\sigma)h = \pi_- Dh, \pi^D h = \pi_- D^{-1}\pi_+ Dh \\ \boxed{\operatorname{Ker} D(\sigma) = \operatorname{Im} \pi^D = X^D \simeq X_D} \end{cases}$$

FACTORIZATIONS AND INVARIANT SUBSPACES

$D(z) \in \mathbb{F}[z]^{m\times m}$ **nonsingular.**

$$\mathcal{V} \subset X_D \& S_D\mathcal{V} \subset \mathcal{V} \\ \Leftrightarrow \\ \mathcal{V} = D_1 X_{D_2}$$

$$\mathcal{V} \subset X^D \& S^D\mathcal{V} \subset \mathcal{V} \\ \Leftrightarrow \\ \mathcal{V} = X^{D_2}$$

$D(z) = D_1(z)D_2(z), D_1(z), D_2(z) \in \mathbb{F}[z]^{m\times m}$

CONNECTION BETWEEN ALGEBRA AND GEOMETRY

DIRECT LINK TO GEOMETRIC CONTROL AND BEHAVIORS

INVARIANT SUBSPACES

$A : \mathcal{X} \longrightarrow \mathcal{X}, \mathcal{V} \subset \mathcal{X}$

$\exists \mathcal{W}; \mathcal{X} = \mathcal{W} \oplus \mathcal{V}$

$$A \simeq \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}$$

$$\begin{aligned} zI - A &= \begin{pmatrix} zI - A_{11} & 0 \\ -A_{21} & zI - A_{22} \end{pmatrix} \\ &= \begin{pmatrix} zI - A_{11} & 0 \\ -A_{21} & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & zI - A_{22} \end{pmatrix} \end{aligned}$$

MODEL HOMOMORPHISMS

Let $D_1(z) \in \mathbb{F}[z]^{m\times m}$ **and** $D_2(z) \in \mathbb{F}[z]^{p\times p}$ **be nonsingular.**

$$\boxed{S^D = \sigma | X^D}$$

$Z : X^{D_1} \longrightarrow X^{D_2}$ **is an** $\mathbb{F}[z]$**-homomorphism, i.e.**

$$\boxed{Z S^{D_1} = S^{D_2} Z}$$

if and only if there exist $N_1(z), N_2(z) \in \mathbb{F}[z]^{p\times m}$ **such that**

$$\boxed{\begin{array}{rcl} N_2(z)D_1(z) & = & D_2(z)N_1(z) \\ Zh & = & \pi_- N_1 h = N_1(\sigma)h. \end{array}}$$

INVERTIBILITY AND COPRIMENESS

SHIFT REALIZATION

$$G(z) = V(z)T(z)^{-1}U(z) + W(z) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

$$\begin{cases} A = S_T \\ B\xi = \pi_T U\xi, \\ Cf = (VT^{-1}f)_{-1} \\ D = G(\infty). \end{cases}$$

$CA^{j-1}B\xi = (VT^{-1}\pi_T z^{j-1}\pi_T U\xi)_{-1} = (V\pi_- T^{-1}z^{j-1}U\xi)_{-1} = (z^{j-1}(VT^{-1}U + W)\xi)_{-1}$

Realization is reachable $\Leftrightarrow$ $T(z)$ **and** $U(z)$ **left coprime**

Realization is observable $\Leftrightarrow$ $T(z)$ **and** $V(z)$ **right coprime.**

SYSTEM EQUIVALENCE

Rosenbrock, Fuhrmann

$G(z) = V_i(z)T_i(z)^{-1}U_i(z) + W_i(z), i = 1, 2$ **(no coprimeness assumptions);** Σ_i **the associated shift realizations**

$$P_i = \begin{pmatrix} T_i(z) & -U_i(z) \\ V_i(z) & W_i(z) \end{pmatrix}$$

$P_1 \simeq P_2$ **if** $\Sigma_1 \simeq \Sigma_2$ **(FSE)**

$P_1 \simeq P_2 \Leftrightarrow \exists M(z), X(z), N(z), Y(z)$**, such that**
$M(z) \wedge_L T_2(z) = I, \;\&\; N(z) \wedge_R T_1(z) = I$

$$\begin{pmatrix} M(z) & 0 \\ X(z) & I \end{pmatrix} \begin{pmatrix} T_1(z) & -U_1(z) \\ V_1(z) & W_1(z) \end{pmatrix} = \begin{pmatrix} T_2(z) & -U_2(z) \\ V_2(z) & W_2(z) \end{pmatrix} \begin{pmatrix} N(z) & Y(z) \\ 0 & I \end{pmatrix}$$

