Scattering theory of Sturm-Liouville operator and a theory of vessels

FUNCTION THEORY AND OPERATOR THEORY: INFINITE DIMENSIONAL AND FREE SETTINGS, Beer Sheva 2011

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Classical Scattering theory on the half line.

Consider the following differential equation with the spectral parameter λ, defined on an interval $\mathcal{I}$, where $q(x)$ is called potential

$$-\frac{d^2}{dx^2}y(x) + q(x)y(x) = -i\lambda y(x), \quad -i\lambda = s^2 \tag{1}$$

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It was studied by C. Sturm [Stu36], R. Liouville [Lio95] in connection to dynamics, heat equation. Using monodromy preserving deformation problem of Linear Differential Equations (LDE) by L. Schlezinger [Sch08], R. Fuchs [Fuc07] and Garnier [Fuc12]. Using Riemann transformations by Marchenko [Mar] and using the scattering theory by Lax–Phillips [LxPh], Gelfand-Levitan [GL].

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Under condition $\int_0^\infty x|q(x)|dx < \infty$ [F] introduce Jost solutions

$$\phi(x,s) \quad : \quad \phi(0,s) = 0, \quad \phi'(0,s) = 1, \tag{2}$$

$$f(x,s) \quad : \quad \lim_{x\to\infty} e^{-isx} f(x,s) = 1. \tag{3}$$

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Define $M(s) = \phi'(x,s)f(x,s) - f'(x,s)\phi(x,s)$ and

$$\Omega(x,y) = 2/\pi \int_0^\infty \frac{\sin(kx)}{k}[\frac{1}{M(k)M(-k)} - 1]\frac{\sin(ky)}{k} k^2 dk.$$

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Solve the Gelfand-Levitan equation [F, (8.5)]

$$K(x,y) + \Omega(x,y) + \int_0^x K(x,t)\Omega(t,y)dt = 0, \quad x > y. \tag{4}$$

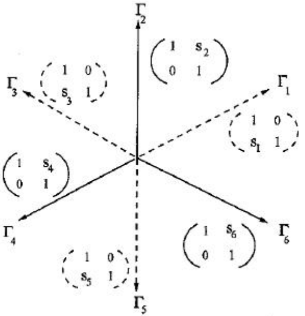
from where $q(x) = 2\frac{d}{dx}K(x,x)$.

Airy equation.

For the solution of $u_{xx} = xu$, Consider the collection

$$\Gamma_k = \{\lambda \mid \arg\lambda = \frac{2k-1}{6}\pi, \}$$

oriented towards infinity.



Let $\Gamma = \Gamma_2 \cup \Gamma_4 \cup \Gamma_6$. The classical solution of the Airy equation is

$$u(x) = \frac{i}{\pi}\{s_2 \int_{\Gamma_2} + s_4 \int_{\Gamma_4} + s_6 \int_{\Gamma_6}\} e^{-\frac{8i}{3}\lambda^3 - 2ix\lambda} d\lambda,$$

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where $s_2 + s_4 + s_6 = 0$. On the other hand, $u(x) = 2 \lim\limits_{\lambda\to\infty} \lambda Y_{12}(\lambda)$, where $Y(\lambda, x)$ is the solution of the **abelian** RH problem with contour Γ and $G(\lambda)$ given by

$$G(\lambda) = G(\lambda, x) = \begin{bmatrix} 1 & s_k e^{-\frac{8i}{3}\lambda^3 - 2ix\lambda} \\ 0 & 1 \end{bmatrix}, \quad \lambda \in \Gamma_k.$$

Second Painlevé equation

Augment the contour Γ by $\Gamma_1, \Gamma_3, \Gamma_5$ and jump matrix there:

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Add the following cyclic relations

$$s_{k+3} = -s_k, k = 1, 2, 3; \quad s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0. \qquad (5)$$

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Proof of this fact is implicit: Bolibruch, Its, Kapaev [BIK].

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Correspondence is implemented through Gelfand-Levitan equation (1.) or Riemann Hilbert problem (2.,3.). Proof of (3.) involves completely integrable systems (Lax Pair).

