Tensor Algebras

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Outline

1. Background: What is an Operator Algebra?
2. Operator Tensor Algebras
   - The Setting
   - Functions
3. Multiplier Spaces
4. Morita Equivalence

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- Pimsner: *A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by $\mathbb{Z}$*, Fields Institute 1997. The Cuntz-Pimsner algebra of a C^*-correspondence.

Our Starting Point

Theorem (M and Solel, JFA 1998)

If E is a C^*-correspondence and if $\mathscr{T}_+(E)$ is its tensor algebra, then the C^*-envelope of $\mathscr{T}_+(E)$ is the Cuntz-Pimsner algebra of E, $\mathscr{O}(E)$.

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- I want to indicate how this function theory transforms under Morita equivalence.

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1. E is a (right) Hilbert C^*-module over M.
2. E is self-dual.
3. E has a left action of M given by a normal representation φ of M in $\mathscr{L}(E)$.

Examples

- (Basic Example) $M = \mathbb{C}$, $E = \mathbb{C}^d$, $1 \leq d \leq \infty$. $(\mathbb{C}^\infty := \ell^2(\mathbb{N}))$

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Note: Every W^*-correspondence over a finite dimensional commutative W^*-algebra comes from a finite directed graph.

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- If Φ is a (not-necessarily-unital) normal endomorphism of M, then $M \otimes_\Phi M$ is naturally isomorphic to ${}_\Phi M$ - the identity Hilbert M-module with left action determined by Φ.

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- The Fock space of E, $\mathscr{F}(E) := \sum_{n\geq 0} E^{\otimes n}$ - the self-dual completion of the C^*-Hilbert module direct sum.

More Definitions

- $\mathscr{F}(E)$ is a W^*-correspondence over M, with left action denoted φ_∞. Thus $\varphi_\infty : M \to \mathscr{L}(\mathscr{F}(E))$ is given by $\varphi_\infty(a) = \sum^{\oplus} \varphi_n(a)$, where $\varphi_n(a)(\xi_1 \otimes \xi_2 \otimes \cdots) = (\varphi(a)\xi_1) \otimes \xi_2 \otimes \cdots$.

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- For $\xi \in E$, and $\eta \in \mathscr{F}(E)$,

$$T_\xi \eta := \xi \otimes \eta.$$

- $T_\xi \in \mathscr{L}(\mathscr{F}(E))$ and is called the (left) *creation operator* determined by ξ.

Definition

The norm-closed subalgebra of $\mathscr{L}(\mathscr{F}(E))$ generated by $\varphi_\infty(M)$ and $\{T_\xi \mid \xi \in E\}$ is called the *tensor algebra* of E and is denoted $\mathscr{T}_+(E)$.

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- When $E = E(G)$, $G = (G^0, G^1, r, s)$, then $\mathscr{T}_+(E)$ is the norm closure of (a faithful representation of) the path algebra of G.
- If $E = {}_\Phi M$, then $\mathscr{T}_+(E)$ and $H^\infty(E)$ are called *analytic crossed products* - first considered by Kadison and Singer, and then by Arveson. Every E is Morita equivalent to a ${}_\Phi M$ in a sense to be described below.

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Intertwiners and Discs

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Given a W^*-representation $\sigma : M \to B(H_\sigma)$, the map σ^E from $\mathscr{L}(E)$ to $E \otimes_\sigma H_\sigma$ defined by $\sigma^E(T) = T \otimes I$ is called *the representation of* $\mathscr{L}(E)$ *induced by* σ (in the sense of Rieffel.)