FROM POLYNOMIALS TO BEHAVIORS

Polynomial Arithmetic:

$$R(z) \in \mathbb{F}[z]^{p\times m}$$
$$R(\sigma) : z^{-1}\mathbb{F}[[z^{-1}]]^m \longrightarrow z^{-1}\mathbb{F}[[z^{-1}]]^p$$
$$R(\sigma)h = \pi_- Rh$$

Geometry: Behaviors:
$\mathcal{B}$ **a linear, shift invariant and closed subspace of** $z^{-1}\mathbb{F}[[z^{-1}]]^m \Leftrightarrow$

$$\mathcal{B} = X^R = \operatorname{Ker} R(\sigma)$$

FACTORIZATIONS and SUBBEHAVIORS

Polynomial Arithmetic:
Factorization:

$$\boxed{R(z) = R_1(z)R_2(z)}$$

Geometry:
Behavior Inclusion:

$$\boxed{\mathcal{B} = X^R = \operatorname{Ker} R(\sigma) \supset X^{R_2} = \operatorname{Ker} R_2(\sigma) = \mathcal{B}_2}$$

BEHAVIOR HOMOMORPHISMS

THEOREM: Let $M(z) \in \mathbb{F}[z]^{p\times m}$ **and** $\overline{M}(z) \in \mathbb{F}[z]^{\overline{p}\times\overline{m}}$ **be of full row rank. Then**

$Z : \mathrm{Ker}\, M(\sigma) \longrightarrow \mathrm{Ker}\, \overline{M}(\sigma)$ **is a <u>continuous</u>** $\mathbb{F}[z]$**-homomorphism, i.e. satisfies** $ZS^M = S^{\overline{M}}Z$ **(and ...)** $\Leftrightarrow$ **there exist** $\overline{U}(z) \in \mathbb{F}[z]^{\overline{p}\times p}$ **and** $U(z)$ **in** $\mathbb{F}[z]^{\overline{m}\times m}$ **such that**

$$\overline{U}(z)M(z) = \overline{M}(z)U(z)$$

$$\boxed{Zh = U(\sigma)h} \qquad h \in \mathrm{Ker}\, M(\sigma)$$

B HOMOMORPHISM INVERTIBILITY

1. Z **is injective** $\Leftrightarrow U(z), M(z)$ **are right coprime.**
2. Z **is surjective** $\Leftrightarrow \overline{U}(z), \overline{M}(z)$ **are left coprime**

and $\text{Ker} \begin{pmatrix} -\tilde{U}(z) & \tilde{M}(z) \end{pmatrix} = \text{Im} \begin{pmatrix} \widetilde{\overline{M}}(z) \\ \widetilde{\overline{U}}(z) \end{pmatrix}$

3. Z **is invertible if and only if there exists a doubly unimodular embedding**

$$\begin{pmatrix} \overline{X}(z) & -\overline{Y}(z) \\ -\overline{U}(z) & \overline{M}(z) \end{pmatrix} \begin{pmatrix} M(z) & Y(z) \\ U(z) & X(z) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$\begin{pmatrix} M(z) & Y(z) \\ U(z) & X(z) \end{pmatrix} \begin{pmatrix} \overline{X}(z) & -\overline{Y}(z) \\ -\overline{U}(z) & \overline{M}(z) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

CONT.

4. **If** Z **is invertible, then in terms of the doubly unimodular embedding, its inverse**

Z^{-1} : Ker $\overline{M}(\sigma) \longrightarrow$ Ker $M(\sigma)$

is given by

$$\boxed{Z^{-1} = -\overline{Y}(\sigma)}$$

Two behaviors $\mathcal{B}_1, \mathcal{B}_2$ **are** **equivalent** **if there exists an invertible B-homomorphism** $Z : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$**.**

PENCIL CHARACTERIZATION

- The pencil $(zI - F) \in \mathbb{F}[z]^{n \times n}$ is called monic.
- A square pencil $(zE - F)$ is called regular if $\det(zE - F)$ is not the zero polynomial.
- A pencil of the form $\begin{pmatrix} zI - A & B \end{pmatrix} \in \mathbb{F}[z]^{n \times (n+m)}$ is called an input pencil.
- A pencil $(zE - F) \in \mathbb{F}[z]^{m \times n}$ that is not regular is called singular.