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3. Vessels with prescribed singularities (on curves)

Overdetermined 2D systems and their transfer functions

$2D$ systems, invariant in one direction (Phd thesis of A. M., V. Vinnikov, 2009 [M])

Overdetermined t_1-invariant $2D$ system is a linear input-state-output (i/s/o) system of the form

$$\Sigma : \begin{cases} \frac{\partial}{\partial t_1} x(t_1,t_2) = A_1(t_2)x(t_1,t_2) + B(t_2)\sigma_1(t_2)u(t_1,t_2) \\ \frac{\partial}{\partial t_2} x(t_1,t_2) = A_2(t_2)x(t_1,t_2) + B(t_2)\sigma_2(t_2)u(t_1,t_2) \\ y(t_1,t_2) = u(t_1,t_2) - B^*(t_2)x(t_1,t_2) \end{cases} \quad (7)$$

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where for some Hilbert spaces $\mathcal{H}, \mathcal{E}$

$$\begin{array}{ll} A_1(t_2), A_2(t_2) & : \mathcal{H} \to \mathcal{H}, \quad B(t_2) \quad : \mathcal{E} \to \mathcal{H}, \\ \sigma_1(t_2), \sigma_2(t_2) & : \mathcal{E} \to \mathcal{E} \end{array}$$

are (bounded or not) operators. $u(t_1,t_2) \in \mathcal{E}$ and $y(t_1,t_2) \in \mathcal{E}$ are called the *input* and the *output*, $x(t_1,t_2) \in \mathcal{H}$ is called the *state*.

Demanding 1. complete integrability:

$$\frac{\partial}{\partial t_1}\left(\frac{\partial}{\partial t_2}x(t_1,t_2)\right) = \frac{\partial}{\partial t_2}\left(\frac{\partial}{\partial t_1}x(t_1,t_2)\right). \qquad (8)$$

2. Mapping of the input $u(t_1,t_2)$

$$\sigma_2(t_2)\frac{\partial}{\partial t_1}u - \sigma_1(t_2)\frac{\partial}{\partial t_2}u + \gamma(t_2)u = 0. \qquad (9)$$

to the output $y(t_1,t_2)$

$$\sigma_2(t_2)\frac{\partial}{\partial t_1}y - \sigma_1(t_2)\frac{\partial}{\partial t_2}y + \gamma_*(t_2)y = 0 \qquad (10)$$

for some $\gamma(t_2), \gamma_*(t_2) : \mathcal{E} \to \mathcal{E}$.

3. Energy balances

$$\frac{\partial}{\partial t_1} < x, x > + < \sigma_1 y, y > = < \sigma_1 u, u >,$$
$$\frac{\partial}{\partial t_2} < x, x > + < \sigma_2 y, y > = < \sigma_2 u, u >$$

Overdetermined 2D systems and their transfer functions

t_1-invariant vessel, A.M.-V. Vinnikov [MV1, MVc]

A t_1-**invariant conservative vessel** as a collection of operators and spaces:

$$\mathfrak{V} = (A_1, A_2, B; \sigma_1, \sigma_2, \gamma, \gamma_*; \mathcal{H}, \mathcal{E})$$

which are all operator-functions of t_2 and satisfy certain regularity assumptions and the following axioms:

$$\begin{aligned}
&\tfrac{d}{dt_2} A_1 = A_2 A_1 - A_1 A_2,\\
&A_1 + A_1^* + B\sigma_1 B^* = 0,\\
&A_2 + A_2^* + B\sigma_2 B^* = 0,\\
&\tfrac{d}{dt_2}(B\sigma_1) - A_2 B\sigma_1 + A_1 B\sigma_2 + B\gamma = 0,\\
&\tfrac{d}{dt_2}(\sigma_1 B^*) + \sigma_1 B^* A_2 - \sigma_2 B^* A_1 - (\gamma_* + \tfrac{d}{dt_2}\sigma_1) B^* = 0,\\
&\gamma = \sigma_2 B^* \widetilde{B} \sigma_1 - \sigma_1 B^* \widetilde{B} \sigma_2 + \gamma_*.
\end{aligned}$$

Remarks: 1. The first equation is the Lax equation, which plays an important role in completely integrable non-linear PDEs. It follows from the Lax equation that the spectrum of $A_1(t_2)$ is independent of t_2. Defining the fundamental solution

$$\frac{d}{dt_2}F(t_2,t_2^0)=A_2(t_2)F(t_2,t_2^0), \qquad F(t_2^0,t_2^0)=I,$$

we obtain

$$A_1(t_2)=F(t_2,t_2^0)A_1(t_2^0)F(t_2,t_2^0)^{-1}. \tag{11}$$

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2. This object is interesting, because it is time varying on the one hand, but has all the advantages of the time-invariant case on the other hand: transfer function, functional model.