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Notation

$\mathfrak{I}(\sigma^E \circ \varphi, \sigma) := \{\mathfrak{z} : E \otimes_\sigma H_\sigma \to H \mid \mathfrak{z}\sigma^E \circ \varphi(\cdot) = \sigma(\cdot)\mathfrak{z}\}$ - *the intertwiner space.*

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$\mathbb{D}(E, \sigma) := \{\mathfrak{z} \in \mathfrak{I}(\sigma^E \circ \varphi, \sigma) \mid \|\mathfrak{z}\| < 1\}$ - *the open unit disc.*

Tensors as Functions (von Neumann's Inequality)

Theorem (M-Solel J.F.A. **158** (1998))

Given $\mathfrak{z} \in \overline{\mathbb{D}(E,\sigma)}$, *define* $\mathfrak{z} \times \sigma$ *by* $\mathfrak{z} \times \sigma(\varphi_\infty(a)) := \sigma(a)$ *and* $\mathfrak{z} \times \sigma(T_\xi)(h) := \mathfrak{z}(\xi \otimes h)$, $a \in M$, $\xi \in E$, *and* $h \in H_\sigma$. *Then* $\mathfrak{z} \times \sigma$ *extends to a completely contractive (c.c.) representation of* $\mathscr{T}_+(E)$ *on* H_σ.

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Tensors as Functions - $H^\infty(E)$

Theorem (M-Solel Math. Ann. 330 (2004))

If $\mathfrak{z} \in \mathbb{D}(E,\sigma)$, then $\mathfrak{z} \times \sigma$ extends to an ultra-weakly continuous, completely contractive representation of $H^\infty(E)$ in $B(H_\sigma)$ and for $F \in H^\infty(E)$, the $B(H_\sigma)$-valued function $\widehat{F}_\sigma$, defined on $\mathbb{D}(E,\sigma)$ by $\widehat{F}_\sigma(\mathfrak{z}) := \mathfrak{z} \times \sigma(F)$, is bounded and analytic.

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Remark (M-Solel on line at IEOT)

Some $\mathfrak{z} \times \sigma$, with $\|\mathfrak{z}\| = 1$, may extend to $H^\infty(E)$. These are called absolutely continuous.

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Tensors as Functions - the Basic Example

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$M = \mathbb{C}$, $E = \mathbb{C}^d$, and assume σ represents $\mathbb{C}$ on $\mathbb{C}^n$. Then $\mathbb{D}(E,\sigma) = \{(Z_1, Z_2, \cdots, Z_d) \in M_n(\mathbb{C})^d \mid \|\sum_{i=1}^d Z_i Z_i^*\| < 1\}$ - the set of all strict row contractions of length d in $M_n(\mathbb{C})$. If $F \in \mathscr{T}_+(\mathbb{C}^d)$, then $F = \sum_{w \in \mathbb{F}_d^+} a_w S_w$, where S_w is the word in the creation operators, S_i, $S_i\xi := e_i \otimes \xi$, $\xi \in \mathscr{F}(\mathbb{C}^d)$ - $\{e_i\}_{i=1}^d$ an o.n. basis for $\mathbb{C}^d$. The function $\widehat{F_\sigma}$ on $\mathbb{D}(E,\sigma)$ is given by the formula $\widehat{F_\sigma}(Z_1, Z_2, \cdots, Z_d) = \sum_{w \in \mathbb{F}_d^+} a_w Z_w$, where the Z_w are the appropriate words in the Z_i and the series converges uniformly and absolutely on each ball of radius < 1;

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Basic Problem

Let
$\Sigma = \{\sigma : M \to B(H_\sigma) \mid \sigma$ is a normal representation on $H_\sigma\}$.
The set of *all* completely contractive representations of $\mathscr{T}_+(E)$ is $\Omega := \bigcup_{\sigma \in \Sigma} \overline{\mathbb{D}(E,\sigma)}$ - a generalization of what Voiculescu, following Taylor, calls a *fully matricial set*

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Problem

What sort of properties do the functions $\widehat{F}$, $F \in \mathscr{T}_+(E)$, have? How should we think about them?

Completely Positive Definite Kernels

- Let Ω be a set, let A and B be C^*-algebras, and let $\mathscr{B}(A, B)$ be all bounded operators from A to B.