EQUIVALENCES

- **Two pencils** $(zE - F), (z\overline{E} - \overline{F})$ **in** $\mathbb{F}[z]^{m\times n}$ **are called unimodularly equivalent if there exists unimodular** $U(z), \overline{U}(z)$ **such that**

$$\boxed{(zE - F)\overline{U}(z) = U(z)(z\overline{E} - \overline{F})}$$

- **Two pencils** $(zE - F), (z\overline{E} - \overline{F})$ **in** $\mathbb{F}[z]^{m\times n}$ **are called strict equivalent if** $\exists L, R \in GL_{\bullet}(\mathbb{F})$ **such that**

$$\boxed{(zE - F)R = L(z\overline{E} - \overline{F})}$$

CANONICAL FORMS MAY DIFFER!!

THE WEIERSTRASS CANONICAL FORM

Assume $(zE - F) \in \mathbb{F}[z]^{n\times n}$ **is a regular pencil, i.e.** $\det(zE - F)$ **is a nonzero polynomial.**

There exist, up to similarity transformations, unique matrices $A \in \mathbb{F}^{r\times r}$ **and a nilpotent** $N \in \mathbb{F}^{(n-r)\times(n-r)}$ **such that**

$$(zE - F) \simeq_{se} \begin{pmatrix} zI_r - A & 0 \\ 0 & I_{n-r} - zN \end{pmatrix}$$

PROOF: Use realization theory and Möbius transformatons.

PROOF:

$$(zE-F)^{-1} = \frac{\text{adj}\,(zE-F)}{\det(zE-F)} = H(z) + P(z)$$

$H(z) \in z^{-1}\mathbb{F}[[z^{-1}]]^{n\times n}$, $P(z) \in \mathbb{F}[z]^{n\times n}$

$H(z) = C(zI-A)^{-1}B$,

$z^{-1}P(z^{-1}) = C_\infty(zI-N)^{-1}B_\infty$

$$(zE-F)^{-1} = \begin{pmatrix} C & C_\infty \end{pmatrix} \begin{pmatrix} zI-A & 0 \\ 0 & I-zN \end{pmatrix}^{-1} \begin{pmatrix} B \\ B_\infty \end{pmatrix}$$

$$(zE-F) = \begin{pmatrix} B \\ B_\infty \end{pmatrix}^{-1} \begin{pmatrix} zI-A & 0 \\ 0 & I-zN \end{pmatrix} \begin{pmatrix} C & C_\infty \end{pmatrix}^{-1}$$

Further reduction to (generalized) Jordan form.

A REDUCTION

$zE - F$ a singular pencil.

There exist a right prime pencil $zE_1 - F_1$ and a pencil $zE_2 - F_2$, with E_2 of full row rank for which we have

$$\boxed{zE - F \simeq_{se} \begin{pmatrix} zE_1 - F_1 & 0 \\ 0 & zE_2 - F_2 \end{pmatrix}}$$

PENCIL REPRESENTATIONS

$(zE - F) \in \mathbb{F}[z]^{n\times n}$ **is unimodular** $\Leftrightarrow$

$$\boxed{(zE - F) \simeq_{se} (I - zN)}$$

E **is of full row rank** $\Leftrightarrow$

$$\boxed{zE - F \simeq_{se} \left(\begin{array}{c|cc} zI - M & 0 & 0 \\ \hline 0 & zI - A & -B \end{array}\right)}$$

$(zE - F)$ **is left prime** $\Leftrightarrow$

$$\boxed{(zE - F) \simeq_{se} \left(\begin{array}{c|cc} I - zN & 0 & 0 \\ \hline 0 & zI - A & -B \end{array}\right)}$$

(A, B) **reachable.**

DUAL PENCIL REPRESENTATIONS

E **has full column rank** $\Leftrightarrow$

$$(zE - F) \simeq_{se} \left(\begin{array}{c|c} zI - M & 0 \\ \hline 0 & zI - \overline{A} \\ 0 & \overline{C} \end{array}\right)$$