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2. This object is interesting, because it is time varying on the one hand, but has all the advantages of the time-invariant case on the other hand: transfer function, functional model.

3. We shall always assume that $\sigma_1(t_2)$ is invertible for all t_2.

Overdetermined 2D systems and their transfer functions

Frequency domain analysis

Performing a partial separation of variables for the system (7),

$$u(t_1, t_2) = u_\lambda(t_2) e^{\lambda t_1},$$
$$x(t_1, t_2) = x_\lambda(t_2) e^{\lambda t_1},$$
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Compatibility PDEs for $u(t_1,t_2), y(t_1,t_2)$ become ODEs for $u_\lambda(t_2), y_\lambda(t_2)$ with the spectral parameter λ,

$$\lambda\sigma_2(t_2)u_\lambda(t_2) - \sigma_1(t_2)\frac{\partial}{\partial t_2}u_\lambda(t_2) + \gamma(t_2)u_\lambda(t_2) = 0,$$
$$\lambda\sigma_2(t_2)y_\lambda(t_2) - \sigma_1(t_2)\frac{\partial}{\partial t_2}y_\lambda(t_2) + \gamma_*(t_2)y_\lambda(t_2) = 0.$$

The corresponding i/s/o system becomes

$$\begin{cases} \lambda x_\lambda(t_2) = A_1(t_2) x_\lambda(t_2) + B(t_2)\sigma_1(t_2) u_\lambda(t_2) \\ \frac{d}{dt_2} x_\lambda(t_2) = A_2(t_2) x_\lambda(t_2) + B(t_2)\sigma_2(t_2) u_\lambda(t_2). \end{cases}$$

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The output $y_\lambda(t_2) = u_\lambda(t_2) - B^*(t_2)x_\lambda(t_2)$ may be found from the first i/s/o equation:

$$y_\lambda(t_2) = S(\lambda, t_2)u_\lambda(t_2),$$

using the **transfer function**

$$S(\lambda, t_2) = I - B^*(t_2)(\lambda I - A_1(t_2))^{-1}B(t_2)\sigma_1(t_2). \qquad (12)$$

It turns out (A. M. 2011 [MSL]) that under certain conditions, one can obtain a simplified form. Defining $\mathbb{X}^2(t_2) = F(t_2, t_2^0)F^*(t_2, t_2^0)$, and using transformation of the inner space, one can obtain that A_1 is constant and $A_2 = 0$. As a result there is obtained a simplified form of a vessel

$$\mathfrak{V} = (A, \mathbb{X}(t_2) = \mathbb{X}^*(t_2), B(t_2); \sigma_1, \sigma_2, \gamma, \gamma_*; \mathcal{H}, \mathcal{E})$$

where operators satisfy

$$0 = \frac{d}{dt_2}(B(t_2)\sigma_1(t_2)) + AB(t_2)\sigma_2(t_2) + B(t_2)\gamma(t_2), \qquad (13)$$

$$A\mathbb{X}(t_2) + \mathbb{X}(t_2)A^* + B(t_2)\sigma_1(t_2)B^*(t_2) = 0, \qquad (14)$$

$$\frac{d}{dt_2}\mathbb{X}(t_2) = B(t_2)\sigma_2(t_2)B^*(t_2), \qquad (15)$$

$$\gamma_*(t_2) = \gamma(t_2) + \sigma_2(t_2)B^*(t_2)\mathbb{X}^{-1}(t_2)B(t_2)\sigma_1(t_2) - \sigma_1(t_2)B^*(t_2)\mathbb{X}^{-1}(t_2)B(t_2)\sigma_2(t_2) \qquad (16)$$

Review of a simplified vessel

Definition

A vessel in a simplified form is a collection:

$$\mathfrak{K}_{\mathfrak{Y}} = (A, B(t_2), \mathbb{X}(t_2); \sigma_1, \sigma_2, \gamma, \gamma_*(t_2); \mathcal{H}, \mathcal{E}, \mathrm{I} = [a, b]), \qquad (17)$$

where $A, \mathbb{X}(t_2) = \mathbb{X}(t_2)^* : \mathcal{H} \to \mathcal{H}$, $B(t_2) : \mathcal{H} \to \mathcal{E}$ *bounded,* $\mathbb{X}(t_2)$ *invertible on* I