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- $K : \Omega \times \Omega \to \mathscr{B}(A, B)$ is called *completely positive definite (c.p.d.)* in case for each finite set of points $\{\omega_1, \omega_2, \ldots, \omega_n\}$ in Ω, for each choice of n elements in A, $a_1, a_2, \ldots, a_n$, and for each choice of n elements in B, $b_1, b_2, \ldots, b_n$, the inequality

$$\sum_{i,j=1}^{n} b_i^* K(\omega_i, \omega_j)[a_i^* a_j] b_j \geq 0$$

holds in B.

RKCs

Theorem (Barreta, Bhat, Liebscher, and Skeide, JFA 212 (2004))

Every c.p.d. $K : \Omega \times \Omega \to \mathscr{B}(A, B)$ *has a unique minimal Kolmogorov decomposition:* $({}_A\mathscr{X}_B, v(\cdot))$*, where* ${}_A\mathscr{X}_B$ *is an* A, B*-correspondence and* $v : \Omega \to {}_A\mathscr{X}_B$ *is a function such that*

$$K(\zeta, \omega)[a] = \langle v(\zeta), a \cdot v(\omega) \rangle.$$

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Ball, Biswas, Fang and ter Horst have done work along this line. Our approach is similar in some respects, but adds another perspective.

The Szegö Kernel

Definition

If σ is a normal representation of M on the Hilbert space H_σ, then the kernel $K_S : \mathbb{D}(E,\sigma) \times \mathbb{D}(E,\sigma) \to \mathscr{B}(\sigma(M)', B(H_\sigma))$, defined by the formula

$$K_S(\mathfrak{z},\mathfrak{w}) = (\imath - \theta_{\mathfrak{z},\mathfrak{w}})^{-1},$$

where $\imath$ denotes the identity map on $\sigma(M)'$ and where $\theta_{\mathfrak{z},\mathfrak{w}}(a) = \mathfrak{z}(I_E \otimes a)\mathfrak{w}^*$, $a \in \sigma(M)'$, is called the *Szegö kernel* for $\mathbb{D}(E,\sigma)$.

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Observation

$\theta_{\mathfrak{z},\mathfrak{w}}$ takes its values in $\sigma(M)'$ so K_S is makes sense.

The σ-dual

Definition

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$$a \cdot \xi \cdot b := (I_E \otimes a)\xi b$$

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$\theta_{\mathfrak{z},\mathfrak{w}}(a) = \langle \mathfrak{z}^*, a \cdot \mathfrak{w}^* \rangle$, $a \in \sigma(M)'$, $\mathfrak{z}, \mathfrak{w} \in \mathbb{D}(E, \sigma)$.

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The Dual Transform

Lemma

Define $U : \mathscr{F}(E^{\sigma}) \to \mathscr{F}(E)^{\sigma}$ *by*

$$U(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) := (I_{E^{\otimes(n-1)}} \otimes \xi_1)(I_{E^{\otimes(n-2)}} \otimes \xi_2) \cdots (I_E \otimes \xi_{n-1})\xi_n.$$

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Note, in particular, that for $\xi \in \mathscr{F}(E^\sigma)$,

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Remark

$\mathscr{F}(E)^\sigma \subseteq B(H_\sigma, \mathscr{F}(E) \otimes_\sigma H_\sigma)$ *- a* $B(\mathscr{F}(E) \otimes_\sigma H_\sigma), B(H_\sigma)$*-correspondence with inner product* $\langle \Xi_1, \Xi_2 \rangle := \Xi_1^* \Xi_2$.

The Cauchy Kernel

Definition

For $\mathfrak{z} \in \mathbb{D}(E, \sigma)$ the Cauchy kernel, $c_{\mathfrak{z}}$, is defined by

$$c_{\mathfrak{z}} := (I_{H_\sigma} \oplus \mathfrak{z}^* \oplus \mathfrak{z}^{*(2)} \oplus \mathfrak{z}^{*(3)} \oplus \cdots).$$

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$c_{\mathfrak{z}}$ is an operator from H_σ to $\mathscr{F}(E) \otimes_\sigma H_\sigma$ with $\|c_{\mathfrak{z}}\| \leq (1 - \|\mathfrak{z}\|)^{-1}$.