$(zE - F)$ **is right prime** $\Leftrightarrow$

$$(zE - F) \simeq_{se} \left(\begin{array}{c|c} I - zN & 0 \\ \hline 0 & zI - \overline{A} \\ 0 & \overline{C} \end{array}\right)$$

$(\overline{C}, \overline{A})$ **observable.**

DIRECT SUM DECOMPOSITION

$zE - F \in \mathbb{F}[z]^{q\times s}$. $\exists$ **an essentially unique, matrix** M**, a nilpotent matrix** N**, a reachable pair** (A, B)**, with** B **of full column rank, and an observable pair** $(\overline{C}, \overline{A})$**, with** $\overline{C}$ **of full row rank, such that the following equivalence holds**

$$zE - F \simeq_{se} \left(\begin{array}{c|c|c|c|c} I - zN & & & & \\ \hline & \begin{matrix} zI - \overline{A} \\ \overline{C} \end{matrix} & & & \\ \hline & & zI - M & & \\ \hline & & & zI - A \quad -B & \\ \hline & & & & 0 \end{array}\right)$$

FACTORIZATIONS

$$\begin{pmatrix} zI - M & 0 & 0 \\ \hline 0 & zI - A & -B \end{pmatrix}$$

$$= \begin{pmatrix} zI - M & 0 \\ \hline 0 & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ \hline 0 & zI - A & -B \end{pmatrix}$$

$$= \begin{pmatrix} I & 0 & 0 \\ \hline 0 & zI - A & -B \end{pmatrix} \begin{pmatrix} zI - M & 0 & \\ \hline 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

$$\mathcal{B} = \operatorname{Ker} \begin{pmatrix} \sigma I - M & 0 & 0 \\ \hline 0 & \sigma I - A & -B \end{pmatrix}$$

$$= \operatorname{Ker} \begin{pmatrix} I & 0 & 0 \\ \hline 0 & \sigma I - A & -B \end{pmatrix} \oplus \operatorname{Ker} \begin{pmatrix} \sigma I - M & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

$$= \mathcal{B}_r \oplus \mathcal{B}_a$$

FEEDBACK AND
STRICT EQUIVALENCE

$$R \begin{pmatrix} A_2 & B_2 \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \end{pmatrix} \begin{pmatrix} R & 0 \\ K & P \end{pmatrix}$$

$$(A_1, B_1) \simeq_{fb} (A_2, B_2)$$
$$\Leftrightarrow$$
$$\begin{pmatrix} zI - A_1 & -B_1 \end{pmatrix} \simeq_{se} \begin{pmatrix} zI - A_2 & -B_2 \end{pmatrix}$$

$$(C_1, A_1) \simeq_{oi} (C_2, A_2)$$
$$\Leftrightarrow$$
$$\begin{pmatrix} zI - A_1 \\ C_1 \end{pmatrix} \simeq_{se} \begin{pmatrix} zI - A_2 \\ C_2 \end{pmatrix}$$

REACHABILITY AND OBSERVABILITY INDICES, BRUNOVSKY FORMS

LEFT WIENER-HOPF FACTORIZATIONS

$D(z) \in \mathbb{F}[z]^{m\times m}$ **NONSINGULAR**
$U_-(z) \in \mathbb{F}[[z^{-1}]]^{m\times m}, U_+(z) \in \mathbb{F}[z]^{m\times m}$
UMIMODULAR

$$\boxed{U_-(z)D(z)U_+(z) = D_\beta(z) = \text{diag}\,(z^{\mu_1}, \ldots, z^{\mu_m})}$$

REACHABILITY INDICES: $\mu_1 \geq \cdots \geq \mu_m \geq 0$

BRUNOVSKY FORM

(A, B) **reachable,** B **full column rank.**

$$\begin{aligned}(zI - A)^{-1}B &= H(z)D(z)^{-1}\\ (zI - A - BK)^{-1}B &= H(z)D_\beta(z)^{-1}\end{aligned}$$

$$D_\beta(z) = U_-(z)D(z)U_+(z) = \mathrm{diag}\,(z^{\mu_1}, \ldots, z^{\mu_m})$$

USE SHIFT REALIZATION:

$$(A, B) \simeq_{fb} (A_\beta, B_\beta)$$

$$\begin{pmatrix} zI - A_\beta & -B_\beta \end{pmatrix} \simeq_{se} \mathrm{diag}\,(L_{\kappa_1}, \ldots, L_{\kappa_m})$$