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$$0 = \frac{d}{dx}(B(t_2)\sigma_1) + AB(t_2)\sigma_2 + B(t_2)\gamma, \quad (18)$$

$$A\mathbb{X}(t_2) + \mathbb{X}(t_2)A^* + B(t_2)\sigma_1 B(t_2)^* = 0, \quad (19)$$

$$\frac{d}{dx}\mathbb{X}(t_2) = B(t_2)\sigma_2 B(t_2)^*, \quad (20)$$

$$\gamma_*(t_2) = \gamma + \sigma_1 B(t_2)^* \mathbb{X}^{-1}(t_2) B(t_2)\sigma_2 - \sigma_2 B(t_2)^* \mathbb{X}^{-1}(t_2) B(t_2)\sigma_1 \quad (21)$$

Review of a simplified vessel (continued)

The vessel is associated to the completely integrable system

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And the transfer function

$$S(\lambda, t_2) = I - B^*(t_2)\mathbb{X}^{-1}(t_2)(\lambda I - A_1)^{-1} B(t_2)\sigma_1(t_2)$$

maps solutions $u_\lambda(t_2)$ to $y_\lambda(t_2)$ $(= S(\lambda, t_2)u_\lambda(t_2))$:

$$\lambda\sigma_2(t_2)u_\lambda(t_2) - \sigma_1(t_2)\frac{\partial}{\partial t_2}u_\lambda(t_2) + \gamma(t_2)u_\lambda(t_2) = 0,$$
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Construction of a vessel

Starting from a function

$$S(\lambda, t_2^0) = I - B_0{}^* \mathbb{X}_0{}^{-1} (\lambda I - A)^{-1} B_0 \sigma_1,$$

fir which $\mathbb{X}_0{}^* = \mathbb{X}_0$ and Lyapunov equation $A\mathbb{X}_0 + \mathbb{X}_0 A^* + B_0 \sigma_1 B_0{}^* = 0$ holds, we solve first (18)

$$0 = \frac{d}{dx}(B(t_2)\sigma_1) + AB(t_2)\sigma_2 + B(t_2)\gamma, \quad B(x_0) = B_0.$$

Then we solve (20)

$$\frac{d}{dx}\mathbb{X}(t_2) = B(t_2)\sigma_2 B(t_2)^*, \quad \mathbb{X}(t_2) = \mathbb{X}_0$$

and define $\gamma_*(t_2)$ from γ using (21). Thus a vessel is created.

Sturm Liouville vessel parameters

Since variable t_1 disappeared in the equations, we use x from now on for t_2.

Definition

Sturm Liouville parameters are

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \gamma = \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix},$$
$$\gamma_*(x) = \begin{bmatrix} -i(\beta'(x) - \beta^2(x)) & -\beta(x) \\ \beta(x) & i \end{bmatrix}$$

for a real valued differentiable function $\beta(x)$, defined on an interval I.

Transfer function of a SL vessel

$$S(\lambda, x) = I - B(x)^* \mathbb{X}^{-1}(x)(\lambda I - A)^{-1} B(x)\sigma_1$$

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$$\begin{cases} -\frac{\partial^2}{\partial x^2} u_1(\lambda, x) = -i\lambda u_1(\lambda, x) \\ u_2(\lambda, x) = -i\frac{\partial}{\partial x} u_1(\lambda, x) \end{cases}$$

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to solutions $\begin{bmatrix} y_1(\lambda, x) \\ y_2(\lambda, x) \end{bmatrix} = S(\lambda, x) \begin{bmatrix} u_1(\lambda, x) \\ u_2(\lambda, x) \end{bmatrix}$ of

$$\begin{cases} -\frac{\partial^2}{\partial x^2} y_1(\lambda, x) + 2\beta'(x) y_1(\lambda, x) = -i\lambda y_1(\lambda, x) \\ y_2(\lambda, x) = -i[\frac{\partial}{\partial x} y_1(\lambda, x) - \beta(x) y_1(\lambda, x)]. \end{cases}$$

Construction of $S(\lambda, x_0)$ for a given potential

Fixing x_0, for which the potential $q(x)$ is locally integrable in a small neighborhood,

Main Theorem

There exists a vessel on $x_0 \in \mathrm{I}_0 \subseteq \mathrm{I}$ realizing this potential (i.e. $2\beta'(x) = q(x)$.