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Observation

$c_{\mathfrak{z}}$ is an operator from H_σ to $\mathscr{F}(E) \otimes_\sigma H_\sigma$ with $\|c_{\mathfrak{z}}\| \leq (1 - \|\mathfrak{z}\|)^{-1}$. If $M = E = \mathbb{C}$ and if σ is the 1-dimensional representation of $\mathbb{C}$ on $\mathbb{C} = H_\sigma$, then $\mathscr{F}(E)^\sigma$ may be identified with $\ell^2(\mathbb{Z}_+)$ and for $\xi \in H^2$, $\xi(\mathfrak{z}) = \langle c_{\mathfrak{z}}, \check{\xi} \rangle$, where $\check{\xi}$ is the sequence of Taylor coefficients of ξ.

The Szegö Correspondence

Theorem

For $a \in \sigma(M)'$ *and* $\mathfrak{z}, \mathfrak{w} \in \mathbb{D}(E, \sigma)$,

$$K_S(\mathfrak{z}, \mathfrak{w})(a) = \langle c_{\mathfrak{z}}, a \cdot c_{\mathfrak{w}} \rangle$$

where the inner product is in $\mathscr{F}(E)^\sigma$.

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where the inner product is in $\mathscr{F}(E)^{\sigma}$. *Consequently,* K_S *is c.p.d., and the Kolmogorov decomposition of* K_S *may be realized in* $B(H_\sigma, \mathscr{F}(E) \otimes_\sigma H_\sigma)$ *as the ultra-weakly closed* $\sigma(M)', B(H_\sigma)$*-bimodule,* $\mathfrak{F}(E,\sigma)$*, generated by* $\{c_{\mathfrak{z}} \mid \mathfrak{z} \in \mathbb{D}(E,\sigma)\}$.

The Proof

Proof.

$$K_S(\mathfrak{z},\mathfrak{w})[a] = (\iota - \theta_{\mathfrak{z},\mathfrak{w}})^{-1}[a] = \sum_{n=0}^{\infty} \theta_{\mathfrak{z},\mathfrak{w}}^{n}[a]$$

$$= \sum_{n=0}^{\infty} \underbrace{\theta_{\mathfrak{z},\mathfrak{w}} \circ \theta_{\mathfrak{z},\mathfrak{w}} \circ \cdots \circ \theta_{\mathfrak{z},\mathfrak{w}}}_{n}[a] = \sum_{n=0}^{\infty} \langle \mathfrak{z}^{*(n)}, a \cdot \mathfrak{w}^{*(n)} \rangle = \langle c_{\mathfrak{z}}, a \cdot c_{\mathfrak{w}} \rangle$$

□

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- $\widehat{\xi}$ is an analytic $B(H_\sigma)$-valued function on $\mathbb{D}(E,\sigma)$.
- We *define* $\mathbb{H}^2(E,\sigma) := \{\widehat{\xi} \mid \xi \in \mathfrak{F}(E,\sigma)\}$ and give $\mathbb{H}^2(E,\sigma)$ the structure of a $\sigma(M)', B(H_\sigma)$-correspondence, so that $\xi \to \widehat{\xi}$ is isometric.

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Example

Let $M = \mathbb{C}$, let $E = \mathbb{C}^d$, and let $\sigma : \mathbb{C} \to \mathbb{C}$ be the one dimensional representation.

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Multipliers

Definition

A function $\Phi : \mathbb{D}(E, \sigma) \to B(H_\sigma)$ is a *multiplier* of $\mathbb{H}^2(E, \sigma)$ if and only if

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for all $\xi \in \mathbb{H}^2(E, \sigma)$.