BRUNOVSKY FORM

$$A_\kappa = \begin{pmatrix} 0 & & & & & & \\ 1 & 0 & & & & & \\ 0 & 1 & 0 & & & & \\ & & \ddots & \ddots & & & \\ & & & \ddots & \ddots & & \\ & & & & \ddots & \ddots & \\ & & & & & 1 & 0 \end{pmatrix}, \quad B_\kappa = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$L_\kappa = \left(\begin{array}{c|ccccc} -1 & z & & & & \\ 0 & -1 & \ddots & & & \\ \vdots & & \ddots & \ddots & & \\ \vdots & & & \ddots & \ddots & \\ 0 & & & & -1 & z \end{array}\right) \in \mathbb{F}[z]^{\kappa \times (\kappa+1)}$$

MORSE GROUP

$$(T, J, S, K, R)(A, B, C) = (SAS^{-1} - JCS^{-1} - SBR^{-1}KS^{-1}, SBR^{-1}, TCS^{-1})$$

$$\begin{pmatrix} zI - A_1 & -B_1 \\ C_1 & 0 \end{pmatrix} \simeq_{se} \begin{pmatrix} zI - A_2 & -B_2 \\ C_2 & 0 \end{pmatrix}$$

$$\Leftrightarrow$$

$$\begin{pmatrix} S & J \\ 0 & T \end{pmatrix} \begin{pmatrix} zI - A_1 & -B_1 \\ C_1 & 0 \end{pmatrix} = \begin{pmatrix} zI - A_2 & -B_2 \\ C_2 & 0 \end{pmatrix} \begin{pmatrix} S & 0 \\ K & R \end{pmatrix}$$

KRONECKER FORM

$\mathbb{F}$ algebraically closed.

Given a singular pencil $zE - F \in \mathbb{F}[z]^{m\times p}$, then the pencil is strict equivalent to a block diagonal pencil of the form

$$\boxed{\operatorname{diag}\left(J_{(\lambda_1,n_1)},\ldots,J_{(\lambda_t,n_t)},N_{\mu_1},\ldots,N_{\mu_k},L_{\kappa_1},\ldots,L_{\kappa_m},\tilde{L}_{\nu_1},\ldots,\tilde{L}_{\nu_p},0\right)}$$

where $J_{(\lambda_i,n_i)}$ are the Jordan blocks corresponding to the eigenvalues λ_i, N_{μ_j} the Jordan blocks corresponding to the infinite zeros, κ_i the reachability indices of the pair (A,B) and ν_j the observability indices of the pair $(\overline{C},\overline{A})$. The canonical pencil is uniquely determined up to the reordering of the blocks.

MODIFIED KRONECKER FORM

$$N_\mu = \begin{pmatrix} 1 & & & \\ -z & \ddots & & \\ & \ddots & \ddots & \\ & & -z & 1 \end{pmatrix}$$

$$\simeq \left(\begin{array}{cccc|c} z & -1 & & & 0 \\ & \ddots & \ddots & & \vdots \\ & & \ddots & -1 & \vdots \\ & & & z & 1 \\ \hline 1 & \cdot & \cdot & 0 & 0 \end{array}\right) = \begin{pmatrix} zI - N_\infty & b_4 \\ c_4 & 0 \end{pmatrix}$$

MODIFIED KRONECKER FORM

$$\mathcal{C} = \begin{pmatrix} 0 & 0 & \mathcal{C}_3 & 0 \\ 0 & 0 & 0 & \mathcal{C}_4 \end{pmatrix},$$

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & 0 & 0 & 0 \\ 0 & \mathcal{A}_2 & 0 & 0 \\ 0 & 0 & \mathcal{A}_3 & 0 \\ 0 & 0 & 0 & \mathcal{A}_4 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ \mathcal{B}_2 & 0 \\ 0 & 0 \\ 0 & \mathcal{B}_4 \end{pmatrix}$$

$\mathcal{A}_1$ is the Jordan form of M, $(\mathcal{A}_2, \mathcal{B}_2)$ is the Brunovsky form of (A, B), $(\mathcal{C}_3, \mathcal{A}_3)$ the dual Brunovsky form of $(\overline{C}, \overline{A})$ and $\mathcal{A}_4 = N_\infty$ the modified Jordan form of N.