Gelfand-Levitan equation for vessels

Defining

$$\Omega(x,y) = \begin{bmatrix} 1 & 0 \end{bmatrix} B(x) B^*(y) \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$K(x,y) = -\begin{bmatrix} 1 & 0 \end{bmatrix} B^*(x) \mathbb{X}^{-1}(x) B(y) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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one finds that Gelfan-Levitan (4) equation holds and
$q(x) = 2\beta'(x) = 2\frac{d}{dx}K(x,x)$.

tau function

Definition

For a given realization

$$S(\lambda, x) = I - B^*(x)\mathbb{X}^{-1}(x)(\lambda I - A)^{-1}B(x)\sigma_1$$

tau function $\tau(x)$ is defined as

$$\tau = \det(\mathbb{X}^{-1}(x_0)\mathbb{X}(x)) \tag{23}$$

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Theorem

The following formula holds

$$q(x) = 2\beta'(x) = -2(\ln \tau(x))''$$

Let us choose a Jordan curve $\Gamma = -\Gamma^* = \{\mu(\ell) \mid \ell \in \mathrm{J}\}$ and define

$$\mathcal{H} = L^2(\Gamma) = \{f(\mu) \mid \int_{\mathrm{J}} |f(\mu(\ell)|^2 d\ell < \infty\}.$$

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Define the operator A as multiplication on μ: $Af(\mu) = -i\mu f(\mu)$. For a bounded interval it is a well defined operator. When Γ is unbounded, it is unbounded operator, with an obvious domain. Define $B(x)$ as a solution of (18)

$$0 = \frac{d}{dx}(B(x)\sigma_1) + AB(x)\sigma_2 + B(x)\gamma,$$

Then it turns out that (without loss of generality) $B(x) : \mathbb{C}^2 \to L^2(\Gamma)$ is an operator of multiplication on

$$B(\mu, x) = c(\mu) \begin{bmatrix} \frac{\sin(tx)}{t} & -i\cos(tx) \end{bmatrix}, \mu = it^2.$$

Finally, we can define

$$\mathbb{X}(x)f(\mu) = \int_{\mathrm{J}} \frac{B(\mu, x)\sigma_1 B^*(\delta, x)}{i(\mu - \delta^*)} f(\delta) d\ell$$

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$$\mathbb{X}(x)f(\mu) = \int_{\mathrm{J}} \frac{B(\mu,x)\sigma_1 B^*(\delta,x)}{i(\mu-\delta^*)} f(\delta)d\ell$$

and $\gamma_*(x)$ by (21)

$$\gamma_*(x) = \gamma + \sigma_1 B(x)^*\mathbb{X}^{-1}(x)B(x)\sigma_2 - \sigma_2 B(x)^*\mathbb{X}^{-1}(x)B(x)\sigma_1$$

Lemma

The collection

$$\mathfrak{K}_{\mathfrak{d}} = (A, B(\mu,x), \mathbb{X}(x); \sigma_1, \sigma_2, \gamma, \gamma_*(x), L^2(\Gamma), \mathbb{C}^2, \mathrm{I}),$$

is a vessel.

Choosing a vessel of this kind, one finds that

$$\begin{aligned}\int q(x) &= Tr(\mathbb{X}'(x)\mathbb{X}^{-1}) = Tr(B(x)\sigma_2 B^*(x)\mathbb{X}^{-1} = \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} B^*(x)\mathbb{X}^{-1}B(x)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \\ &= \int_{\mathrm{J}} c^*(\mu)\frac{\sin(tx)}{t}\mathbb{X}^{-1}(c(\mu)\frac{\sin(tx)}{t})d\ell\end{aligned}$$

Classical inverse scattering and its possible generalizations.

Construction of $S(\lambda, x_0)$ for a given $q(x)$ locally integrable potential on interval $[x_0, L]$ $(x_0 > 0)$, uses inverse scattering theory for the potential

$$\widetilde{q}(x) = \begin{cases} q(x), & x \in \mathrm{I}, \\ 0, & x \notin \mathrm{I}. \end{cases}$$

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In this case the obtained matrix $S(\lambda, x_0)$ and a corresponding vessel are defined on the curve $i\mathbb{R}_+$ (usually the whole curve).

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In this case the obtained matrix $S(\lambda, x_0)$ and a corresponding vessel are defined on the curve $i\mathbb{R}_+$ (usually the whole curve). The generalization of the classical inverse scattering theory will be created, by studying the properties of the potential, obtained for other curves.

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Thank you