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Proposition

$\mathfrak{M}(E,\sigma) \subseteq \mathscr{L}(\mathbb{H}^2(E,\sigma))$ and $X \in \mathscr{L}(\mathbb{H}^2(E,\sigma))$ is a multiplier given by $\Phi \in \mathfrak{M}(E,\sigma)$ iff $X^*\widehat{c}_{\mathfrak{z}} = \widehat{c}_{\mathfrak{z}}\Phi(\mathfrak{z})^*$, for all $\mathfrak{z} \in \mathbb{D}(E,\sigma)$.

Functional Representation

Since $\mathbb{H}^2(E,\sigma)_{B(H_\sigma)}$ is isomorphic to $\mathfrak{F}(E,\sigma)$ - an ultraweakly closed W^*-submodule of $B(H_\sigma, \mathscr{F}(E)\otimes_\sigma H_\sigma)_{B(H_\sigma)}$, there is a $\sigma(M)', B(H_\sigma)$-correspondence coisometry V from $B(H_\sigma, \mathscr{F}(E)\otimes_\sigma H_\sigma)_{B(H_\sigma)}$ onto $\mathbb{H}^2(E,\sigma)$.

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Theorem

Define $\rho : \mathscr{L}(\mathscr{F}(E)) \to \mathscr{L}(B(H_\sigma, \mathscr{F}(E)\otimes_\sigma H_\sigma)_{B(H_\sigma)})$ *by* $\rho(A)\Xi = (A\otimes I)\Xi$ *and define* $\pi : H^\infty(E) \to \mathscr{L}(\mathbb{H}^2(E,\sigma))$ *by* $\pi(F) := V\rho(F)V^*$.

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1. *ρ is a normal $*$-representation.*
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3. *π is an ultraweakly continuous c.c. representation of $H^\infty(E)$ on $\mathbb{H}^2(E,\sigma)$.*
4. *$\pi(F)$ is a multiplication operator with symbol $\widehat{F_\sigma}$.*

Functional Representation

Since $\mathbb{H}^2(E,\sigma)_{B(H_\sigma)}$ is isomorphic to $\mathfrak{F}(E,\sigma)$ - an ultraweakly closed W^*-submodule of $B(H_\sigma, \mathscr{F}(E)\otimes_\sigma H_\sigma)_{B(H_\sigma)}$, there is a $\sigma(M)', B(H_\sigma)$-correspondence coisometry V from $B(H_\sigma, \mathscr{F}(E)\otimes_\sigma H_\sigma)_{B(H_\sigma)}$ onto $\mathbb{H}^2(E,\sigma)$.

Theorem

Define $\rho : \mathscr{L}(\mathscr{F}(E)) \to \mathscr{L}(B(H_\sigma, \mathscr{F}(E)\otimes_\sigma H_\sigma)_{B(H_\sigma)})$ by $\rho(A)\Xi = (A\otimes I)\Xi$ and define $\pi : H^\infty(E) \to \mathscr{L}(\mathbb{H}^2(E,\sigma))$ by $\pi(F) := V\rho(F)V^*$. Then

1. *ρ is a normal $*$-representation.*
2. *For $F \in H^\infty(E)$, $\rho(F)^*$ leaves $\mathfrak{F}(E,\sigma)$ invariant.*
3. *π is an ultraweakly continuous c.c. representation of $H^\infty(E)$ on $\mathbb{H}^2(E,\sigma)$.*
4. *$\pi(F)$ is a multiplication operator with symbol $\widehat{F_\sigma}$.*

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Theorem (M - Solel, Doc. Math. 13 (2008))

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is a completely positive definite kernel on $\mathbb{D}(E,\sigma)\times\mathbb{D}(E,\sigma)$ *with values in* $\mathscr{B}(\sigma(M)', B(H_\sigma))$*, where, in general, for any Hilbert space* H *and any* $a,b\in B(H)$*,* $Ad(a,b) : B(H)\to B(H)$ *is defined by* $Ad(a,b)(X) := aXb^*$.