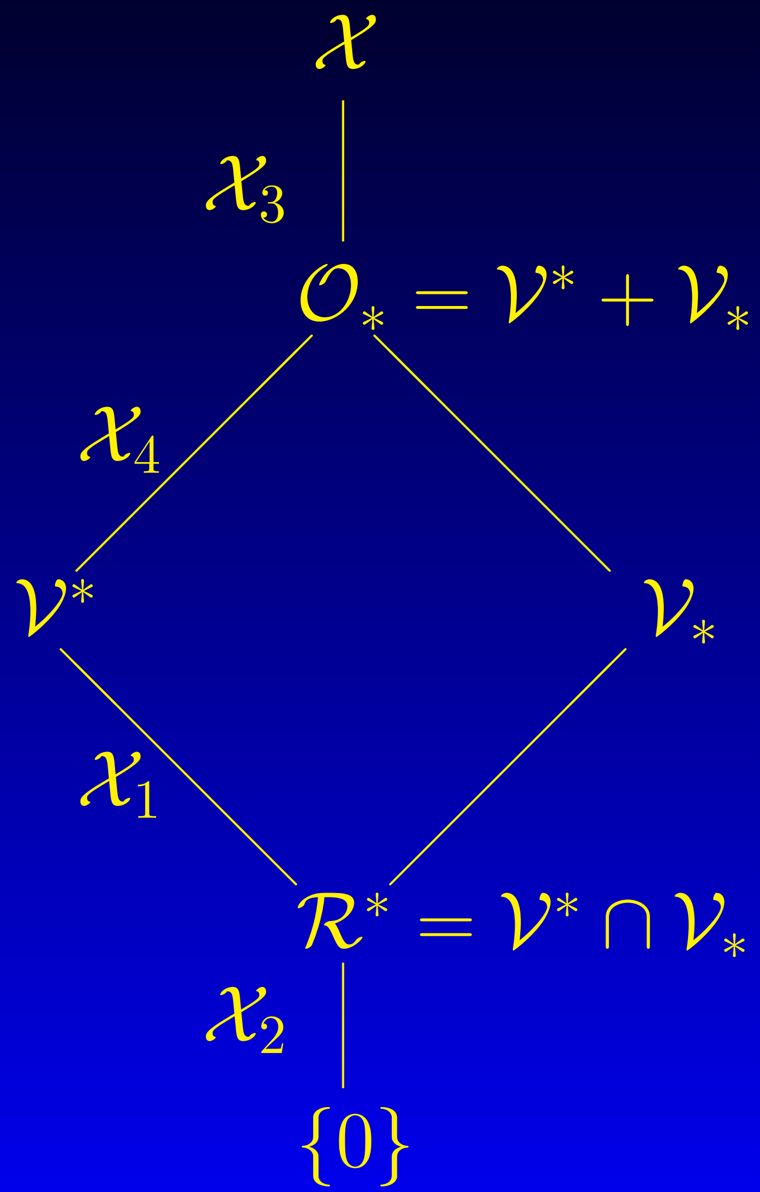
DIRECT SUM DECOMPOSITION

$$\begin{aligned} \mathcal{X} &= \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4 \\ &\simeq \mathcal{V}^*/\mathcal{R}^* \oplus \mathcal{R}^* \oplus \mathcal{X}/\mathcal{O}_* \oplus \mathcal{O}_*/\mathcal{V}^* \end{aligned}$$

$$\begin{aligned} \mathcal{R}^* &= \mathcal{X}_2 \\ \mathcal{V}^* &= \mathcal{X}_1 \oplus \mathcal{X}_2 \\ \mathcal{V}_* &= \mathcal{X}_2 \oplus \mathcal{X}_4 \\ \mathcal{O}_* &= \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_4 \end{aligned}$$

$$\begin{aligned} \mathcal{R}^* &= \mathcal{V}^* \cap \mathcal{V}_* \\ \mathcal{O}_* &= \mathcal{V}^* + \mathcal{V}_* \\ \mathcal{O}_*/\mathcal{V}^* &\simeq \mathcal{V}_*/\mathcal{R}^* \\ \mathcal{O}_*/\mathcal{V}_* &\simeq \mathcal{V}^*/\mathcal{R}^*. \end{aligned}$$

MORSE DIAGRAM



THANKS

THANKS FOR YOUR ATTENTION