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The Quotient

Theorem (Gurevich-Meyer)

Let $J = \bigcap_{\mathfrak{z}\in\mathbb{D}(E,\sigma)} \ker(\mathfrak{z}\times\sigma)$. Then J is an ultra-weakly closed ideal in $H^\infty(E)$ and π passes to an ultra-weakly continuous, completely isometric representation of $H^\infty(E)/J$ in $\mathscr{L}(\mathbb{H}^2(E,\sigma))$. Its image is $\mathfrak{M}(E,\sigma)$;

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Gurevich and Meyer show how to represent $H^\infty(E)/J$ completely isometrically on Hilbert space *a la* Davidson and Pitts, for any homogeneous ultraweakly closed two-sided ideal in $H^\infty(E)$. Viselter has similar results for ideals in $\mathscr{T}_+(E)$ using subproduct systems.

Functional Models

Definition

For $\mathfrak{z} \in \mathbb{D}(E, \sigma)$, let $\Delta_*(\mathfrak{z}) = I_{\mathscr{F}(E)} \otimes (I_{H_\sigma} - \mathfrak{z}\mathfrak{z}^*)^{\frac{1}{2}}$. The *Poisson Kernel* (evaluated at $\mathfrak{z}$) is

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$$K(\mathfrak{z})^* \rho(F) K(\mathfrak{z}) = K(\mathfrak{z})^* \pi(F) K(\mathfrak{z}) = \widehat{F_\sigma}(\mathfrak{z}) = \mathfrak{z} \times \sigma(F).$$

Morita Equivalence for Correspondences

Definition

Suppose (E_i, M_i) are W^*-correspondences, $i = 1, 2$. We say that E_1 is (strongly) Morita equivalent to E_2 in case there is a (strong) Morita equivalence bimodule $\mathscr{X} = {}_{M_1}\mathscr{X}_{M_2}$ such that $E_1 \otimes_{M_1} \mathscr{X} \simeq \mathscr{X} \otimes_{M_2} E_2$.

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Theorem (M-Solel, Proc. London Math. Soc. 81 (2000))

Suppose (E_1, M_1) and (E_2, M_2) are S.M.E. via $\mathscr{X} = {}_{M_1}\mathscr{X}_{M_2}$ and that $W : E_1 \otimes_{M_1} \mathscr{X} \to \mathscr{X} \otimes_{M_2} E_2$ is a correspondence isomorphism. Then W extends to $\widetilde{W} : \mathscr{F}(E_1) \otimes_{M_1} \mathscr{X} \to \mathscr{X} \otimes_{M_2} \mathscr{F}(E_2)$ so that if $X := \mathscr{F}(E_1) \otimes_{M_1} \mathscr{X}$ and $Y := \mathscr{F}(E_2) \otimes_{M_2} \mathscr{X}^*$, then $(\mathscr{T}_+(E_1), \mathscr{T}_+(E_2), X, Y, (\cdot,\cdot), [\cdot,\cdot])$ is a Morita context in the sense of Blecher-M-Paulsen, where $(\cdot,\cdot)$ and $[\cdot,\cdot]$ are implemented via $\widetilde{W}$.

Morita Equivalence Pairings

$$
\begin{aligned}
X \otimes_{h\mathcal{T}_+(E_2)} Y &= (\mathscr{F}(E_1) \otimes_{M_1} \mathscr{X}) \otimes_{h\mathcal{T}_+(E_2)} (\mathscr{F}(E_2) \otimes_{M_2} \mathscr{X}^*) \\
&\xrightarrow{\widetilde{W}} (\mathscr{X} \otimes_{M_2} \mathscr{F}(E_2)) \otimes_{h\mathcal{T}_+(E_2)} (\mathscr{F}(E_2) \otimes_{M_2} \mathscr{X}^*) \\
&\to \mathscr{X} \otimes_{hM_2} \mathcal{T}_+(E_2) \otimes_{hM_2} \mathscr{X}^* \\
&\to \mathcal{T}_+(E_1)
\end{aligned}
$$

Morita Equivalence and Discs

Theorem (M - Solel, NYJM 17a (2011))

Suppose (E_1, M_1) and (E_2, M_2) are S.M.E. via $\mathscr{X} = {}_{M_1}\mathscr{X}_{M_2}$ and that $W : E_1 \otimes_{M_1} \mathscr{X} \to \mathscr{X} \otimes_{M_2} E_2$ is a correspondence isomorphism. Given $\sigma : M_2 \to B(H_\sigma)$, let $\sigma^{\mathscr{X}} : M_1 \to B(\mathscr{X} \otimes_\sigma H_\sigma)$ be the induced representation of M_1. Then, for $\mathfrak{z} \in \mathbb{D}(E_2, \sigma)$, the map $\mathfrak{z}^{\mathscr{X}} := (I_{\mathscr{X}} \otimes \mathfrak{z})(W \otimes I_{H_\sigma})$ lies in $\mathbb{D}(E_1, \sigma^{\mathscr{X}})$ and the map $\mathfrak{z} \to \mathfrak{z}^{\mathscr{X}}$ is an isometry mapping $\mathbb{D}(E_2, \sigma)$ onto $\mathbb{D}(E_1, \sigma^{\mathscr{X}})$.

Morita Equivalence and Multipliers

Theorem

Suppose (E_1, M_1) and (E_2, M_2) are S.M.E. via $\mathscr{X} = {}_{M_1}\mathscr{X}_{M_2}$, then for every normal representation σ of M_2, $\mathfrak{M}_c(E_2, \sigma)$ and $\mathfrak{M}_c(E_1, \sigma^{\mathscr{X}})$ are Morita equivalent in the sense of Blecher-M-Paulsen.

Stabilization

Example

$(\mathbb{C}^d, \mathbb{C})$ is S.M.E. to $({}_\alpha B(H), B(H))$ where $H := \ell^2(\mathbb{Z}_+)$, and α is the endomorphism of $B(H)$ determined by a d-tuple of Cuntz isometries, $\{S_i\}_{i=1}^d$.

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$(\mathbb{C}^d, \mathbb{C})$ is S.M.E. to $({}_\alpha B(H), B(H))$ where $H := \ell^2(\mathbb{Z}_+)$, and α is the endomorphism of $B(H)$ determined by a d-tuple of Cuntz isometries, $\{S_i\}_{i=1}^d$. Indeed, $\mathbf{C}_d(B(H)) \otimes_{B(H)} \mathbf{C}_\infty \simeq \mathbf{C}_\infty \otimes_{\mathbb{C}} \mathbb{C}^d$, where $\mathbf{C}_\infty = \ell^2(\mathbb{Z}_+)$ viewed as $B(\mathbb{C}, \ell^2(\mathbb{Z}_+))$, $\mathbf{C}_d(B(H))$ is the space of $d \times 1$ block matrices with entries in $B(H)$, and $V : \mathbf{C}_d(B(H)) \to {}_\alpha B(H)$ is the correspondence isomorphism defined by

$$V\left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{bmatrix}\right) := \sum_{i=1}^d S_i a_i.$$

Free Algebras and Analytic Crossed Products

Theorem (M - Solel, NYJM 17a (2011))

$\mathscr{T}_+(\mathbb{C}^d)$ and $\mathscr{T}_+({}_\alpha B(H))$ are Morita equivalent, where α is the endomorphism induced by a d-tuple of Cuntz isometries.

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$\mathscr{T}_+(\mathbb{C}^d)$ and $\mathscr{T}_+({}_\alpha B(H))$ are Morita equivalent, where α is the endomorphism induced by a d-tuple of Cuntz isometries. This equivalence induces Morita equivalences between appropriate pairs of multiplier algebras.